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# Coordination of distributed model predictive controllers using price-driven coordination and sensitivity analysis

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**Abstract:** In this paper, a coordination control algorithm based on hierarchical scheme is presented to coordinate several non-linear model predictive controllers (NMPC) working in parallel, with an upper layer, where a price-driven coordination technique is used to drive the controllers in such a way that some global constraints are satisfied. To coordinate the lower layers, it is used a price-adjustment algorithm based on Newton's method, in which a reformulation of Fiacco's work is used in order to obtain the sensitivity analysis for a nonlinear system no matter the set of active constraints. The efficiency of the scheme is evaluated using a simulation of a four-tank benchmark.

*Keywords:* Dual composition control, Predictive control, Price-driven Coordination, Sensitivity analysis

## 1. INTRODUCTION

Distributed model predictive control (DMPC) has recently experienced a renewed interest because of the benefits that can be obtained when optimal decisions driven by economics or technical aims can be made in large scale or networked systems. Centralized solutions to these problems are quite often impossible to obtain and some kind of distribution in the computations of the control or optimization algorithm must be implemented taking into account, the interactions and dependencies among different parts of a process. An excellent review of these techniques can be seen in (Scattolini R., 2007). Essentially, the main goal of the DMPC is to overcome computational and communication limitations of centralized approaches, by splitting the control or optimization task among several controllers and coordinating their actions in such a way that the behaviour of the whole system approaches to the one that would be the same which has been obtained with a centralized approach.

There are many approaches to solve this problem. In hierarchical control, this is addressed modifying the set points. On the other hand, distributed control acts in the interchange of information, while price coordination methods modify the objective function to ensure that global constraints are fulfilled. In particular, in price coordination, several controllers, acting each one on a subset of the process, arrive to a global optimum using a market-like mechanism for coordination: each controller modifies the cost function that defines its control or optimization aims according to a set of prices assigned from an upper layer to the common process resources that represents the process interactions, as in (Jose and Ungar, 1998).

Several approaches based on price coordination methods have been proposed for optimization of large-scale and networked systems, being some of them based on price (Lagrangean) coordination methods or dual decomposition (Jose and Ungar, 1998) and (Cheng, R. et al, 2007). More recently (Marcos N. et al 2009) and (Negenborn et al, 2008), dual decomposition has been applied also in the distributed Model Predictive Control (MPC). Other approach is shown in (Marti et al, 2013): the assignment of prices is seen as a control problem using a non-iterative method, where the manipulated variables are the prices to be sent to the NMPC controllers.

In this paper, dual decomposition is used to implement a distributed control scheme for a nonlinear dynamical system. Secondly, to estimate the update policy of the prices, it has been used the reformulation of Fiacco's work presented by Biegler and co-workers (Ganesh and Biegler, 1987) in order to obtain the sensitivity of the optimal point with respect to the shadow prices without the need to take into account possible changes in the set of active constraints (this technique to calculate sensitivity has been used in RTO works). With this information the Lagrangean multipliers are updated using a full step Newton's method. Therefore, the paper presents a procedure where a set of NMPC controllers act independently on different parts of a process and the assignment of prices is based on a sensitivity analysis in the coordination layer. The scheme used is shown in Figure 1.

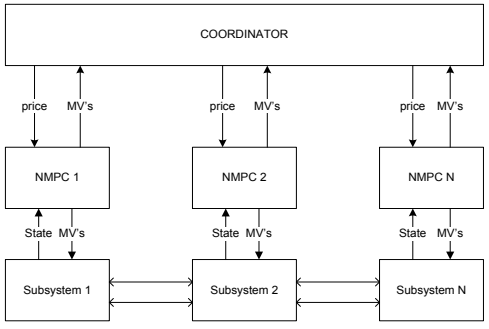


*Figure 1. Price coordination method scheme.*

The paper is organized as follows: in section 2 the price-driven coordination decomposition is explained. Section 3 describes the sensitivity method to estimate the derivative of the optimal point with respect to the prices. In section 4 the steps for a correct implementation of the proposed price-coordinated NMPC are presented. Section 5 shows the results when the algorithm is applied on a four-tank benchmark, and finally, section 6 summarizes the conclusions.

## 2. PRICE-DRIVEN COORDINATION

In this section, the generic formulation of the NMPC problem for large-scale systems with shared resources is shown. The optimization problem (1) can be summarized as an economic-control objective function subject to nonlinear models $h_i(.)$ and $g_i(.)$, corresponding to the dynamics of each subsystem and local constraints and a set of shared resource constraints or global constraints linking all subsystems. In (1), $n$ corresponds to the total number of subsystems, $N$ represents the number of shared resources, $RT_j$ is the availability of the shared resource $j$, and $R_{ji}(u_{ji})$ represents the consumption of the shared resource $j$ in each subsystem $i$, which is a function of the corresponding manipulated variable $u_{ji}$. For simplicity, in (1) all manipulated variables become part of the global constraints. In addition, each subsystem can have its own, independent, manipulated variables $v_i$.

$$\begin{aligned}
&\min_{\{u,v\}} J = \sum_{i=1}^{n} J_{local,i} \\
&\text{st :} \\
&\quad h_i(\dot{x}_i, x_i, v_i, u_{1i}, ..., u_{Ni}, t) = 0 \quad \forall\, i = 1,..,n \\
&\quad g_i(\dot{x}_i, x_i, v_i, u_{1i}, ..., u_{Ni}, t) \le 0 \quad \forall\, i = 1,..,n \\
&\quad \sum_{i=1}^{n} R_{ji}(u_{ji}) \le RT_j \quad \forall\, j = 1,..., N
\end{aligned} \tag{1}$$

The solution of the problem using a centralized architecture is complex if the number of manipulated variables or subsystems is large, which sometimes implies that the solution cannot be applied in real time due to the excess of computational time. On the other hand, the decentralized architecture approach is the most commonly used technique in the industry because of its simplicity of development and maintenance. However, when common resources are limited the decentralized approach, which does not take into account the behaviour of the entire system, may lead non optimal and sometimes infeasible solutions.

A compromise between centralized and decentralized architecture is therefore desired in order to improve the system performance. A hierarchical architecture based on price coordination provides a good compromise between performance and ease of implementation.

### 2.1 Lagrangean decomposition

The principle of the price coordination method consists in assigning prices $\boldsymbol{p}$ to the resources consumed in each individual subsystem, in such a way that each subsystem tries to optimize its objective function by accepting an amount of the resources at certain prices such that the global constraints are satisfied (Findeisen et al., 1980).

Supposing an optimization problem of the form (1) where a price coordination method is used, each subsystem $i$ can be decomposed into $n$ subproblems (2).

$$\begin{aligned}
&\min_{\{u_{1i},...,u_{Ni},v_i\}} J_i = J_{local,i} + \sum_{j=1}^{N} p_j R_{ji}(u_{ji}) \\
&\text{st :} \\
&\quad h_i(\dot{x}_i, x_i, v_i, u_{1i}, ..., u_{Ni}, t) = 0 \\
&\quad g_i(\dot{x}_i, x_i, v_i, u_{1i}, ..., u_{Ni}, t) \le 0
\end{aligned} \tag{2}$$

where $p_j$ represents the price or Lagrangean multipliers related to each share resource $j$. Provided the equations (3) are fulfilled, the outcomes of the decomposed problem (2) are equivalent to the ones from the initial global problem (1) (Jose and Ungar, 1998). That is to say, when the price meets (3), then the sum of the local optimal solutions is equal to the centralized optimal solution.

$$\begin{aligned}
&\sum_{i=1}^{n} R_{ji}(u_{ji}) - RT_j \le 0 \quad \forall\, j = 1,..., N \\
&p_j \left( \sum_{i=1}^{n} R_{ji}(u_{ji}) - RT_j \right) = 0 \quad \forall\, j = 1,..., N \\
&p_j \ge 0 \quad \forall\, j = 1,..., N
\end{aligned} \tag{3}$$

The role of the coordinator consists in updating the prices $p_j$ until equation (3) is satisfied. The system behaves in the same way as a market, where the prices $\boldsymbol{p}$ of goods are related to the behaviour of the consumers. As prices rise, supply will increase and demand will decrease. In the same way, as prices decrease, demand will increase.

Notice that, one benefit of this implementation is that the coordination-based MPC will not provide worse plant operations than the completely decentralized MPC without any kind of coordination (Cheng, R. et al, 2007).

### 2.2 Price updating policy

The technique selected to update the price vector is the same one proposed in Cheng's work (Cheng, R. et al, 2007). In this work, the authors presented an updating technique based on Newton's method (4).

$$\begin{aligned}
&\mathbf{p}_{[k+1]} = \mathbf{p}_{[k]} - \alpha \mathbf{Q}^{-1} \mathbf{T}_{[k]} \quad \mathbf{p}^T = [p1, p2, \ldots, p_N] \\
&\mathbf{T} = \left[ \sum_{i=1}^{n} R_{1i}(u_{1i}) - RT_1, ..., \sum_{i=1}^{n} R_{Ni}(u_{Ni}) - RT_{Ni} \right]^T
\end{aligned} \tag{4}$$

This technique is an iterative method, where the index $k$ denotes the iteration step, $\alpha$ is the step size in Newton's method and $\boldsymbol{Q}$ is the sensibility matrix with dimension N x N which can be calculated as:

$$\mathbf{Q} = \left. \frac{d \sum_{i=1}^{n} R_{ji}(u)}{d\mathbf{p}} \right|_{[k]} = \sum_{i=1}^{n} \left. \frac{dR_{ji}}{du_{ji}} \frac{du_{ji}}{d\mathbf{p}} \right|_{[k]} \tag{5}$$

In (5), information of sensitivity $d\boldsymbol{u}/d\boldsymbol{p}$ is needed to calculate $\boldsymbol{Q}$. In (Cheng, R. et al, 2007) a sensitivity analysis for an algebraic optimizations process and active set change identification was done. In our paper, it is used a reformulation of Fiacco work (Fiacco, 1983) as a QP problem (Ganesh and Biegler, 1987) in order to obtain the sensitivity analysis for a nonlinear system. This technique will be fully explained in the next point.

## 3. SENSITIVITY ANALYSIS

Let us consider the nonlinear programming (NLP) problem (6) for sensitivity analysis. The objective function $\phi$: $\Re^{n_z} \times \Re^{n_p} \to \Re$ and constraints $g(\cdot): \Re^{n_z} \times \Re^{n_p} \to \Re^{n_g}$.

$$\begin{aligned} &\min_{u} \varphi(\mathbf{u},\mathbf{p}) \\ &\text{s.t. } g(\mathbf{u},\mathbf{p}) \geq 0 \end{aligned} \tag{6}$$

The Lagrangean function of NLP problem (6) can be defined as $L(\mathbf{u},\mathbf{p},\boldsymbol{\lambda}) = \varphi(\mathbf{u},\mathbf{p}) - \boldsymbol{\lambda}^T g(\mathbf{u},\mathbf{p})$, where $\boldsymbol{\lambda} \in \Re^{n_g}$ is the vector of Lagrangean multipliers and $\boldsymbol{u}_0$, $\boldsymbol{\lambda}_0$ is the optimal solution of problem (6) at $\boldsymbol{p}_0$. At the optimal solution, the constraints $g$ are divided into active constraints $g^a$ and the inactive ones $g^{ina}$ of dimension $n_g$ and $n_g$ - $n_g{}^a$, respectively. The corresponding Lagrange multipliers $\boldsymbol{\lambda}_0$ are divided into $\boldsymbol{\lambda}_0{}^a$ and $\boldsymbol{\lambda}_0{}^{ina}$. Assuming that $\phi$ and $g$ are at least twice continuously differentiable in $\boldsymbol{u}$, the first order necessary conditions of optimally (NCO) of (6) are:

$$\begin{aligned} &L_u(\mathbf{u}_0,\mathbf{p}_0,\boldsymbol{\lambda}_0) = 0, \\ &g_i^a(\mathbf{u}_0,\mathbf{p}_0) = 0; \quad \lambda_{0,i}^a > 0; i = 1,...,n_g^a, \\ &g_j^{ina}(\mathbf{u}_0,\mathbf{p}_0) > 0; \quad \lambda_{0,i}^{ina} > 0; \quad j = 1,...,n_g^{ina}, \end{aligned} \tag{7}$$

Assuming that the NCO holds at $u_0$ and $\lambda_0$ with strict complementarity (i.e., $\lambda_{0,i}^a \neq 0, \quad i = 1...n_g^a$), the sensitivity of the optimal point with respect of the parameter $\boldsymbol{p}$ ($\partial\mathbf{u}/\partial\mathbf{p}$) can be obtained deriving equation (7). Under the assumption that the set of active constraints remains constant for a change in the parameter $\boldsymbol{p}=\boldsymbol{p}_0+\Delta\boldsymbol{p}$, the derivative of the NCO is reduced to solve the linear system from equation (8) (Fiacco, 1983), provided the functions $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{p})$ and $\boldsymbol{\lambda}^a = \boldsymbol{\lambda}^a(\boldsymbol{p})$ are at least once differentiable in $\boldsymbol{p}$.

$$\begin{bmatrix} L_{\mathbf{uu}} & -g_{\mathbf{u}}^a(\cdot) \\ g_{\mathbf{u}}^a(\cdot) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial\mathbf{u}}{\partial\mathbf{p}} \\ \frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{p}} \end{bmatrix} = -\begin{bmatrix} L_{\mathbf{up}}(\cdot) \\ g_{\mathbf{p}}^a(\cdot) \end{bmatrix} \tag{8}$$

### 3.1 Changing active set

The equation (8) can be used when a perturbation $\Delta\boldsymbol{p}$ does not cause a changing active set. Therefore, to estimate the optimal solution of (6) at $\boldsymbol{p}$ from $\boldsymbol{u}_0$, a full Newton step (4) can be implemented, if and only if, there is no active set change in each subproblem is identified (Ganesh and Biegler, 187) and (Cheng, R. et al, 2007).

For a moderate perturbation $\Delta\boldsymbol{p}$, the set of active constraints can change. This means that at the new optimal solution, some of the nominally active set constraints can become inactive and viceversa, implying that the estimate of $\boldsymbol{u}$ at $\boldsymbol{p}$ must be obtained using the sensitivity information that takes into account this change. Ganesh and Biegler reformulated equation (8) as a QP problem, to take into account the scenario of change in the active set. Equation (9) shows their generalization (Ganesh and Biegler, 1987).

$$\begin{aligned} &\min_{\Delta\mathbf{u}} \frac{1}{2}\Delta\mathbf{u}^T L_{\mathbf{uu}}\Delta\mathbf{u} + \Delta\mathbf{p}^T L_{\mathbf{up}}^T\Delta\mathbf{u} + \frac{\partial\varphi^T}{\partial\mathbf{u}}\Delta\mathbf{u} \\ &\text{s.t} \\ &g_{\mathbf{u}}(\cdot)\Delta\mathbf{u} \geq -g_{\mathbf{u}}(\cdot)\Delta\mathbf{p} - g(\cdot) \end{aligned} \tag{9}$$

Where all the functions are evaluated at $\boldsymbol{u}_0$, $\boldsymbol{p}_0$ and $\boldsymbol{\lambda}_0$. The solution of this QP problem (9) detects a new active set which is a better estimation of true active set. To use this method, the user has to provide as constant data $\Delta\boldsymbol{p}$ and as a result is obtained $\Delta\boldsymbol{u}$. Therefore, $\partial\boldsymbol{u}/\partial\boldsymbol{p}$ is approximated as $\Delta\mathbf{u}/\Delta\boldsymbol{p}$.

If the active set does not change, the solution of problem (8) and (9) are the same. Therefore, this technique is used to determinate sensitivity matrix which is needed in equation (5) for updating the prices (4).

## 4. IMPLEMENTATION PRICE-DRIVEN ALGORITHM

Implementation of the DCMPC coordinator is carried out each sampling time according to the following steps and Figure 2:

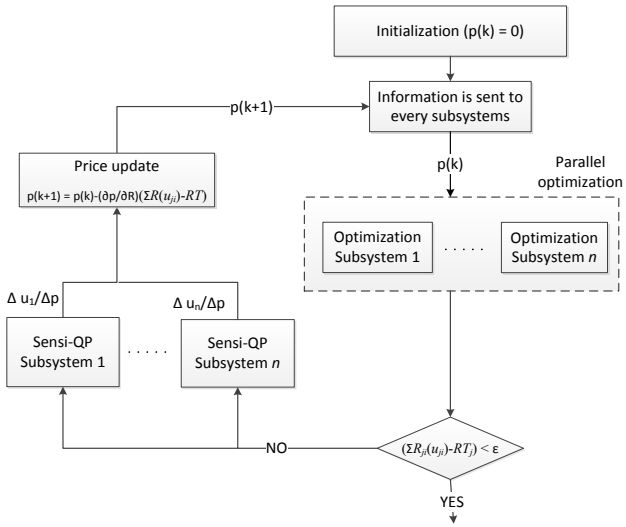


*Figure 2. Price-driven coordination algorithm with sensitivity analysis.*

**1. Initialization**: The coordination sets up an initial price vector $\boldsymbol{p}_{[k]}$ and sends that information to every subsystem.
**2. Optimization performed by each subsystem**: Based on the price given by the coordinator, each subsystem solves its own optimization problem and calculates the resource $R_i(u_{ij})$. In addition, each subsystem solves its QP problem in order to determine $\Delta\boldsymbol{u}/\Delta\boldsymbol{p}$ (9). This information is communicated back to the coordinator
**3. Price update**: The coordinator gathers the information from each subsystem, it calculates $\Sigma R_i(u_{ij})$-$R_T$ and $Q$ given by (5). Then, the coordinator updates the price vector $\boldsymbol{p}_k$ using equation (4). The new price vector is informed to each subsystem.
**4. Iteration until convergence**: Step (2)-(3) are repeated until convergence of the price-driven decomposition method. The convergence of the method is achieved when $(\Sigma R_{ji}(u_{ji})\text{-}RT_j) < \varepsilon$, being ε a tolerance error.

## 5. CASE OF STUDIES

In this section, two cases of studies are shown to illustrate the performance of the proposed coordination approach and QP reformulation (sensitivity analysis). The first example describes the computational efficiency and compares this one with the results that have been obtained using different approaches. On the other hand, the second case illustrates the efficacy of the technique in a benchmark.

### 5.1 Case 1: Different numerical efficiency

The proposed algorithm is compared with the Newton's (Fiacco) price-update scheme discussed in Cheng's work (Cheng et al, 2007) and with the P-control price-update described in Jose's work (Jose et al, 1998) for a quadratic system with linear constraints (10).

In the Newton's price-update scheme, the price is updated using equation (4), in which sensitivity analysis (8) and active set change identification techniques are employed. On the other hand, in the P-control approach, the price is updated using equation (4), but the term $\alpha Q^{-1}$ is taken as a constant $K_c$. In the proposed method, the price is updated using equation (4), in which sensitivity analysis is done using (9).

The global QP problem which has been considered is the following one:

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} - \mathbf{c}^{\mathrm{T}}\mathbf{x}$$
s.t
$$\mathbf{A}\mathbf{x} \le \mathbf{b} \quad \mathbf{x} \ge 0 \tag{10}$$
$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 7 & 3 \\ 3 & 5 & 3 & 4 \\ 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1.5 & 4 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 14 \\ 10 \\ 6 \\ 5 \\ 12 \\ 6 \end{bmatrix}$$
$$\mathbf{Q} = \mathrm{diag}\{2,4,3,8\}, \quad \mathbf{c}^{\mathrm{T}} = \begin{bmatrix} 2 & 5 & 6 & 8 \end{bmatrix}$$

This example can be decomposed into two subproblems (11) and (12) with two linking constraints. By using the price-driven coordination each subproblem has to be defined as follows:

$$\min_{\mathbf{x}_1} \mathrm{f}_1 = \frac{1}{2}\mathbf{x}_1{}^{\mathrm{T}}\mathbf{Q}\mathbf{x}_1 - \mathbf{c}_1{}^{\mathrm{T}}\mathbf{x}_1$$
s.t
$$\mathbf{A}_1\mathbf{x}_1 \le \mathbf{b}_1 \tag{11}$$
$$\mathbf{x}_1 \ge 0$$
$$\mathbf{A}_1 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$
$$\mathbf{Q}_1 = \mathrm{diag}\{2,4\}$$
$$\mathbf{c}_1{}^{\mathrm{T}} = \begin{bmatrix} 2 - 2\mathrm{p}_1 - 3\mathrm{p}_2 & 5 - 5\mathrm{p}_1 - 5\mathrm{p}_2 \end{bmatrix}$$

Where, $p_1$ and $p_2$ are the prices which are manipulated by the coordinator.

To use the expression (4) for updating the price vector, it is necessary to determinate $d\boldsymbol{x}i/d\boldsymbol{p}$.

$$\min_{\mathbf{x}_2} \mathrm{f}_2 = \frac{1}{2}\mathbf{x}_2{}^{\mathrm{T}}\mathbf{Q}\mathbf{x}_2 - \mathbf{c}_2{}^{\mathrm{T}}\mathbf{x}_2$$
s.t
$$\mathbf{A}_2\mathbf{x}_2 \le \mathbf{b}_2 \ \mathbf{x}_2 \ge 0 \tag{12}$$
$$\mathbf{A}_2 = \begin{bmatrix} 1.5 & 4 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$
$$\mathbf{Q}_2 = \mathrm{diag}\{3,8\}$$
$$\mathbf{c}_2{}^{\mathrm{T}} = \begin{bmatrix} 6 - 7\mathrm{p}_1 - 3\mathrm{p}_2 & 8 - 3\mathrm{p}_1 - 4\mathrm{p}_2 \end{bmatrix}$$

Using Fiacco (8) is necessary to know if some change in active set exists, meanwhile using QP reformulation is not necessary. In this case for example ($\boldsymbol{p}_0$ = [-1 0]), if it is used $\Delta\mathbf{p}$ = [0.1 0] both methods give the same sensibility value (there are not change in active sets). On the other hand, if it is used $\Delta\mathbf{p}$ = [1.5 0], a change in active set occurs. Fiacco's method cannot be applied because it is not known which active constraints form equation (8), initial active constraints ($\mathbf{p}_0$) or final active constraints ($\boldsymbol{p}_0$+$\Delta\mathbf{p}$) (using $\boldsymbol{p}_0$ all constraints are inactive, but using $\boldsymbol{p}_0$+$\Delta\mathbf{p}$ two constraints are active), so an incorrect sensibility value will be obtained.

Therefore, to calculate $d\boldsymbol{x}i/d\boldsymbol{p}$ which is obtained from equation (9), two QP problems are defined which are represented in equations (13) and (14), to estimate $d\boldsymbol{x}_1/d\boldsymbol{p}$ and $d\boldsymbol{x}_2/d\boldsymbol{p}$ respectively.

$$\min_{\Delta\mathbf{x}_1} \frac{1}{2}\Delta\mathbf{x}_1{}^{\mathrm{T}}\mathbf{L}_{\mathrm{xx}}\Delta\mathbf{x}_1 + \Delta\mathbf{p}^{\mathrm{T}}\mathbf{L}_{\mathrm{xp}}^{\mathrm{T}}\Delta\mathbf{x}_1 + \frac{\partial \mathrm{f}_1^{\mathrm{T}}}{\partial \mathbf{x}_1}\Delta\mathbf{x}_1$$
s.t
$$\mathbf{g}_{\mathbf{x}_1}(\cdot)\Delta\mathbf{x}_1 \ge -\mathbf{g}_{\mathbf{p}}(\cdot)\Delta\mathbf{p} - \mathbf{g}(\cdot)$$
$$\mathbf{L}_{\mathrm{xx}} = \mathbf{Q}_1, \mathbf{L}_{\mathrm{xp}}^{\mathrm{T}} = \begin{bmatrix} 2 & 5 \\ 3 & 5 \end{bmatrix} \tag{13}$$
$$\frac{\partial \mathrm{f}_1^{\mathrm{T}}}{\partial \mathbf{x}_1} = \begin{bmatrix} 2\mathbf{x}_1(1) - (2 - 2\mathrm{p}_1 - 3\mathrm{p}_2) \\ 4\mathbf{x}_1(2) - (5 - 5\mathrm{p}_1 - 5\mathrm{p}_2) \end{bmatrix}$$
$$\mathbf{g}_{\mathbf{x}_1}(\cdot) = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \mathbf{g}_{\mathbf{p}}(\cdot) = 0$$
$$\mathbf{g}(\cdot) = -\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\mathbf{x}_1 - \begin{bmatrix} 6 & 5 \end{bmatrix}$$

$$\min_{\Delta\mathbf{x}_2} \frac{1}{2}\Delta\mathbf{x}_2{}^{\mathrm{T}}\mathbf{L}_{\mathrm{xx}}\Delta\mathbf{x}_2 + \Delta\mathbf{p}^{\mathrm{T}}\mathbf{L}_{\mathrm{xp}}^{\mathrm{T}}\Delta\mathbf{x}_2 + \frac{\partial \mathrm{f}_2^{\mathrm{T}}}{\partial \mathbf{x}_2}\Delta\mathbf{x}_2$$
s.t
$$\mathbf{g}_{\mathbf{x}_2}(\cdot)\Delta\mathbf{x}_2 \ge -\mathbf{g}_{\mathbf{p}}(\cdot)\Delta\mathbf{p} - \mathbf{g}(\cdot)$$
$$\mathbf{L}_{\mathrm{xx}} = \mathbf{Q}_2, \mathbf{L}_{\mathrm{xp}}^{\mathrm{T}} = \begin{bmatrix} 7 & 3 \\ 3 & 4 \end{bmatrix} \tag{14}$$
$$\frac{\partial \mathrm{f}_2^{\mathrm{T}}}{\partial \mathbf{x}_2} = \begin{bmatrix} 3\mathbf{x}_2(1) - (6 - 7\mathrm{p}_1 - 3\mathrm{p}_2) \\ 8\mathbf{x}_2(2) - (8 - 3\mathrm{p}_1 - 4\mathrm{p}_2) \end{bmatrix}$$
$$\mathbf{g}_{\mathbf{x}_2}(\cdot) = \begin{bmatrix} 1.5 & 4 \\ 2 & 1 \end{bmatrix}, \mathbf{g}_{\mathbf{p}}(\cdot) = 0$$
$$\mathbf{g}(\cdot) = -\begin{bmatrix} 1.5 & 4 \\ 2 & 1 \end{bmatrix}\mathbf{x}_2 - \begin{bmatrix} 12 & 6 \end{bmatrix}$$

Table 1 shows the performance of these three algorithms. The comparison is based on the number of iterations and convergence rate. For all cases an initial guess $\boldsymbol{p}$ = 0 is used. The proposed sensitivity analysis provides the same number of iterations and convergence rate than Cheng's approach,

moreover it is not necessary to determinate the Newton step size because the sensitivity analysis takes into account the active set change and can be applied on nonlinear systmes.

**Table 1. Performance of price-update technique**

| Methods | Tuning | Iterations | Convergence |
|---|---|---|---|
| Newton's (QP) | $\Delta \boldsymbol{p} = 0.1$ | 2 | Yes |
| Newton's (QP) | $\Delta \boldsymbol{p} = 1$ | 2 | Yes |
| Newton's (Fiacco) | NA | 2 | Yes |
| P-control | $K_c = 0.02$ | 400 | Yes |
| P-control | $K_c = 0.1$ | NA | No |

5.2 Case 2: Coordination of Four-Tank Benchmark

This plant is based on the one used by Alvarado in (Alvarado et al, 2011). It consists of four equal interconnected tanks as in Figure 3. Two pumps take water from a storage tank located under the four tanks. The first one sends a flow of water $q_A$ to tanks 1 and 4. The second pump takes a flow $q_B$ and sends it to the tanks 2 and 3. Finally, tank 3 empties its content into tank 1 and tank 4 does the same into tank 2.

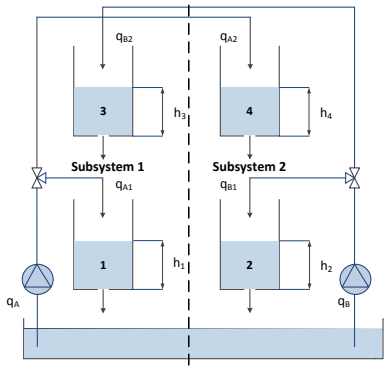


*Figure. 3. The four-tank process diagram.*

A simulation model of the plant has been developed using EcoSimPro simulation software. The model is given by the following non-linear ODE set.

$$\frac{dh_1}{dt} = -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}q \tag{15.1}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_b}{S}q_b \tag{15.2}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{S}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{S}q_b \tag{15.3}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{S}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{S}q_a \tag{15.4}$$

Here $h_i$ ($i \in \{1, 2, 3, \text{and } 4\}$) refers to the water level of tank $i$, $a_i$ is discharge parameters and $S$ is the cross section of the tanks. On the other hand, $q_j$ and $\gamma_j$ ($j \in \{A, B\}$) denote the flow and the ratio of the three-way valve of pump $j$ and $g$ is the gravitational acceleration.

In the plant, the goal of the control system is to maintain levels $h_1$ and $h_2$ close to specified set-points, $Refh_1$ and $Refh_2$, manipulating the flows $q_A$ and $q_B$.

In order to test price-driven coordination strategies, the plant was split in two subsystems: Subsystem 1 includes tank 1 and 3, while subsystem 2 comprises tanks 2 and 4, as indicated in Figure 3. In the benchmark, one NMPC controller was assigned to each subsystem, but both of them were able to manipulate flows $q_A$ and $q_B$. Denoting as $q_{A1}$, $q_{B1}$ or $q_{A2}$, $q_{B2}$ the ones seen by each controller, this means that two global constraints have to be satisfied:

$$q_{A1} = q_{A2} \tag{16.1}$$

$$q_{B1} = q_{B2} \tag{16.2}$$

The centralized NMPC architecture implies solving the optimization problem (17) each sampling time ($T_s$=90 seconds), taking into account the complete nonlinear model of the process (15) and the constraints. A sequential approach using a control vector parameterization on the manipulated variables $q_A$ and $q_B$ and a SQP algorithm have been used to solve this dynamic optimization problem on-line. This controller has been successfully tested on the simulated plant and the results of the controlled variables for several step changes in references are shown in blue in Fig 4.

$$\min_{\substack{q_A(j),\ldots,q_A(j+Nu-1)\\ q_B(j),\ldots,q_B(j+Nu-1)}} J = \int_0^{N_2 \cdot T_s} \left((h_1 - Refh_1)^2 + (h_2 - Refh_2)^2\right) dt$$

s.t. (17)

Equation set (15)

$$h_{i\min} \le h_i(t) \le h_{i\max} \quad \forall i \in \{1,2,3,4\} \quad t \in [0, N_2]$$

$$q_{k\min} \le q_k(j) \le q_{k\max} \quad \forall k \in \{A, B\} \quad j = 0,\ldots, N_u - 1$$

$$q_k(j) = q_k(N_u - 1) \; \forall k \in \{A, B\} \quad j = Nu,\ldots, N_2 - 1$$

The coordinated NMPC based on price-driven coordination scheme was also tested and results of the controlled variables for the same step changes in references are shown in black in Figure 3. This scheme achieves a similar behaviour of centralized one.

The price coordinated optimization problem is composed of two layers: In the lower one, two separated $NMPC_i$ $\forall i \in \{1, 2\}$ (18) and (19), performs the level control of the corresponding subsystem. The cost function on theses NMPCs include a first term corresponding to control objective, a second and a third term corresponding to Lagrangean decomposition in order to achieve fulfil global constraints (16.1) and (16.2). The first $NMPC_1$, manipulates the variables $q_{A1}$ and $q_{B1}$ and the second $NMPC_2$, manipulates $q_{A2}$ and $q_{B2}$. In the upper coordinating layer, two QP problems are used to obtain sensitivity analysis and Newton's method is implied to assign prices $p_1$ and $p_2$ to the NMPCs in order to fulfil global constraints (16).

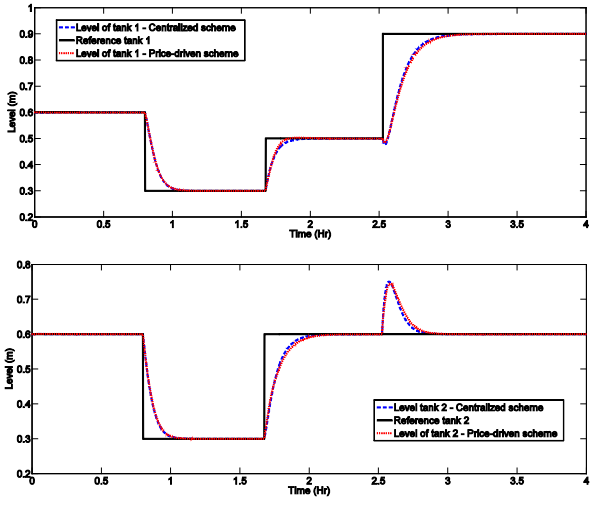


*Figure 4. Controlled variables (levels $h_1$ and $h_2$) using a centralized and price-driven coordination schemes.*

$$\min_{\substack{\{q_{A1}(j),\ldots,q_{A1}(j+Nu-1)\}\\ \{q_{B1}(j),\ldots,q_{B1}(j+Nu-1)\}}} J_1 = \int_{t=0}^{N_2\cdot T_s} \left((h_1(t) - Refh_1(t))^2\right)dt$$

$$+ \sum_{j=0}^{N_u-1}\left(p_1(q_{A1}(j) - q_{A2}(j))^2\right) + \sum_{j=0}^{N_u-1}\left(p_2(q_{B1}(j) - q_{B2}(j))^2\right) \quad (18)$$

s.t.

Equation set subsystem 1

$$q_{k1\min} \le q_{k1}(j) \le q_{k1\max} \quad \forall k \in (A,B) \;\; j = 0,\ldots,N_u - 1$$
$$q_{k1}(j) = q_{k1}(N_u - 1) \quad \forall k \in (A,B) \;\; j = N_u,\ldots,N_2 - 1$$

$$\min_{\substack{\{q_{A2}(j),\ldots,q_{A2}(j+Nu-1)\}\\ \{q_{B2}(j),\ldots,q_{B2}(j+Nu-1)\}}} J_2 = \int_{t=0}^{N_2\cdot T_s} \left((h_2(t) - Refh_2(t))^2\right)dt$$

$$+ \sum_{j=0}^{N_u-1}\left(p_1(q_{A2}(j) - q_{A1}(j))^2\right) + \sum_{j=0}^{N_u-1}\left(p_2(q_{B2}(j) - q_{B1}(j))^2\right) \quad (19)$$

s.t.

Equation set subsystem 2

$$q_{k2\min} \le q_{k1}(j) \le q_{k2\max} \quad \forall k \in (A,B) \;\; j = 0,\ldots,N_u - 1$$
$$q_{k2}(j) = q_{k2}(N_u - 1) \quad \forall k \in (A,B) \;\; j = N_u,\ldots,N_2 - 1$$

Figure 5 shows the manipulated variable obtained by the centralized and price-driven coordination schemes (the dashed blue lines are the MV's using centralized scheme, meanwhile the dotted red lines are the MV's using price-driven coordination scheme). As it is shown, approximately 10 sampling time are spent to achieve the centralized solution.

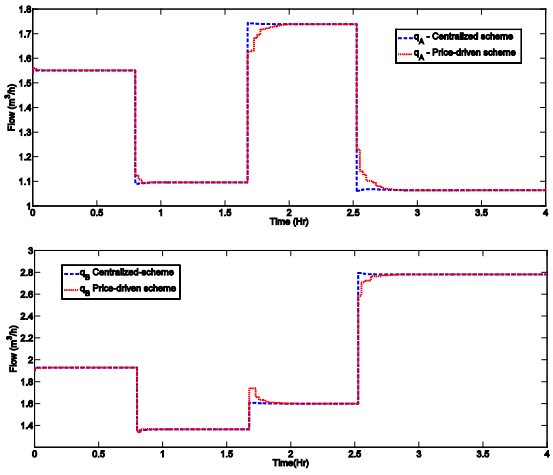


*Figure 5. Manipulated variable calculated with centralized and price-driven coordination schemes.*

Figure 6 shows the number of iterations in each sampling time needed to find the optimal price, it means, to fulfil with the global constraints.

As it is shown in Figure 6, the QP formulation method used to obtain the sensitivity analysis, provides a fast convergence of Newton's method (price-update technique). Each sampling time, the coordinator converges in a feasible solution, which $q_{A1}^* = q_{A2}^*$ and $q_{B1}^* = q_{B2}^*$ is achieved. With this method each subsystem can manipulate its own coupled manipulated variables.

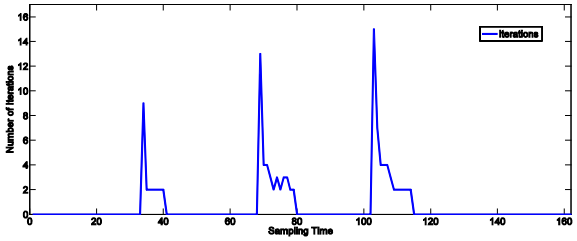


*Figure 6. Number of iterations using Newton's method for updating the prices.*

## 6. CONCLUSIONS

In this paper, a sensitivity analysis for nonlinear systems has been presented using a sensitivity analysis that take into account possible changes in the set of active constraints. This algorithm allows using price-driven coordination scheme to coordinate different NMPCs. One advantage of this scheme is that does require a new reconfiguration of the decentralized MPC controllers, just an extra term to modify the control formulation is needed. Second, it was shown with a four-tank benchmark that the QP sensitivity analysis and Newton's method provide a fast convergence towards the plat optimal performance, allowing calculating the sensitivity matrix for nonlinear systems and avoiding the problem of changes in active set of constraints.

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