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On the Numerical Solution of Discounted Economic NMPC on Infinite Horizons

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Abstract: In this work, two numerical solution methods are presented for discounted economic nonlinear model predictive control on infinite horizons without terminal constraints. While the first formulation simply replaces the infinite by a finite horizon, the second formulation uses a time transformation function to project the infinite to a finite horizon. For the first formulation, an algorithm is presented which heuristically determines a sufficiently long final time with the help of the turnpike property in order to ensure good closed-loop control performance. For the second formulation, a two-stage formulation is introduced to deal with large differences in the dynamics of the objective function and the states. The solution accuracy is improved for both formulations by using a control vector adaptation strategy such that an adequate number of decision variables is obtained. Both solution methods are compared in a case study.

Keywords: economic model predictive control, dynamic real-time optimization, infinite horizon, finite horizon, turnpike

1. INTRODUCTION

In recent years, the objective of process control changed from the mere control task to economically optimal process operation at any time. While Tvrzská de Gouvêa and Odloak (1998) suggested to add an economic term to the setpoint tracking objective, Helbig et al. (2000) were the first to solve an economic optimal control problem using nonlinear model predictive control (NMPC) with a pure economic objective function. NMPC with an economic objective function is nowadays simply called economic NMPC or eNMPC (cf. Rawlings and Amrit (2009)). eNMPC can be applied for the control of batch processes as well as continuously operated processes. Since no fixed final time exists for continuously operated processes, it is favorable to employ an infinite horizon formulation. However, the eNMPC problem on an infinite horizon cannot be solved directly by numerical solution methods. Consequently, the infinite horizon eNMPC problem must be reformulated to a finite horizon problem. In order to achieve this, two methods can be employed which will be presented next in detail: the finite horizon formulation and the transformed infinite horizon formulation.

When applying the finite horizon (FH) formulation, the infinite horizon is replaced by a finite horizon with a heuristically determined final time in a moving horizon setting. It was pointed out by Bitmead (1990) that the finite horizon formulation might become unstable in closed loop. Whereas a lot of stability proofs exist for regulatory NMPC (cf. Mayne et al. (2000)), just few stability proofs are currently available for eNMPC which are typically based on the results developed in mathematical economics (cf. Carlson et al. (1991)). Diehl et al. (2011) have been the first to proof stability for an arbitrary economic objective function using Lyapunov arguments and terminal constraints. Extensions of the proof can be found in the works of Huang et al. (2011) and Angeli et al. (2012). Just recently, Grüne (2013) proved stability without using terminal constraints by employing the so-called turnpike property of finite horizon economic optimal control problems. Dorfman et al. (1958) stated that the turnpike is the fastest route of travel between any two points. It will pay off to get on to the turnpike and to add a little mileage at either end, if the start and final time of the horizon are far enough apart. We refer the interested reader to Würth et al. (2009) and Hartwich and Marquardt (2010) for an illustration of a turnpike. With the help of the turnpike property, Grüne (2013) also proved that the finite horizon solution converges to the infinite horizon solution with growing horizon length. As the finite horizon formulation is easy to implement, it has been widely applied, for example, by Tvrzská de Gouvêa and Odloak (1998), Engell (2007), Rawlings and Amrit (2009), Diehl et al. (2011), Huang et al. (2011) and Grüne (2013). However, in most works, the final time of the finite horizon has been chosen arbitrarily and was not (heuristically) determined such that the horizon is as short as possible to limit computational effort but long enough to guarantee good closed-loop performance as well as stability (cf. Grüne (2013)).

When applying the transformed infinite horizon (TIH) formulation, the infinite horizon is transformed to a finite horizon with the help of a time transformation function. As pointed out by Würth et al. (2009), if disturbances do not arise and model mismatch does not exist, the recurrent solutions of the economic optimal control problem on a (transformed) infinite horizon fulfill Bellmann's optimality principle (cf. Bellman (1957)). As the discrepancy between the open-loop and the closed-loop solution is removed, nominal stability can be guaranteed, at least if perfect numerical solutions are assumed. Würth et al. (2009) first applied the TIH formulation in a receding horizon setting using a transformation function suggested by Kunkel and Hagen (2000) in the context of mathematical economics. However, the solution quality obtained was not satisfactory because the transformation function projected the transient interval of the infinite horizon on to a very small interval of the transformed infinite horizon.

In this work, we will suggest several algorithmic improvements for the numerical solution of eNMPC with a discounted objective on infinite horizons based on the FH as well as the TIH formulation. This work presents and reinterprets the results on infinite horizons of Würth (2013). For the FH formulation, a heuristic algorithm is suggested to determine a sufficiently long final time. The algorithm makes use of the turnpike property and is closely related to the theoretical results of Grüne (2013) though it was developed independently. For the TIH formulation, we apply a transformation function as described by Würth and Marquardt (2013) for which the projection can be adapted in order to guarantee a good resolution of the transient part of the control profiles on the transformed infinite horizon. In this contribution, we introduce a multi-stage formulation for the TIH formulation to deal with the different dynamics of the system and the discounted objective function.

The paper is organized as follows. In Section 2, we introduce the open-loop economic optimal control problem on an infinite horizon and consider the discounted objective function in more detail. In Section 3, the solution strategy for the open-loop economic optimal control problem is presented. First, the infinite horizon is reformulated to a finite horizon by either applying the FH formulation or the TIH formulation using the novel two-stage formulation. Hereafter, a single shooting strategy is outlined. In Section 4, we introduce adaptation strategies to improve the numerical solution presented in Section 3. A heuristic algorithm is suggested to determine a sufficiently long final time for the FH formulation. Furthermore, we introduce an adaptation strategy for control vector parameterization such that the numerical solution is close to the optimal continuous-time solution with a well-tuned number of degrees of freedom for both formulations. In Section 5, the closed-loop strategy is described. Finally, we compare both formulations in a benchmark case study in Section 6.

2. PRELIMINARIES

2.1 Open-loop economic optimal control problem on infinite horizon

The continuous-time open-loop economic optimal control problem on an infinite horizon is given by

$$\min_{u(t)} \quad \int_{t_0}^{\infty} \hat{\Phi}(x(t), z(t), u(t), t)\, dt, \tag{1a}$$

$$s.t. \quad \dot{x}(t) = f(x(t), z(t), u(t)), \tag{1b}$$

$$0 = g(x(t), z(t), u(t)), \tag{1c}$$

$$x(t_0) = x_0, \tag{1d}$$

$$\hat{c}(x(t), z(t), u(t)) \preceq 0, \tag{1e}$$

$$t \in \mathcal{I} := [t_0, \infty)\,, \tag{1f}$$

where $x : \mathcal{I} \to \mathbb{R}^{n_x}$ and $z : \mathcal{I} \to \mathbb{R}^{n_z}$ represent the trajectories of the differential and algebraic variables on time horizon $\mathcal{I}$, respectively. $u : \mathcal{I} \to \mathbb{R}^{n_u}$ represents the trajectories of the control variables. The economic objective function in (1a) is subject to the differential-algebraic equation (DAE) model of index one, (1b) and (1c), with consistent initial conditions (1d) as well as path constraints (1e). The symbol $\preceq$ denotes component-wise inequality. $\hat{\Phi}$, f, g, $\hat{c}$ are assumed to be at least once continuously differentiable. We assume that a feasible optimal solution to Eq. (1) exists which is not cyclic.

2.2 Objective function of open-loop problem

From an economic point of view, the time value of money shall be accounted for in the objective function of an economic optimal control problem on a long or infinite time horizon. This can be accomplished by discounting the future profit to the present value. In a discrete-time formulation, the objective function then corresponds to the net present value. An equivalent objective function with continuous-time discounting is given by

$$\int_{t_0}^{\infty} e^{-\rho t} \phi(x(t), z(t), u(t))\, dt, \tag{2}$$

where $\rho \in \mathbb{R}^+ \backslash \{0\}$ is the discount factor and can be chosen to equal the annual market rate. The function ϕ reflects the negative profit per time unit calculated, for example, based on real costs of products and reactants. The function ϕ is bounded by a minimum achievable negative profit value ϕ_{min} which is defined as the minimum of $\phi(x(t), z(t), u(t))$ satisfying (1b)-(1f). Consequently, the objective function (2) is also bounded:

$$\int_{t_0}^{\infty} e^{-\rho t} \phi(x(t), z(t), u(t))\, dt \geq \frac{\phi_{min}}{\rho} e^{-\rho t_0}. \tag{3}$$

A bounded objective function is a prerequisite to make use of Lyapunov-based stability proofs and to apply numerical solution methods such as single shooting, multiple shooting or collocation for the TIH formulation. Though other formulations with bounded objective functions exist such as regulatory objective functions (cf. Würth and Marquardt (2013) using a TIH formulation) and objective functions reduced to finite rewards (cf. Carlson et al. (1991) and Diehl et al. (2011)), we restrict ourselves to economic objective functions with continuous discounting due to the economic motivation presented above.

3. SOLUTION STRATEGY FOR OPEN-LOOP ECONOMIC OPTIMAL CONTROL PROBLEM

3.1 Reformulation of infinite to finite horizon problem

We reformulate the infinite horizon problem (1) to a finite horizon problem in order to facilitate numerical solution. In the sequel, we will introduce the finite horizon (FH) and the transformed infinite horizon (TIH) formulation in detail.

Finite horizon formulation. For the FH formulation, the infinite horizon $\mathcal{I}$ is simply replaced by the finite horizon $\mathcal{I}_t := [t_0, t_f]$ by introducing the final time $t_f \in \mathbb{R}$ with

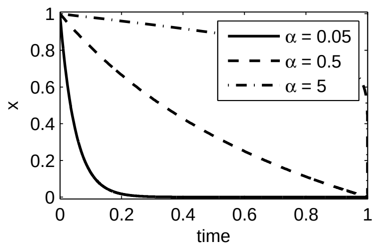

Fig. 1. Trajectory $x(t)$ of autonomous system $\dot{x}(t) = -x(t)$ with initial condition $x_0(0) = 1$ on $\mathcal{I}_\tau$ for different values of α (cf. Würth and Marquardt (2013)).

$t_f > t_0$. The open-loop economic optimization problem on a finite horizon is then given by

$$\min_{u(t)} \quad \int_{t_0}^{t_f} \hat{\Phi}(x(t), z(t), u(t), t)\, dt, \tag{4a}$$

$$s.t. \quad \dot{x}(t) = f(x(t), z(t), u(t)), \tag{4b}$$

$$0 = g(x(t), z(t), u(t)), \tag{4c}$$

$$x(t_0) = x_0, \tag{4d}$$

$$\hat{c}(x(t), z(t), u(t)) \preceq 0, \tag{4e}$$

$$t \in \mathcal{I}_t. \tag{4f}$$

We assume that the turnpike property holds for the open-loop economic optimal control problem (4). The choice of a sufficiently long final time t_f is crucial to reach the neighborhood of the turnpike ensuring that the closed-loop performance of the FH formulation approximates the closed-loop performance of an ideal controller solving (1) (cf. Grüne (2013)). In Section 4.1, we will introduce a heuristic procedure which determines a sufficiently long final time t_f for the FH formulation.

Transformed infinite horizon formulation. For the TIH formulation, the infinite time horizon $\mathcal{I}$ is converted into a transformed infinite time horizon $\mathcal{I}_\tau := [0, 1]$ with the help of a transformation function. In this work, we use the transformation function suggested by Würth and Marquardt (2013),

$$\tau = \tanh(\alpha(t - t_0)), \quad \alpha \in \mathbb{R}^+ \setminus \{0\}, \tag{5}$$

where $\lim_{t\to\infty} \tau = 1$. It has the advantage that the projection of the infinite time horizon to the transformed infinite time horizon can be adapted with the help of the parameter α. By adapting α, we can predefine the length of the time interval on $\mathcal{I}_\tau$ which represents the transient part of the system's trajectories as shown in Figure 1. When choosing a sufficiently small α, it can be guaranteed that the steady state is reached well before the end of $\mathcal{I}_\tau$ as it is the case for $\alpha = 0.5$ and $\alpha = 0.05$ in Figure 1. Furthermore, a good resolution of the transient region can be achieved, when α is chosen large enough, as it is the case for $\alpha = 0.5$. To this end, an equidistant control grid defined on $\mathcal{I}_\tau$ leads to more decision variables in the transient part and an improved closed-loop performance (cf. Würth et al. (2009) and Würth and Marquardt (2013)).

If the system's behavior comprises fast and slow dynamics, it is not sufficient to adapt α, but a sequence of different transformation functions must be applied. This is especially true if a discounted objective function is used, since $\hat{\Phi}(x(t), z(t), u(t), t)$ would usually decay very slowly compared to the system's state. For this case, we suggest the formulation of an open-loop two-stage economic optimal control problem where the first stage corresponds to the transient region of the system's dynamics and the second stage corresponds to the steady-state region of the system's dynamics. The two-stage problem allows defining different values of $\alpha_k \in \mathbb{R}^+ \setminus \{0\}$ on each stage k and is given by

$$\min_{u_k(\tau)} \quad \sum_k \int_{\tau_{k-1}}^{\tau_k} \frac{\hat{\Phi}(x_k(\tau), z_k(\tau), u_k(\tau), \operatorname{artanh}(\tau)\alpha_k^{-1})}{\alpha_k(1-\tau^2)}\, d\tau, \tag{6a}$$

$$s.t. \quad \dot{x}_k(\tau) = \frac{f_k(x_k(\tau), z_k(\tau), u_k(\tau))}{\alpha_k(1-\tau^2)}, \tag{6b}$$

$$0 = g_k(x_k(\tau), z_k(\tau), u_k(\tau)), \tag{6c}$$

$$x_k(\tau_{k-1}) = x_{k-1}(\tau_{k-1}), \tag{6d}$$

$$\hat{c}_k(x_k(\tau), z_k(\tau), u_k(\tau)) \preceq 0, \tag{6e}$$

$$\tau \in \mathcal{I}_{\tau,k} := [\tau_{k-1}, \tau_k], \tag{6f}$$

$$k = 1, 2, \tag{6g}$$

where index k denotes the quantities of stage k and $\{\tau | \tau \in \mathcal{I}_{\tau,k},\ k = 1, 2\} = \mathcal{I}_\tau$. The multi-stage transformation function is thus defined as

$$\tau = \tau_{k-1} + \tanh(\alpha_k(t - t_{k-1})), \quad k = 1, 2. \tag{7}$$

In order to achieve a high resolution in the transient region, α_1 should be chosen relatively high. On the other hand, in the second steady-state region, a small value for α_2 should be chosen such that the system and the objective function reach their steady states before the end of $\mathcal{I}_\tau$. The choice of the switching time τ_1 determines the weighting of the transient region compared to the steady-state region. Hence a late switching time puts more weight on the transient part which improves closed-loop performance as described above. The choice of the tuning parameters α_1, α_2 and τ_1 can be performed based on offline optimizations.

3.2 Single shooting strategy

We approximate the solution to (4) and (6) by a sequential strategy using control vector parametrization. Consequently, the infinite number of decision variables is reduced to a finite number. As pointed out by Würth and Marquardt (2013), a sequential solution strategy can only be applied if the open-loop system is asymptotically stable for any $u(t)$. If the system is open-loop unstable, another solution strategy such as collocation may be used.

Each control $u_{l,k}$, with $l = 1, \ldots, n_u$, is represented by piecewise constant basis functions Ψ on each stage k. The number of basis functions may differ for each control and each stage. For the FH formulation, we have

$$u_{l,k}(t) = \sum_{\kappa=1}^{K_{l,k}} \bar{u}_{l,k}^{\kappa} \Psi_{l,k}^{\kappa}(t), \tag{8}$$

where $t \in \mathcal{I}_t$, $k = 1$, $K_{l,k} \in \mathbb{N}$ and $\bar{u}_{l,k}^{\kappa}$ is a scalar control parameter. For the TIH formulation, we get

$$u_{l,k}(\tau) = \sum_{\kappa=1}^{K_{l,k}} \bar{u}_{l,k}^{\kappa} \Psi_{l,k}^{\kappa}(\tau), \tag{9}$$

where $\tau \in \mathcal{I}_{\tau,k}$ and $k = 1, 2$. For both formulations, we can summarize all decision variables in the vector of control variables $\zeta \in \mathbb{R}^{n_{\bar{u}}}$ with $n_{\bar{u}} = \sum_k \sum_l K_{l,k}$.

The optimal control problems (4) and (6) can then be transcribed into the NLP

$$\begin{aligned} \min_{\zeta} \quad & \Phi(\zeta) \\ s.t. \quad & c(\zeta) \preceq 0, \end{aligned} \tag{10}$$

where the objective function $\Phi(\zeta)$ corresponds to the value of the objective function (4a) and (6a) at the end of the time horizon $\mathcal{I}_t$ and $\mathcal{I}_\tau$, respectively. c are the functions entering the inequality constraints which result from control vector parametrization of (1e).

The objective function, the constraints and their first-order derivatives with respect to ζ are computed by simultaneous integration of the (transformed) nonlinear DAE model and the associated DAE sensitivity equation system by some efficient tailored algorithm such as SLIMEX (cf. Schlegel et al. (2005)). Since the discontinuity occurring at τ_1 for the TIH formulation is explicit, the first-order sensitivities of the objective function Φ and the constraints c with respect to ζ are continuous (cf. Özyurt and Barton (2005)). Consequently, the NLP (10) can be solved by a standard gradient-based method such as SNOPT (cf. Gill et al. (1998)) for both formulations.

4. ADAPTATION STRATEGIES FOR OPEN-LOOP ECONOMIC OPTIMAL CONTROL PROBLEM

In order to obtain a numerical solution of high accuracy, a two-step procedure is suggested for the FH formulation. First, a sufficiently long time horizon is determined for the finite horizon formulation. Second, the control vector discretization is adapted by the signal-based adaptation strategy presented by Schlegel et al. (2005). For the TIH formulation, the signal-based adaptation strategy is also applied to allow for a good numerical solution.

4.1 Horizon length adaptation for finite horizon formulation

The horizon length t_f should be chosen sufficiently long in order to guarantee good closed-loop performance and stability. In contrast to most works, where t_f is chosen arbitrarily, we suggest the heuristic Algorithm 1 to determine t_f by running offline optimizations with different sets of initial conditions such that the turnpike is reached.

Algorithm 1

(1) Generate a grid $t_0, ..., t_I$, where t_i are the grid points with $i = 0, ..., I$ and $I \in \mathbb{N}$. $t_{f,1} := t_I$ is an initial guess for the horizon length.
(2) for $n = 1, ..., N_{IC}$ do
  (a) for $j = 1, ..., N_{steps}$ do
    (i) Solve open-loop problem (4) for the set of initial values $x_{0,n}$ on horizon $[t_0, t_{f,j}]$.
    (ii) Select the set $\mathcal{T}$ of gridpoints t_i for which $\mathcal{T} = \left\{ t_i \in (t_0, t_{f,j}) | \sum_l \frac{(\bar{u}^i_{l,1} - \bar{u}^{i-1}_{l,1})}{|t_i - t_{i-1}|} < \epsilon \right\}$
    (iii) if $|\mathcal{T}| \leq 1$ then
      Set $t_{f,j} := 1.5 \cdot t_{f,j}$.
    else break
    end if
  end for
  (b) Set the horizon length $t_{f,n}$ to $t_{f,j}$.
  end for
(3) Set the horizon length t_f to $t_{f,N_{IC}}$.

The different sets of initial conditions are chosen by random sampling, where N_{IC} is the number of sets. $\mathcal{T}$ denotes the set of grid points where the turnpike is reached. If the set contains less than two elements, the finite horizon length is doubled. N_{step} is the maximal number of iterations of the algorithm, which are performed, until a finite horizon length may be found.

4.2 Signal-based adaptation strategy

A signal-based adaptation strategy has been applied by Würth et al. (2009). It performs a wavelet transformation of the input profile and introduces or deletes grid points by analyzing the magnitude of the wavelet coefficients (cf. Schlegel et al. (2005)). In each iteration of the signal-based adaptation algorithm, the regions of the control profile with relatively high wavelet coefficients are refined, whereas the regions of the control profile wih relatively low wavelet coefficients are merged. In this way, we receive a numerical solution of high accuracy without introducing too many decision variables ζ along the time horizon.

5. CLOSED-LOOP STRATEGY

For simplicity, we consider state feedback and assume that model-mismatch and measurement noise does not exist. At each sampling instant $t_{0,h}$, the current state $x(t_{0,h})$ is measured and sent to the eNMPC. The eNMPC then computes an (approximated) optimal solution to the open-loop problem (1) for the current state $x(t_{0,h})$ using either the FH or the TIH formulation. The optimal controls $u_h(t)$ are immediately sent to the process, where they are implemented for the current sampling interval Δt. Hereafter, the horizon is shifted by the sampling time Δt, i.e.,

$$t_{0,h} := t_{0,h-1} + \Delta t, \tag{11}$$

such that an initial guess for the decision variables ζ_h based on the optimal solution of horizon $h-1$ becomes available.

Finite horizon formulation. As also pointed out by Grüne (2013), $u_h(t)$ should not be prolonged at the end of the finite horizon in a moving horizon setting because the trajectory might be far away from the turnpike. The time horizon is rather prolonged in the time interval of the finite time horizon which is within a small neighborhood around the turnpike. The control algorithm for the FH formulation is summarized in Algorithm 2.

Algorithm 2

(1) Solve the open-loop problem (4) and determine a suitable final time $t_{f,1}$ with the help of Algorithm 1 described in Section 4.1.
(2) for $h = 1, ..., N_h$ do
  (a) Measure current state $x(t_{0,h})$.
  (b) Solve the open-loop problem (4) to obtain the optimal solution. Adapt the grid on finite horizon $[t_{0,h}, t_{f,h}]$ with the signal-based adaptation strategy presented in Section 4.2. Solve (4) again, if the grid was refined.
  (c) Set the optimal solution to $u_h(t)$.
  (d) Implement $u_h(t)$ for the current Δt.

(e) Reduce the time horizon by one sampling time, i.e. set $t_{0,h} := t_{0,h-1} + \Delta t$. For the moving horizon setting, prolong $u_h(t)$ in the neighborhood of the turnpike by one sampling time and set $t_{f,h} := t_{f,h-1} + \Delta t$.
(f) Use the shifted solution $u_h(t)$ as initial guess for the optimization of the next open-loop problem.
end for

Note that the number of iterations for the signal-based adaptation strategy is restricted to one for each horizon in order to reduce computational time online. This is also valid for the control algorithm of the TIH formulation presented next.

Transformed infinite horizon formulation. Algorithm 3 represents the control algorithm for the TIH formulation which is taken from Würth and Marquardt (2013).

Algorithm 3

(1) Solve the open-loop problem (6) and determine suitable values for α_1, α_2 and τ_1 by numerical experiments.
(2) for $h = 1, ..., N_h$ do
  (a) Measure current state $x(t_{0,h})$.
  (b) Transform $u_h(t)$ to $u_h(\tau)$ with $\tau \in [0, 1]$ by performing the time transformation (7).
  (c) Solve the open-loop problem (6) to obtain the optimal solution. Adapt the grid on transformed infinite horizon $[0, 1]$ with the signal-based adaptation strategy. Solve (6) again, if the grid was refined.
  (d) Set the optimal solution to $u_h(\tau)$.
  (e) Transform $u_h(\tau)$ to the original time representation $u_h(t)$.
  (f) Implement $u_h(t)$ for the current sampling interval Δt.
  (g) Reduce the time horizon by one sampling time, i.e. set $t_{0,h} := t_{0,h-1} + \Delta t$.
  (h) Use the shifted solution as initial guess $u_h(t)$ for the optimization of the next open-loop problem.
end for

6. CASE STUDIES

As a case study, we consider the control of the Williams-Otto continuous stirred-tank reactor (WO CSTR) introduced by Forbes (1994). The exothermic reactions taking place in the reactor are $A + B \rightarrow C$, $C + B \rightarrow P + E$ and $P + C \rightarrow G$. The manipulated control variables of this process are the reactor temperature T_r and the inlet mass flowrate $F_{B,in}$ of reactant B. The open-loop economic optimal control problem on an infinite horizon is given by

$$\begin{aligned}\min_{T_r, F_{B,in}} \quad & \int_0^\infty e^{-\rho t}(-1143.38\, F_{P,out}(t) - 25.92\, F_{E,out}(t) \\ & \qquad + 143.34\, F_{B,in}(t) + 23\, F_{A,in}(t))\$/\mathrm{m}^3\, dt, \\ s.t. \quad & \text{process model} \\ & 0\,\mathrm{m}^3/\mathrm{s} \le F_{B,in}(t) \le 10\,\mathrm{m}^3/\mathrm{s}, \\ & 0\,°\mathrm{C} \le T_r(t) \le 150\,°\mathrm{C},\end{aligned}$$

where $t \in \mathcal{I} := [0, \infty)$. $F_{P,out}$ and $F_{E,out}$ are the outlet flowrates of the product P and the side product E. $F_{A,in}$ is the inlet flowrate of reactant A. The discount factor ρ is set to $1.58 \cdot 10^{-9}$ which approximately corresponds to an annual market rate of 5 %.

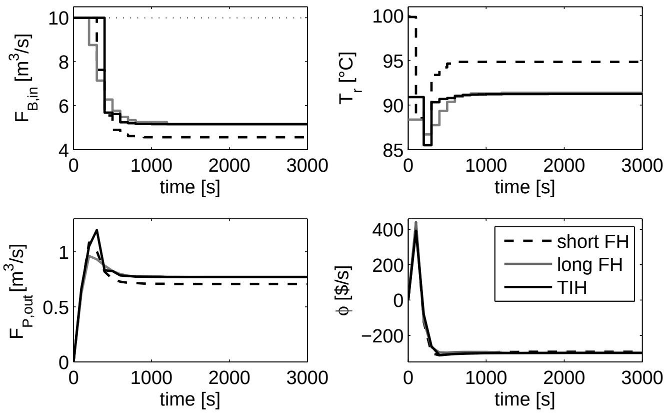

Fig. 2. Closed-loop trajectories for the WO CSTR.

6.1 Offline optimization and adaptation

Finite horizon formulation. For the FH formulation, the length of the finite horizon is determined as 4556 s using the algorithm proposed in Section 4.1 with $\epsilon = 1 \cdot 10^{-5}$ starting from an initial final time horizon of 400 s.

Transformed infinite horizon formulation. For the TIH formulation, α_1 is set to 0.0005 in the first stage, which is determined in offline optimization runs to ensure a suitable discretization of the first stage. α_2 is set to a very small number of $1.5 \cdot 10^{-10}$ in the second stage such that the objective function converges to the steady state before the end of $\mathcal{I}_\tau$. In order to achieve an accurate solution in closed loop, the major weight in the optimization is put on the transient region and the switching time between stage 1 and 2 is set to $\tau_1 = 0.999$.

6.2 Closed-loop results and discussion

Figure 2 shows the closed-loop response for a sampling time of $\Delta t = 100$ s for three different formulations: a FH formulation with a short horizon of 400 s (short FH), a FH formulation with a sufficiently long horizon of 4556 s (long FH) and the TIH formulation (TIH). As discussed above, it can be seen that the length of the horizon influences the closed-loop FH solutions. If a significantly shorter horizon than the sufficiently long horizon is employed, the turnpike is not reached. This can be observed, for example, in the reactor temperature profile: the long FH and the TIH formulation reach the turnpike, a constant path around 92 °C, whereas the short FH formulation approaches a value of approximately 95 °C.

Table 1. Profit $-\Phi$ [$] at sampling instant $t_{0,h}$

$t_{0,h}$ (s)	2000	4000	6000	10000
short FH	$4.020 \cdot 10^5$	$9.88 \cdot 10^5$	$1.58 \cdot 10^6$	$2.747 \cdot 10^6$
long FH	$3.951 \cdot 10^5$	$9.941 \cdot 10^5$	$1.593 \cdot 10^6$	$2.791 \cdot 10^6$
TIH	$4.005 \cdot 10^5$	$9.993 \cdot 10^5$	$1.598 \cdot 10^6$	$2.796 \cdot 10^6$

Table 1 shows that, if a shorter horizon is employed, the profit $-\Phi$ is lower compared to the profit obtained with the longer horizon. However, the profit of the short

FH formulation is higher at the beginning, since the discretization is finer at the beginning of the horizon compared to the long FH formulation. Hence, the results show that on one hand the discretization influences the profit in the transient phase, whereas the length of the horizon determines the profit achieved in the long run, when the state variables are close to the steady state. The profit of the long FH and the TIH formulation is similar. The results show that the TIH and especially the long FH solution should be further refined in the transient region to achieve the same accuracy as the short FH solution at the beginning of the simulation, though a signal based adaption strategy has been applied. This means that a high number of grid refinement iterations and discretization parameters would be required to get the desired accuracy and grid size. The computational time required for the TIH and the long FH formulation is of the same order of magnitude ($20 - 40\,\mathrm{s}$).

7. CONCLUSIONS

In this contribution, two strategies for the numerical solution of discounted economic NMPC on infinite horizons have been presented. The first formulation replaces the infinite horizon by a finite one in a moving horizon setting. We suggested an algorithm which automatically determines a sufficiently long horizon length based on the turnpike property in order to improve control performance. In the second formulation, the infinite horizon is transformed to a finite horizon with the help of an adaptable transformation function. We suggested a two-stage problem formulation such that the transient time intervals are not compressed on the transformed infinite horizon. For both formulations, we further improved solution accuracy by applying a control grid adaptation strategy online.

In future work, we will develop an improved control grid adaptation strategy because the analysis of the wavelet coefficients just heuristically determines regions, where additional grid points are required or should be removed, and does not reflect the influence of a finer grid on the optimal solution (cf. Würth and Marquardt (2013)). In this way, we will further enhance solution accuracy. Furthermore, the adaptation procedure for the transformation function of the second strategy will be automated by developing an algorithm similar to the algorithm suggested by Würth and Marquardt (2013).

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