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Discrete-time Sliding Mode Tracking Control for NMP Systems using Reduced Order Switching Function

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Abstract: In this paper design of reduced order switching function for a discrete-time uncertain nonminimum phase system in special coordinate basis form is proposed. The sliding mode control with this method guarantees the asymptotic stability of all states of system in presence matched disturbance. This problem is further extended to the tracking problem of discrete time uncertain nonminimum phase systems. The results obtained with reduced order sliding surface design are compared with the results of full order sliding surface.

Keywords: Sliding mode control, Reduced order switching surface, Special coordinate basis, Nonminimum phase systems.

1. INTRODUCTION

In nonminimum phase systems, the problem in tracking arises because the inverse of the system is unstable. Another problem associated with nonminimum phase system is that, the system is remained internally unstable even after stabilization of state trajectory. Therefore output may become unbounded in presence of disturbance.

Asymptotic tracking of arbitrary reference signals in a nonminimum phase system is classical control problem and yet it is challenging. In earlier work, Francis and Wonham (1976) solved the tracking problem in a linear systems by incorporating the dynamical model of plant in an exosystem that generate the reference signal . This property is commonly known as Internal Model Principle. Isidori and Byrnes (1990) solved the tracking problem by identifying the acceptable dynamics on a particular center manifold. Gopalswamy and Hedrick (1993) redefined the output variables to obtain a system with a stable zero dynamics. They solved this problem by extending center manifold technique and sliding mode control. Using a method of system center and the dynamic sliding manifold technique the nonminimum phase output tracking problem is solved by Shkolnikov and Shtessel (1999). In Lan (2008) a structural decomposition of a system is done to transform it into special coordinate basis (SCB) for easier tuning of composite nonlinear feedback control. A system in SCB shows zero structure (zero dynamics) of a system in state equation, which is stabilized by virtual gain in second transformation so that bounded reference states can be generated.

A sliding mode control (SMC) has been a topic of great interest for researchers because of the robustness against the matched disturbance, See Utkin (1977); Hung et al. (1993); Decarlo et al. (1988). This motivates the use of SMC for tracking systems. Tracking problem in sliding mode control is solved as a stabilization problem by expressing the system in terms of error between output and reference trajectory. Bandyopadhyay and Fulwani (2009) have solved the tracking problem using nonlinear sliding surface to achieve high performance in terms of transient response specifications for both the continuous time and discrete-time systems. Also refer to Bandyopadhyay et al. (2009) and references therein.

Basic problem in output tracking of nonminimum phase system is to obtain the bounded solution of a unstable zero dynamics. The high accuracy tracking can be achieved by computing noncausal inverse using a preview-based approach, which allows the inversion process to be applied online, see S.Devasia et al. (1996). In the similar manner Jeong and Utkin (1999) have obtained the bounded solution to the internal dynamics and solved the problem using SMC for tracking of predefined smooth arbitrary reference signal without an exosystem. They allowed noncausality to obtain the bounded solution for unstable zero dynamics and convert the tracking problem into stabilization of mismatch dynamics. A multirate output feedback based sliding mode control of nonminimum system is solved by computing noncausal inverse. See Bandyopadhyay and Janardhanan (2006). Output feedback tracking problem in the class of causal nonminimum phase uncertain nonlinear systems has been handled using higher order sliding mode technique by Baev et al. (2008). Recently, sliding mode control for a NMP system in SCB with matched disturbance using full order sliding surface has been addressed in Patil and Bandyopadhyay (2013). They have used virtual stabilization of unstable zeros using certain state transformation with some virtual gain. In Lan (2008), use of virtual stabilization of zero dynamics of the system in SCB form by using the transformation is found.

A sliding mode control design consists of two steps, selection of a sliding surface and design of a discontinuous control that enforces the system into sliding motion (refer, Edwards and Spurgeon (1998)). A dynamical behavior of the system during the sliding motion is governed by the sliding surface parameters, therefore the design of the sliding surface to achieve the stability and improvement in response characteristics in presence of the disturbance has received considerable attention of researchers. A sliding surface design with reduced order states is rarely found in literature. White (1983) proposed the reduced order switching function in the variable structure system via modal synthesis. A similar approach has been used in Paul et al. (1994) for the positioning of the piston in pneumatic actuator. In Bandyopadhyay et al. (2006), it has been shown that reduced order sliding surface can be designed if all eigenvalues of the system are not unstable. However, in former two cases sliding motion was quasi-sliding and in later case approach is limited to the plants with stable eigen values.

In this paper, we propose a method to design sliding surface with reduced order states that guarantees the asymptotic stability of all the states of a system in special coordinate basis form (SCB) for NMP systems. It is shown with the help of numerical simulation that results obtained from reduced order sliding surface are quite comparable with one obtained from full order sliding surface.

2. PROBLEM FORMULATION

Consider a discrete time uncertain nonminimum phase system

$$x(k+1) = Ax(k) + Bu(k) + d(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

Where $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}$ and y(k) $\in \mathbb{R}$ are state, input and output vectors of the system respectively. $d(k) \in \mathbb{R}$ is considered to be matched disturbance. Consider that system has relative degree r at output. Assume that (A, B) pair is stabilizable and (A, C) pair is observable.

2.1 Special Coordinate Basis

The special coordinate basis of linear time-invariant systems was first developed in the seminal work of Sannuti and Saberi (1987) and further unproven facts are rigorously proved by Chen (1998). SCB form displays the minimum phase part (unit delay sequence) and the zero dynamics in diagonal form and latter are driven by output function only. Such a decomposition can be obtained using specially designed software toolbox like one in MATLAB by Liu et al. (2005).

Assume that the disturbance $d(k) = 0$. There exist state and input transformations that transform the system (1) into the special coordinate basis (SCB) form which reveals the structure of the system. Define,

$$x(k) := \Gamma_s \bar{x}(k), \;\; u(k) := \Gamma_i \bar{u}(k) \tag{3}$$

such that

$$\begin{aligned}
x_a^-(k+1) &= A_{aa}^- x_a^-(k) + L_{ad}^- x_1(k)\\
x_a^+(k+1) &= A_{aa}^+(k) x_a^+ + L_{ad}^+ x_1(k)\\
\bar{x}_1(k+1) &= \bar{x}_2(k)\\
\bar{x}_2(k+1) &= \bar{x}_3(k)\\
&\vdots\\
\bar{x}_{r-1}(k+1) &= \bar{x}_r(k)\\
\bar{x}_r(k+1) &= \bar{E}\,\bar{x}(k) + \bar{u}(k)
\end{aligned} \tag{4}$$

and

$$y(k) = \bar{x}_1(k) \tag{5}$$

Where

$$\bar{x}(k) := \begin{bmatrix} x_a^-(k) & x_a^+(k) & \bar{x}_1(k) & \cdots & \bar{x}_r(k) \end{bmatrix}^T$$

$$\bar{E} := \begin{bmatrix} \bar{E}_{da}^- & \bar{E}_{da}^+ & \bar{E}_1 & \cdots & \bar{E}_r \end{bmatrix}$$

In this structure A_{aa}^- and A_{aa}^+ are diagonal matrices that contain stable and unstable invariant zeros of a system, respectively.

Remark 1. For the output tracking, the choice of control law that generate desired trajectory $(y_0(k), x_0(k))$ satisfying (4)-(5) could have the form

$$\bar{u}(k) = y_0(k+r) - \bar{E}\bar{x}_0(k) \stackrel{\Delta}{=} \bar{u}_0(k)$$

However, as A_{aa}^+ contains unstable zeros, x_a^+ go unbounded and so does the desired control $\bar{u}_0$.

2.2 Virtual Zero Placement

As the system (1) is stabilizable, there exists matrix F_a such that eigen values of $(A_{aa}^+ + L_{ad}^+ F_a)$ are placed within unit circle. Define the transformations in following manner that stabilizes the unstable zero dynamics.

$$\tilde{x}(k) = \tilde{\Gamma}_s^{-1}\bar{x}(k), \;\; \bar{u}(k) = \tilde{\Gamma}_i \tilde{u}(k) \tag{6}$$

where

$$\tilde{\Gamma}_s^{-1} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0\\
0 & 1 & 0 & \cdots & 0\\
0 & -F_a & 1 & \cdots & 0\\
0 & -F_a A_{aa}^+ & -F_a L_{ad}^+ & \cdots & 0\\
0 & -F_a (A_{aa}^+)^2 & -F_a A_{aa}^+ L_{ad}^+ & \cdots & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
0 & -F_a (A_{aa}^+)^{r-1} & -F_a (A_{aa}^+)^{r-2} L_{ad}^+ & \cdots & 1
\end{bmatrix}$$

So the system in new coordinate space is represented by

$$\begin{aligned}
\tilde{x}_a^-(k+1) &= A_{aa}^- \tilde{x}_a^- + L_{ad}^- \tilde{x}_1(k) + L_{ad}^+ F_a \tilde{x}_a^+(k)\\
\tilde{x}_a^+(k+1) &= (A_{aa}^+ + L_{ad}^+ F_a)\tilde{x}_a^+(k) + L_{ad}^+ \tilde{x}_1(k)\\
\tilde{x}_1(k+1) &= \tilde{x}_2(k)\\
\tilde{x}_2(k+1) &= \tilde{x}_3(k)\\
&\vdots\\
\tilde{x}_{r-1}(k+1) &= \tilde{x}_r\\
\tilde{x}_r(k+1) &= \tilde{E}\tilde{x}(k) + \tilde{u}(k)
\end{aligned} \tag{7}$$

and output

$$y(k) = F_a \tilde{x}_a^+ + \tilde{x}_1(k) := \tilde{C}\tilde{x}(k) \tag{8}$$

Where

$$\tilde{x}(k) := \left[\, x_a^-(k)\ x_a^+(k)\ \tilde{x}_1(k)\ \cdots\ \tilde{x}_r\,(k)\right]^T$$

$$\tilde{E} := \left[\, \tilde{E}_{da}^-\ \tilde{E}_{da}^+\ \tilde{E}_1\ \cdots\ \tilde{E}_r\,\right]$$

Here the intention for second transformation is clear. For the tracking of a constant reference $R(k)$, bounded solution $\tilde{x}_0$ can be derived by means of some control $\tilde{u}_0(k)$ such that $\|y_0(k) - R(k)\| \to 0$ as $k \to \infty$. Once the nominal trajectories are made available, tracking problem can easily be converted into stabilization of mismatch in the system trajectory and nominal trajectory.

Fact 2. Apparently zero (or internal) dynamics of the system is stable, however, output of the system is modified and keeping the nonminimum phase behavior of the system intact.

3. TRACKING CONTROL

3.1 Nominal Control

Consider that $\tilde{u}_0(k)$ be the nominal control that generates the nominal trajectory

$$\tilde{x}_0 = \left[\, \tilde{x}_{a0}^-\ \tilde{x}_{a0}^+\ \tilde{x}_{10}\ \cdots\ \tilde{x}_{r0}\,\right]^T$$

which satisfy the system in (7).

Lemma 3. For asymptotic tracking of constant reference $R(k)$, there exists a nominal control

$$\tilde{u}_0(k) = -\tilde{E}\tilde{x}_0(k) + G_a R(k+r) \tag{9}$$

that generates the nominal trajectory $\tilde{x}_0(k)$ and nominal output $y_0(k)$ such that $\|y_0(k) - R(k)\| \to 0$ as $k \to \infty$.

Where

$$G_a := [F_a(I - A_{aa}^+ - L_{ad}^+ F_a)^{-1} L_{ad}^+\ + 1]^{-1} \tag{10}$$

Proof. Applying nominal control (9) to the system (7) gives,

$$\tilde{x}_{10}(k) = G_a R(k) \tag{11}$$

and consequently the unstable zero dynamic subsystem is given by

$$\tilde{x}_{a0}^+(k+1) = (A_{aa}^+ + L_{ad}^+ F_a)\tilde{x}_{a0}^+(k) + L_{ad}^+ G_a R(k) \tag{12}$$

Define $A_{as}^+ := (A_{aa}^+ + L_{ad}^+ F_a)$ and rewriting (12),

$$\tilde{x}_{a0}^+(k+1) = (A_{as}^+)\,\tilde{x}_{a0}^+(k) + L_{ad}^+ G_a R(k) \tag{13}$$

For constant reference $R(k)$, the solution to the (13) is given by

$$\begin{aligned}\tilde{x}_{a0}^+(k) &= (A_{as}^+)^k \tilde{x}_{a0}^+(0) + \sum_{j=0}^{k-1} (A_{as}^+)^{k-j-1} L_{ad}^+ G_a R(k)\\ &= (A_{as}^+)^k \tilde{x}_{a0}^+(0) + \sum_{j=0}^{k-2} (A_{as}^+)^{k-j-1} L_{ad}^+ G_a R(k)\\ &\quad + L_{ad}^+ G_a R(k)\end{aligned} \tag{14}$$

Pre-multiplying (14) by A_{as}^+ and rearranging the terms,

$$\begin{aligned}(A_{as}^+)\,\tilde{x}_{a0}^+(k) &= (A_{as}^+)^{k+1}\tilde{x}_{a0}^+(0) + (A_{as}^+)^k L_{ad}^+ G_a R(k)\\ &\quad + \sum_{j=0}^{k-2} (A_{as}^+)^{k-j-1} L_{ad}^+ G_a R(k)\end{aligned} \tag{15}$$

Subtracting (15) from (14), we get

$$\begin{aligned}(I - A_{as}^+)\,\tilde{x}_{a0}^+(k) &= (I - A_{as}^+)(A_{as}^+)^k\,\tilde{x}_{a0}^+(0)\\ &\quad - (A_{as}^+)^k L_{ad}^+ G_a R(k) + L_{ad}^+ G_a R(k)\\ \Rightarrow\ \ \tilde{x}_{a0}^+(k) &= (A_{as}^+)^k\,\tilde{x}_{a0}^+(0)\\ &\quad - (I - A_{as}^+)^{-1}(A_{as}^+)^k L_{ad}^+ G_a R(k)\\ &\quad + (I - A_{as}^+)^{-1} L_{ad}^+ G_a R(k)\end{aligned} \tag{16}$$

As A_{as}^+ is a stable matrix, $(A_{as}^+)^k \to 0$ as $k \to \infty$. so

$$\tilde{x}_{a0}^+(\infty) = (I - A_{as}^+)^{-1} L_{ad}^+ G_a R(k) \tag{17}$$

From (8) and (9), the output of a system under nominal control is given by,

$$y_0(k) = F_a \tilde{x}_{a0}^+(k) + G_a R(k) \tag{18}$$

So with constant reference $R(k)$, as $k \to \infty$ the output of the system tends to

$$\begin{aligned}y_0(\infty) &= F_a(I - A_{as}^+)^{-1} L_{ad}^+\ G_a R(k) + G_a R(k) \qquad (19)\\ &= [F_a(I - A_{as}^+)^{-1} L_{ad}^+\ + 1]\ G_a R(k)\\ &= [F_a(I - A_{aa}^+ - L_{ad}^+ F_a)^{-1} L_{ad}^+\ + 1]\ GR(k) \qquad (20)\end{aligned}$$

Follow from the definition (10), $y_0(k) \to R(k)$ as $k \to \infty$. This completes the proof.

Remark 4. If zeros are not placed on unit circle, there exists such a invertible scalar G_a. Also reference $R(k)$ can be generated from r^{th} unit delay of $R(k+r)$. This can be easily obtained from r^{th} order stable reference system.

3.2 Tracking via Stabilization of mismatch dynamics

Consider that $\tilde{x}_0(k)$ be the nominal trajectory that satisfies the system (7) and $\tilde{u}_0(k)$ be the nominal control as in (9), we can write

$$\tilde{x}_0(k+1) = \tilde{A}\tilde{x}_0(k) + \tilde{B}\tilde{u}_0(k) \tag{21}$$

where $\tilde{A} = \Gamma_s^{-1} A \Gamma_s$ and $\tilde{B} = \Gamma_s^{-1} B$ Define mismatch in the system and nominal trajectories.

$$\begin{aligned}\Delta\tilde{x}(k) &:= \tilde{x}(k) - \tilde{x}_0(k)\\ &:= \left[\, \Delta\tilde{x}_a^-(k)\ \Delta\tilde{x}_a^+(k)\ \Delta\tilde{x}_1(k)\ \cdots\ \Delta\tilde{x}_r(k)\,\right]^T\end{aligned}$$

and

$$\Delta\tilde{u}(k) := \tilde{u}(k) - \tilde{u}_0(k)$$

So the tracking problem is converted into stabilization problem by describing the system into mismatch variables as defined above.

$$\Delta\tilde{x}(k+1) = \tilde{A}\Delta\tilde{x}(k) + \tilde{B}\Delta\tilde{u}(k) \tag{22}$$

As the system is assumed to be stabilizable, there exists a state feedback control

$$\Delta\tilde{u}(k) = -F\Delta\tilde{x}(t) \tag{23}$$

such that $\|\Delta\tilde{x}(k)\| = \|\tilde{x}(k) - \tilde{x}_0(k)\| \to 0$ as $k \to \infty$. It implies that $\|\tilde{y}(k) - \tilde{y}_0(k)\| \to 0$.

However, when high performance transient response in presence of disturbance is desirable, merely state feedback control as in (23) is not enough. In the following sections sliding mode control is designed to achieve the robustness and high performance transient response.

4. SLIDING MODE CONTROL

4.1 Reduced order Switching Function

If the system is transformed into SCB structure, then it is possible to construct the the sliding surface with reduced order state vector such that the state trajectory of the system reaches the sliding surface in finite time and remained thereon thereafter. Define,

$$\tilde{A}_d := \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right],\ \tilde{x}_d(k) := \left[\begin{array}{c} \tilde{X}_1(k) \\ \hline \tilde{X}_2(k) \end{array}\right] = \left[\begin{array}{c} \tilde{x}_1(k) \\ \vdots \\ \tilde{x}_{r-1}(k) \\ \hline \tilde{x}_r(k) \end{array}\right] \quad (24)$$

Input matrix and disturbance $d(k)$ after transformations (3) and (6) is given by

$$\tilde{B} := \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix},\ \tilde{d}(k) := (\Gamma_s \tilde{\Gamma}_s)^{-1} d(k) := \begin{bmatrix} 0 \\ \tilde{d}_2(k) \end{bmatrix}$$

So minimum phase part of system dynamics can be written as ,

$$\tilde{X}_1(k+1) = \tilde{A}_{11}\tilde{X}_1(k) + \tilde{A}_{12}\tilde{X}_2(k) \quad (25)$$

$$\tilde{X}_2(k+1) = \tilde{A}_{21}\tilde{X}_1(k) + \tilde{A}_{22}\tilde{X}_2(k) + \tilde{u}(k) + \tilde{d}_2(k) \quad (26)$$

Note that $(\tilde{A}_{11}, \tilde{A}_{12})$ pair inherits the controllability property from the system.

Theorem 5. If the system (7) in SCB form has relative degree $1 < r < n$ at output, then it is possible to construct the sliding surface,

$$s(k) := c^T \tilde{x}_d(k) := \begin{bmatrix} c_1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{X}_1(k) \\ \tilde{X}_2(k) \end{bmatrix} \quad (27)$$

using at least r^{th} order state vector $x_d(k)$ as in (24) such that motion of the full state vector $\tilde{x}(k)$ along the sliding surface is asymptotically stable.

Proof. Sliding motion along the surface (27) can be evaluated by setting the surface $s(k) = 0$. This implies $\tilde{X}_2(k) = -c_1\tilde{X}_1(k)$. Follow from equation (25), the sliding motion along the surface $s(k)$ is given by

$$\tilde{X}_1(k) = (\tilde{A}_{11} - \tilde{A}_{12}c_1)\tilde{X}_1(k) \quad (28)$$

As $(\tilde{A}_{11} - \tilde{A}_{12}c_1)$ is stable by design, $\|\tilde{X}_1(k)\| \to 0$ as $k \to \infty$. Also, $\tilde{X}_2(k)$ state follows the $\tilde{X}_1(k)$ motion, since they have algebraic relationship via sliding surface.

Let us investing the effect of sliding motion on remaining $(n - r)$ states of the zero dynamics. It is evident from the structure of the system that zero dynamics $[\tilde{x}_a^-(k)^T\ \tilde{x}_a^+(k)^T]^T$ is driven by the state $\tilde{X}_1(k)$. As $\tilde{X}_1(k)$ state vector is bounded and zero dynamics is stable, it follows $\|[\tilde{x}_a^-(k)^T\ \tilde{x}_a^+(k)^T]^T\| \to 0$ as $\|\tilde{X}_1(k)\| \to 0$.

Hence motion of the full state vector of a system (7) is asymptotically stable along the sliding surface $s(k)$. As remaining $(n - r)$ states are stable, some of them can be included to design the sliding surface. So reduced order sliding surface using reduced order state vector of any order higher than or equal to r can be designed. This completes the proof.

4.2 Existence of the sliding mode

Theorem-5 proves the asymptotic stability of the sliding motion of the entire state vector $x(k)$ along the sliding surface $s(k)$. However, the sliding mode exists only when $x_d(k)$ trajectory originated from any initial condition is forced to reach the sliding surface $s(k)$ in finite time.

Proposition 6. If the system in SCB form (7) has relative degree $1 < r < n$ at output, then the control law with $\tilde{d}_2(k) \leq |\tilde{d}_2(k)|_{max}$

$$\tilde{u}(k) = -c^T(k+1)\tilde{A}\tilde{x}_d(k) - \tilde{d}_2(k-1) \quad (29)$$

enforces the trajectory $x_d(k)$ of the system from any initial condition to reach the quasi-sliding band

$$\|s(k)| \leq |\tilde{d}_2(k)|_{max}$$

in finite time and remained therein thereafter.

Proof. The control law for the system in the regular form is derived using reaching law due to Utkin (1993). To reach the sliding surface in one sampling period the reaching law is given by $s(k+1) = 0$. So from (27) control law can be derived as

$$0 = c_1\tilde{X}_1(k+1) + \tilde{X}_2(k+1)$$

$$\Rightarrow \tilde{u}(k) = -c^T(k+1)\tilde{A}\tilde{x}_d(k) - \tilde{d}_2(k) \quad (30)$$

However, the control law (30) can not be realized as it contains the uncertain term. If the disturbance is assumed to be constant during one sampling instant then estimation of the disturbance is possible, see Su et al. (2000).

$$\tilde{d}_2(k-1) = \tilde{X}_2(k) - \tilde{A}_{21}\tilde{X}_1(k-1) - \tilde{A}_{22}\tilde{X}_2(k-1) - \tilde{u}(k-1) \quad (31)$$

Substituting (31) in (30), we get

$$\tilde{u}(k) = -c^T(k+1)\tilde{A}\tilde{x}_d(k) - \tilde{d}_2(k-1)$$

With this control, the sliding surface dynamics is given by,

$$s(k+1) = -\tilde{d}_2(k-1) \Rightarrow |s(k)| \leq |\tilde{d}_2(k)|_{max}$$

Hence proved.

4.3 Tracking problem

Tracking problem can be converted into stabilization of mismatch in dynamics of the system and reference trajectory. Let $(\tilde{x}_{a0}^-(k), \tilde{x}_{a0}^+(k), \tilde{x}_d(k))$ be the trajectory that generate desired output $y_0(k) = R(k)$ by means of a control $\tilde{u}_0(k)$.

Define mismatch variables, $\Delta\tilde{u}(k) = \tilde{u}(k) - \tilde{u}_0(k)$

$$\Delta\tilde{x}(k) := \begin{bmatrix} \Delta\tilde{x}_a^-(k) \\ \Delta\tilde{x}_a^+(k) \\ \Delta\tilde{x}_d(k) \end{bmatrix} := \begin{bmatrix} \tilde{x}_a^-(k) - \tilde{x}_{a0}^-(k) \\ \tilde{x}_a^+(k) - \tilde{x}_{a0}^+(k) \\ \tilde{x}_d(k) - \tilde{x}_{d0}(k) \end{bmatrix}$$

So the mismatch dynamics of an integration subsystem in (7) in presence of disturbance can be written as

$$\Delta\tilde{X}_1(k+1) = \tilde{A}_{11}\Delta\tilde{X}_1(k) + \tilde{A}_{12}\Delta\tilde{X}_2(k) \tag{32}$$

$$\Delta\tilde{X}_2(k+1) = \tilde{A}_{21}\Delta\tilde{X}_1(k) + \tilde{A}_{22}\Delta\tilde{X}_2(k) + \Delta\tilde{u}(k) + \tilde{d}_2(k) \tag{33}$$

As (A, B) pair is assumed to be controllable, there exists matrix F such that $(A_{11} - A_{12}F)$ matrix is Hurwitz. Define the sliding surface with the reduced order state vector $\Delta x_d(k)$,

$$s(k) := c^T\tilde{\Delta}x_d(k) := \begin{bmatrix} c_1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{X}_1(k) \\ \tilde{X}_2(k) \end{bmatrix} \tag{34}$$

By the virtue of Theorem-5, full state of the system in mismatch dynamics will be asymptotically converging to the origin. From proposition-6, the sliding mode control law for the stabilization of mismatch dynamics can be given by

$$\Delta\tilde{u}(k) = -c^T\tilde{A}\Delta\tilde{x}_d(k) - \tilde{d}_2(k) \tag{35}$$

5. NUMERICAL EXAMPLE

Consider the system with sampling time $T_s = 0.1$ sec.

$$A = \begin{bmatrix} 0.1 & -0.7416 & 0.7071 & 0 & 0 \\ 0 & 0.2 & 1.0488 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0 \\ 0 & 0 & 0 & 1.3 & 1 \\ 0 & 0 & 0 & -0.04 & 1.3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.3536 & -0.5244 & 0.5 & 0 & 0 \end{bmatrix}, \; D = 0$$

The relative degree at the output is $r = 3$, and the system has invariant zeros at $z = 0.6, 1.3$ The system is transformed into SCB using transformations in (3) with

$$\Gamma_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1.0595 & 1.0595 & 0 & 0 & 0 \\ -0.4041 & 1.1112 & 2 & 0 & 0 \\ -0.0808 & 1.0001 & 2.4 & 2 & 0 \\ 0.0566 & 0 & 0.16 & -0.2 & 2 \end{bmatrix}; \; \Gamma_i = 1$$

Select the gain $F_a = -0.1473$ to place virtually the unstable zero at $z = 0.8$ and define the transformation (6)

$$\tilde{\Gamma}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -0.1473 & 1 & 0 & 0 \\ 0 & -0.1179 & -0.5 & 1 & 0 \\ 0 & -0.0943 & -0.4 & -0.5 & 1 \end{bmatrix};$$

The system after this transformation takes the desired form with

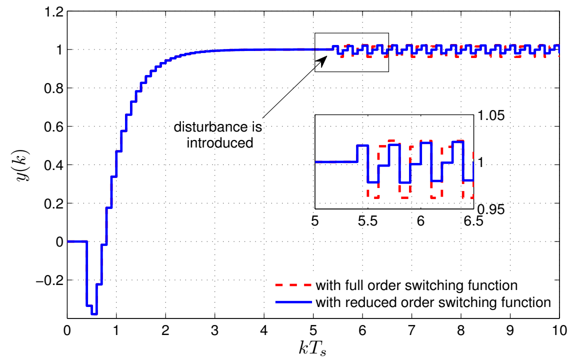

Fig. 1. Comparison of responses with reduced and full order switching function

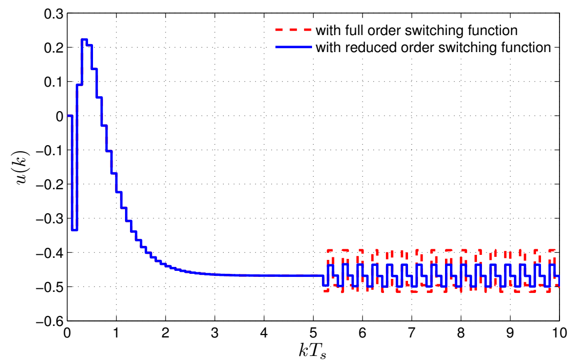

Fig. 2. Control input $u(k)$ for with reduced and full order switching function

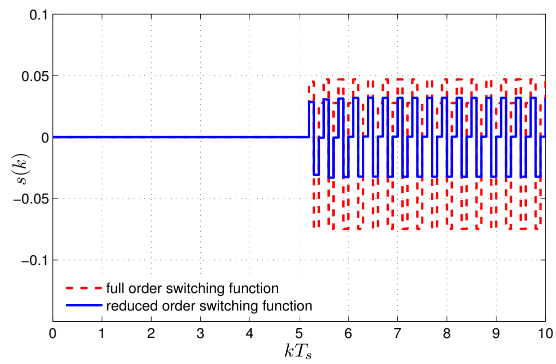

Fig. 3. Reduced and full order switching functions s(k)

$$\tilde{A} = \begin{bmatrix} 0.6 & -0.2083 & 1.4142 & 0 & 0 \\ 0 & 0.8 & 3.3941 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0214 & -0.0495 & -0.099 & -0.55 & 1.9 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & -0.1473 & 1 & 0 & 0 \end{bmatrix}, \; \tilde{D} = 0$$

As per *lemma-3*, choose $G_a = -0.6667$

Full order surface: Selecting Full order sliding surface gain

$$F = \begin{bmatrix} -0.0318 & 0.188 & 0.6775 & 0.7 \end{bmatrix}$$

$$s(k) = \begin{bmatrix} F \,\vdots\, 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_a^-(k) & \tilde{x}_a^+(k) & \tilde{x}_1(k) & \tilde{x}_2(k) \,\vdots\, \tilde{x}_3(k) \end{bmatrix}^T$$

so that dynamics of sliding motion has eigen values at $z = 0.1, 0.15, 0.2, 0.25$.

Reduced order surface: As relative degree of the system is 3, we design the sliding surface with reduced states $\tilde{x}_1(k), \tilde{x}_2(k)$ and $\tilde{x}_3(k)$

$$\tilde{A}_d = \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline -0.099 & -0.55 & 1.9 \end{array}\right]$$

Designing reduced order switching function such that dynamics of sliding motion has eigen values at $z = 0.1, 0.15$

$$s(k) = \left[\begin{array}{c:c} c_1 & 1 \end{array}\right] \left[\begin{array}{c} \tilde{x}_1(k) \\ \tilde{x}_2(k) \\ \hdashline \tilde{x}_3(k) \end{array}\right] \text{ with } \mathrm{c}_1 = [\, 0.015 \;\; -0.25 \,]$$

Fig. 1 shows the responses of a system with full and reduced order sliding surfaces. To verify the robustness property, the disturbance vector

$$\tilde{d}(k) = 0.2 \sin(6\pi \mathrm{k}) \, [\, 0 \;\; 0 \;\; 0 \;\; 0.1 \,]^{\mathrm{T}}$$

is introduced after 5 seconds. Fig. 2 shows the comparison of control efforts while Fig. 3 exhibit the switching functions $s(k)$ using full and reduced order system states.

6. CONCLUSION

It has been shown here that reduced order switching function of order higher than or equal to the output relative degree of the system can be designed effectively for stability and performance of any system. This method is also applicable to the system with unstable internal dynamics and can be used for output tracking of constant reference signal. It is also possible to design control law with only states that are involved in the design of switching function by considering remaining state variables a matched disturbance.

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