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A Bivalued Observer for a Class of Uncertain Reactors

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Abstract: In order to design an observer for a dynamical system it is usually required that the model of the process is observable, or at least detectable. However, in some cases, in particular when unknown uncertainties are present, none of these properties is available. We would be tempted to give up the possibility of constructing an observer. However, in certain situations a multivalued observer, an observer giving multiple possible values of the state can be a reasonable alternative. In this paper we will analyze a realistic reactor model for which this situation is met: the process is unobservable and undetectable but a bivalued observer can be designed that provides a very satisfactory solution to the estimation problem.

Keywords: Observability, Discontinuous Observers, Bioprocesses, Multivalued Observers.

1. INTRODUCTION

It is well-known, that the possibility of constructing an observer is tied to the observability/detectability properties of the system model. When only the initial conditions are unknown, observability corresponds to the (theoretical) possibility of estimating the state in a finite time-horizon, whereas if the system is only detectable the state estimation can only be attained asymptotically. In a more realistic case, besides the uncertainty in the initial conditions, also model parameters or even input uncertainties are usually present. In these cases, the concepts of observability/detectability have to be modified in order to consider the given uncertainties. Observability would then correspond to the possibility of reconstructing the state in a finite-horizon despite of the uncertainties acting on the system, while detectability would allow this asymptotically. All these concepts require that given an input/output pair for a system, there is (asymptotically) a *unique* possible state trajectory corresponding to this input/output behavior [6, 7, 8, 9, 12, 13, 16]. In these cases there are many methods to design observers for such a system, and this is still an active research topic.

However, what happens if the (asymptotic) uniqueness condition is **not** satisfied? What if for an input/output pair there is more than one state trajectory that is compatible with the behavior of the system, and these trajectories are not convergent to each other? In this case the system is certainly neither observable nor detectable, and it is impossible to construct an observer for it, at least not a single-valued observer, that is, an observer giving the estimation of a *single* state trajectory for the system. A natural solution to this problem would be to construct a multivalued observer, giving an estimation of **all** possible trajectories compatible with the input/output behavior of the system. The nice paper [5] proposes such kind of observers from a very general perspective without giving a concrete solution.

Our objective in this paper is to study a (simple) but very important class of (bio)reactors, for which exactly this problem appears. This simple system shows that the unobservability problem described above is realistic, and not only a mathematical curiosity. Moreover, we use a method to study the observability/detectability properties of the nonlinear model, originally proposed in [10] for the induction machine, and also used in [15] for a class of Bioreactors, and we obtain a clear understanding of the properties of the system. Finally, we construct bivalued observers for the states and the unknown input of the system, that solve the estimation problem completely.

2. THE CLASS OF SYSTEMS CONSIDERED

The bioreactor model considered is

$$R : \begin{cases} \dot{X}(t) = \mu(S) X - K_d X - DX \ , \\ \dot{S}(t) = -\beta \mu(S) X + D(S_{in}(t) - S) \ , \end{cases} \tag{1}$$

with $X(t_0) = X_0$, $S(t_0) = S_0$, and where $X \geq 0$ is the biomass concentration in the reactor, $S \geq 0$ is the substrate concentration, $\mu : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is the specific growth rate, $D \geq 0$ is the dilution rate, $K_d \geq 0$ is the mortality rate, $S_{in}(t) \geq 0$ is the (time-varying) substrate concentration present in the inflow and $\beta > 0$ is a yield coefficient. The given reactor model is widely used, for example in a process for the treatment of industrial wastewater (see e.g. [1, 2, 4]). In this case the specific growth rate $\mu(S)$, the only nonlinearity of the system, is of non monotonous type. A typical model for it is given by the Haldane law

$$\mu\left(S\right)=\frac{\mu_0 K_I S}{S^2+K_I S+K_S K_I} \tag{2}$$

with positive and constant kinetic parameters μ_0, K_I and K_S, which are assumed to be known. The maximum value $\mu^*=\mu\left(S^*\right)$ is reached at the point $S^*=\sqrt{K_S K_I}$. Realistic numerical values for the parameters are

$$\begin{array}{lll} \mu_0=0.072\ h^{-1} & K_I{=}50\ mg/l & K_S=2\,mg/l \\ K_d=0 & S_{in}=200\ mg/l & \beta=2 \\ X_0=4000\ mg/l & S_0=50\ mg/l & \end{array} \tag{3}$$

For the bioreactor (1) given an initial condition $(X_0,\,S_0)$ and an input $(D\left(t\right),\,S_{in}\left(t\right))$, the system has a unique solution $X\left(t\right)=\varphi_X\left(t,\,t_0,\left(X_0,\,S_0\right),\left(D\left(t\right),\,S_{in}\left(t\right)\right)\right)$ and $S\left(t\right)=\varphi_S\left(t,\,t_0,\left(X_0,\,S_0\right),\left(D\left(t\right),\,S_{in}\left(t\right)\right)\right)$.

3. OBSERVABILITY ANALYSIS

For the observability analysis it will be assumed that the model parameters and the input D are known, and that the only state available for measurement is X. Moreover, it will be assumed that the inflow substrate concentration $S_{in}\left(t\right)$ is unknown, what is a realistic situation for the particular application we are considering.

The basic (state) observability/detectability question is if the available information is sufficient to determine *uniquely* the state of the system in a finite time horizon (observability) or at least asymptotically (detectability).

Two different pairs of initial conditions and inputs, $\left[\left(X_{10},\,S_{10}\right),\left(D\left(t\right),\,S_{in1}\left(t\right)\right)\right]$ and $\left[\left(X_{20},\,S_{20}\right),\left(D\left(t\right),\,S_{in2}\left(t\right)\right)\right]$, for system (1) are *distinguishable* in a time interval $t\in[0,\,T]$, if their corresponding trajectories,

$$\left(X_1\left(t\right),\,S_1\left(t\right)\right)=\varphi\left(t,\,t_0,\left(X_{10},\,S_{10}\right),\left(D\left(t\right),\,S_{in1}\left(t\right)\right)\right)$$
$$\left(X_2\left(t\right),\,S_2\left(t\right)\right)=\varphi\left(t,\,t_0,\left(X_{20},\,S_{20}\right),\left(D\left(t\right),\,S_{in2}\left(t\right)\right)\right)$$

have different *measurements*, i.e. $X_1\left(t\right)\neq X_2\left(t\right)$. Otherwise they are indistinguishable. We also say that the trajectories are (in)distinguishable.

Observability (in a time interval $t\in[0,\,T]$) is therefore the absence of *Indistinguishable Trajectories (IT)*, since then it is possible to determine the trajectory causing the measurements. System (1) is *Detectable* if the difference of two indistinguishable trajectories vanishes as the time horizon becomes unbounded, i.e.

$$\begin{aligned} lim_{t\to\infty}\,&\|\varphi\left(t,\,t_0,\left(X_{10},\,S_{10}\right),\left(D\left(t\right),\,S_{in1}\left(t\right)\right)\right)-\\ &-\varphi\left(t,\,t_0,\left(X_{20},\,S_{20}\right),\left(D\left(t\right),\,S_{in2}\left(t\right)\right)\right)\|=0\,. \end{aligned}$$

Note that these definitions are basically the same used for the classical observability analysis, but in this case we include the Unknown Input in the definition, that can be different in the two trajectories, since they are unmeasured.

In order to asset the observability/detectability properties for the bioreactor (1) we will calculate the whole set of *Indistinguishable Trajectories (IT)*, that we call the *Indistinguishable Dynamics (ID)*. For this consider two copies of the same system

$$\begin{aligned} R_1&:\begin{cases}\dot X\left(t\right)=\mu\left(S\right)X-K_dX-DX\ ,\\ \dot S\left(t\right)=\ -\beta\mu\left(S\right)X+D\left(S_{in}\left(t\right)-S\right)\ ,\end{cases}\\ R_2&:\begin{cases}\dot X_2\left(t\right)=\mu\left(S_2\right)X_2-K_dX_2-DX_2\ ,\\ \dot S_2\left(t\right)=-\beta\mu\left(S_2\right)X_2+D\left(S_{in2}\left(t\right)-S_2\right)\ .\end{cases} \end{aligned} \tag{4}$$

with $X\left(t_0\right)=X_0$, $S\left(t_0\right)=S_0$, $X_2\left(t_0\right)=X_{20}$, $S_2\left(t_0\right)=S_{20}$. Introducing the deviation variables

$$e_X=X_2-X\ ,\ e_S=S_2-S\ ,\ e_{in}=S_{in2}-S_{in}$$

the system can be written as

$$R:\begin{cases}\dot X\left(t\right)=\mu\left(S\right)X-K_dX-DX\ ,\ X\left(t_0\right)=X_0\\ \dot S\left(t\right)=-\beta\mu\left(S\right)X+D\left(S_{in}\left(t\right)-S\right)\ ,\ S\left(t_0\right)=S_0\end{cases} \tag{5}$$

$$E:\begin{cases}\dot e_X\left(t\right)=\mu\left(S+e_S\right)\left(X+e_X\right)-\mu\left(S\right)X-\\ \qquad -K_de_X-De_X\ ,\ e_X\left(t_0\right)=e_{X0}\\ \dot e_S\left(t\right)=-\beta\mu\left(S+e_S\right)\left(X+e_X\right)+\\ \qquad \beta\mu\left(S\right)X+D\left(e_{in}-e_S\right)\ ,\ e_S\left(t_0\right)=e_{S0}\end{cases} \tag{6}$$

Note that systems R_1 and R_2 in (4) produce two different trajectories of the bioreactor (1), and $(e_X,\,e_S)$ in system E (6) corresponds to the difference of these two trajectories. Consequently, two trajectories are indistinguishable during the interval $t\in[0,\,T]$ if and only if in system (5-6) $e_X\left(t\right)\equiv0$ and $e_S\left(t\right)\neq0$ during that time interval. The set of indistinguishable trajectories corresponds to the set of solutions of the Differential-Algebraic System (DA) obtained from (5-6) by setting $e_X\left(t\right)\equiv0$, i.e.

$$P:\begin{cases}\dot X\left(t\right)=\mu\left(S\right)X-K_dX-DX\ ,\\ \dot S\left(t\right)=-\beta\mu\left(S\right)X+D\left(S_{in}\left(t\right)-S\right)\ ,\\ \dot e_S\left(t\right)=-\beta\left[\mu\left(S+e_S\right)-\mu\left(S\right)\right]X+D\left(e_{in}-e_S\right)\ ,\end{cases} \tag{7}$$

$$A:\{0=\left[\mu\left(S+e_S\right)-\mu\left(S\right)\right]X\ , \tag{8}$$

with $X\left(t_0\right)=X_0$, $S\left(t_0\right)=S_0$, $e_S\left(t_0\right)=e_{S0}$, such that $e_S\left(t\right)\neq0$ during the time interval. This system describes all pairs of indistinguishable trajectories that can be generated by the plant. This is so, since the solutions of this system (7-8) correspond to trajectories of the plant that have the same output with the same (known) input but possibly different initial states and unknown inputs. Note that this system is three dimensional, with states $(X,\,S,\,e_S)$, and three inputs $(D\left(t\right),\,S_{in}\left(t\right),\,e_{in}\left(t\right))$, and it has a one dimensional algebraic restriction.

The algebraic equation (8) describes a (two dimensional) surface on the state space $(X,\,S,\,e_S)$, and the trajectories of the Dynamical System (7) have to stay on that surface (manifold) during the indistinguishability time interval. For this to be possible the initial conditions of (7) have to be selected to lie on the surface (8). However, this is not sufficient, since the surface (8) is not usually a (positive) invariant set for the dynamics (7). Therefore we look for a submanifold (a lower dimensional surface contained in (8)) that can be made (positively) invariant by designing the inputs $(D\left(t\right),\,S_{in}\left(t\right),\,e_{in}\left(t\right))$ adequately. The procedure to do this consists in deriving with respect to the time the algebric restriction repeteadly until an input appears in the obtained equation. When this stage has been reached, the input can (under appropriate conditions) be used to satisfy that algebraic equations and all the ones previously obtained. All algebraic restrictions obtained in this form define the submanifold where the ID (7-8) can "live", and the dynamical system evolving in this manifold is the Indistingushable Dynamics of the plant.

Note first that (8) can be satisfied if $X\left(t\right)=0$ for $t\in[0,\,T]$, i.e. if there is no biomass in the bioreactor. In this case, $X\left(t\right)$ will stay in $X\left(t\right)=0$ for all future times, and all trajectories of the bioreactor (1) will be indistinguishable. Since this is an undesirable situation of no interest, we will consider that $X(t) \neq 0$ for $t \in [0, T]$. Under this assumption the algebraic restriction (8) becomes

$$\mu(S + e_S) = \mu(S) \ . \tag{9}$$

Remark 1. Note that, when μ is a monotonic function equation (9) implies that $e_S = 0$, so that one concludes that there is no indistinguishable dynamics, and consequently the system is globally observable (with unknown inputs).

Deriving (9) w.r.t. t ($\dot{\mu}(S + e_S) = \dot{\mu}(S)$) one obtains

$$\partial_S \mu(S + e_S)\left[-\beta\mu(S + e_S) X + D(S_{in}(t) - S) + \right. \\ \left. + D(e_{in} - e_S)\right] = \partial_S \mu(S)\left[-\beta\mu(S) X + D(S_{in}(t) - S)\right] ,$$

where ∂_S represents the partial derivative with respect to S. Using (9) this can be rewritten as

$$\partial_S \mu(S + e_S) D(e_{in} - e_S) = \left[\partial_S \mu(S) - \partial_S \mu(S + e_S)\right] \times \\ \left[-\beta\mu(S) X + D(S_{in}(t) - S)\right] \ . \tag{10}$$

If we consider the case of a continuous or fed batch reactor, i.e. $D(t) \neq 0$, then (10) will be satisfied if the input $S_{in}(t)$ is selected so that

$$S_{in}(t) = S + \frac{\beta\mu(S) X}{D} + \frac{\partial_S \mu(S + e_S)(e_{in} - e_S)}{\left[\partial_S \mu(S) - \partial_S \mu(S + e_S)\right]} \ . \tag{11}$$

Since for the Haldane Law (2)

$$\partial_S \mu(S) = \frac{1}{\mu_0}\left(\frac{K_S}{S^2} - \frac{1}{K_I}\right)\mu^2(S)$$

one can write

$$\frac{\partial_S \mu(S + e_S)}{\partial_S \mu(S) - \partial_S \mu(S + e_S)} = \frac{1 - \frac{1}{K_S K_I}(S + e_S)^2}{(2S + e_S) e_S} S^2 ,$$

when (9) is satisfied, the input (11) can be expressed as

$$S_{in} = S + \frac{\beta\mu(S) X}{D} + \frac{1 - \frac{1}{K_S K_I}(S + e_S)^2}{(2S + e_S) e_S} S^2 (e_{in} - e_S) \ . \tag{12}$$

Since this input is feasible for many situations (recall that $e_{in}(t)$ is also an input to be selected), we can conclude that the bioreactor 1 is globally not (state) observable (with unknown input S_{in}).

When the trajectory of the system (7-8) starts on the manifold (8), i.e. $\mu(S_0 + e_{S0}) = \mu(S_0)$ and the input (12) is applied, then the trajectory remains for the future on the manifold (8), that is the relationship $\mu(S(t) + e_S(t)) = \mu(S(t))$ is satisfied. For the Haldane Law (2) this can be expressed also as

$$\frac{(S + e_S)}{(S + e_S)^2 + K_I(S + e_S) + K_S K_I} = \frac{S}{S^2 + K_I S + K_S K_I} ,$$

or solving for e_S (when $S \neq 0$) we find two solutions

$$e_S(t) = \begin{cases} 0 \\ \dfrac{K_S K_I}{S(t)} - S(t) \end{cases} .$$

A further simplification of the expression (12) can be achieved by using this last equation

$$S_{in}(t) = S + \frac{\beta\mu(S) X}{D} + \frac{S^{*2} - S^2}{S^4 - S^{*4}} S^2 \left(e_{in} + S - \frac{S^{*2}}{S}\right) , \tag{13}$$

where we have used the definition $S^* = \sqrt{K_S K_I}$ for value of S at which the growth rate $\mu(S)$ achieves its maximum value. Replacing the input (13) in the indistinguishable dynamics system (7-8) one obtains

$$ID : \begin{cases} \dot{X}(t) = \mu(S) X - K_d X - DX \ , \\ \dot{S}(t) = D\dfrac{S^{*2} - S^2}{S^4 - S^{*4}}\left(e_{in} + S - \dfrac{S^{*2}}{S}\right) S^2 \ , \\ e_S(t) = \begin{cases} 0 \\ \dfrac{K_S K_I}{S(t)} - S(t) \end{cases} \end{cases} \tag{14}$$

with $X(t_0) = X_0$, $S(t_0) = S_0$. Moreover, replacing the input (12) in the indistinguishable dynamics system (7-8) one obtains

$$P : \begin{cases} \dot{X}(t) = \mu(S) X - K_d X - DX \ , \\ \dot{S}(t) = D\dfrac{1 - \frac{1}{K_S K_I}(S + e_S)^2}{(2S + e_S) e_S} S^2 (e_{in} - e_S) \ , \\ \dot{e}_S(t) = D(e_{in} - e_S) \ , \end{cases}$$

$$A : \{0 = [\mu(S + e_S) - \mu(S)] X \ .$$

with $X(t_0) = X_0$, $S(t_0) = S_0$, $e_S(t_0) = e_{S0}$. Note that in general $\lim_{t\to\infty} e_S(t) \neq 0$ (except when $\lim_{t\to\infty} e_{in}(t) = 0$), showing that the bioreactor model (1) is globally not (state) detectable as well.

The interpretation of the Indistinguishable Dynamics (14) is as follows: Give an initial condition (X_0, S_0), a dilution rate $D(t) > 0$, and an input $e_{in}(t)$. The solution of (14) and (13) provides two indistinguishable trajectories of the bioreactor (1) given by: i) Trajectory $(X(t), S(t), S_{in}(t), D(t))$ and ii) trajectory $(X(t), S(t) + e_S(t), S_{in}(t) + e_{in}(t), D(t))$.

In Fig. 1 and 2 two examples of such pairs of indistinguishable trajectories are presented. In Fig. 1 the data for (14) were $(X_0, S_0, D, e_{in}) = (4000, 50, 0.2, 0)$, and in Fig. 2 they were $(X_0, S_0, D, e_{in}) = (4000, 50, 0.2, 100\sigma(t - 10))$, where $\sigma(t)$ is a step function. Note that in Fig. 1 the values of the substrate concentrations of the two indistinguishable trajectories converge to each other, i.e.

$$\lim_{t\to\infty} S_1(t) = \lim_{t\to\infty} S_2(t) = S^* ,$$

where $S^* = \sqrt{K_I K_S}$ is the values at which $\mu(S)$ achieves its maximum value. This corresponds to a detectable behavior. Incidentally, detectable behavior does always occur if $e_{in}(t) \to 0$.

The indistinguishable trajectories presented in Fig. 2 are however not "detectable", since they meet at the point S^* , and then they diverge.

It is important to notice that there is a kind of bifurcation phenomenon in the possible indistinguishable trajectories for the plant. Recall that for every instant of time there are two possible values of S causing the measurements. Since the trajectories of the plant are (absolutely) continuous functions of time it follows that the possible indistinguishable trajectories have to be continuous. If the indistinguishable trajectories never meet (see for example Fig. 2), then only two (indistinguishable) trajectories are possible. However, when the trajectories meet at some time point (see for example Fig. 2 at time about 10h), they can be continued in different manners. And this happens every time they meet again. For example, in Fig. 2 there exist four (instead of two) possible indistinguishable trajectories (see Fig. 2 on the Upper Left) : i) Start with the red dash-dot line, and after 10h continue with the red dash-dot line.

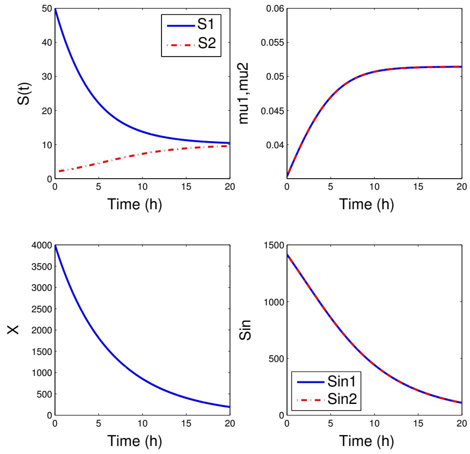

Fig. 1. Indistinguishable trajectories with $(X_0, S_0, D, e_{in}) = (4000, 50, 0.2, 0)$. Upper Left: The two different values of $S(t)$. Upper right: Both indistinguishable trajectories have the same value of $\mu(S(t))$. Lower Left: both indistinguishable trajectories have the same value of $X(t)$. Lower right: The value of $S_{in}(t)$ required to cause indistinguishable trajectories.

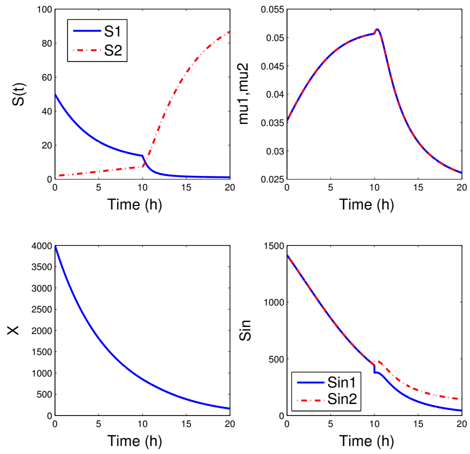

Fig. 2. Indistinguishable trajectories with $(X_0, S_0, D, e_{in}) = (4000, 50, 0.2, 100\sigma(t-10))$. Upper Left: The two different values of $S(t)$. Upper right: Both indistinguishable trajectories have the same value of $\mu(S(t))$. Lower Left: both indistinguishable trajectories have the same value of $X(t)$. Lower right: The value of $S_{in}(t)$ required to cause indistinguishable trajectories.

ii) Start with the red dash-dot line, and after 10h continue with the blue solid line. iii) Start with the blue solid line, and after 10h continue with the red dash-dot line. iv) Start with the blue solid line, and after 10h continue with the blue solid line. And for every possible indistinguishable trajectory there is a corresponding unknown input $S_{in}(t)$.

Our conclusion leads us to a very negative situation with respect to the possibility of designing a (state) observer for the bioreactor (1): There is no observer (however it is designed) able to estimate in finite time or asymptotically the unmeasured state S.

4. A MULTIVALUED OBSERVER FOR THE BIOREACTOR

In view of the negative results of the previous section it seems impossible to construct an observer for the bioreactor (1), when the substrate concentration in the inflow S_{in} is time-varying, arbitrary and unknown. This is indeed the case, if one looks for a univalued observer, that is, an observer giving only one possible value of the state variable.

Notice from (14) that the indistinguishable trajectories of (1) are given in pairs. That is, for every set of measured variables $(X(t), D(t))$ there exists *at most* a pair of possible values of the substrate concentration, given by $(S(t), S(t)+e_S(t))$, solutions of (14). In fact, for every pair of measurements $(X(t), D(t))$ (with $D(t)>0$) the pair of indistinguishable trajectories $(S(t), S(t)+e_S(t))$, solutions of (14), *does always exist.* So it is impossible to determine from the measurements which of the two is the right one.

In this situation it seems reasonable to construct an observer that, using the measurements and the (known) model of the process provides both (equally) possible (indistinguishable) trajectories. We will call it a *bivalued observer*, in contraposition to the classical *univalued* one, that does not exist in our case.

4.1 A bivalued state observer

We first design a bivalued observer providing the two possible values of the state.

Proposition 2. The system

$$BSO:\begin{cases}\dot{\hat{X}}(t) = -k_1\phi_1(e_X)+\hat{\mu}X-K_dX-DX\ ,\\ \dot{\hat{\mu}}(t) = -k_2X\phi_2(e_X)\ ,\\ \hat{S}_1(t) = \dfrac{K_I(\mu_0-\hat{\mu}(t))-\xi}{2\hat{\mu}(t)}\\ \hat{S}_2(t) = \dfrac{K_I(\mu_0-\hat{\mu}(t))+\xi}{2\hat{\mu}(t)}\end{cases} \tag{15}$$

with $\xi=\sqrt{K_I^2(\mu_0-\hat{\mu}(t))^2-4K_SK_I\hat{\mu}^2(t)}$, $\hat{X}(t_0)=\hat{X}_0$, $\hat{\mu}(t_0)=\hat{\mu}_0$, $e_X=\hat{X}-X$ and where

$$\phi_1(e_X)=\gamma_1\sqrt{|e_X|}sign(e_X)+\gamma_2e_X\ ,\ \gamma_1>0\ ,\ \gamma_2\geq 0\ ,$$

$$\phi_2(e_X)=\frac{\gamma_1^2}{2}sign(e_X)+\frac{3}{2}\gamma_1\gamma_2\sqrt{|e_X|}sign(e_X)+\gamma_2^2e_X$$

and for sufficiently high gains $k_1>0$, $k_2>0$ provides in finite time an estimate of both possible indistinguishable states S_1 and S_2.

PROOF. We give a sketch of the proof. From the plant model (1) we can write

$$\dot{X}(t)=\mu(S)X-K_dX-DX\ ,$$
$$\dot{\mu}(t)=\frac{\partial\mu(S)}{\partial S}\left[-\beta\mu(S)X+D(S_{in}(t)-S)\right]\ .$$

The dynamics of the error e_X and $e_\mu=\hat{\mu}-\mu$ is given by

$$\dot{e}_X(t)=-k_1\phi_1(e_X)+Xe_\mu\ ,$$
$$\dot{e}_\mu(t)=-k_2X\phi_2(e_X)-\rho\ ,$$

where $\rho = \frac{\partial \mu(S)}{\partial S}\left[-\beta\mu(S)X + D(S_{in}(t) - S)\right]$, and it is bounded in a bounded region of the state space. The finite-time stability of this system can be analyzed in the same form as in [14], so that we conclude that $\hat{\mu} \to \mu$ in finite time. One concludes easily that $\left\{\hat{S}_1(t), \hat{S}_2(t)\right\} \to \{S_1(t), S_2(t)\}$ also in finite time. ■

An important issue in the implementation of the observer is related with the domain of the nonlinear map used in (15) to calculate the possible values of S. Its domain is given by $\hat{\mu} \in (0, \mu^*]$. Outside from this domain the function has unbounded values (e.g. when $\hat{\mu} = 0$) or complex values (e.g. when $\hat{\mu} > \mu^*$). Although the range of the function $\mu(S)$ of the system lie in this set, due to estimation and numerical errors, and due to noise, the values of $\hat{\mu}$ can lie outside its domain. To avoid the problems associated with this it is important to force the values of $\hat{\mu}$ to belong to its domain (this can be achieved by a saturation function).

It is important to note that the convergence concept used in the proposition is not a pointwise but a setwise convergence. This means that the set of estimated values of the substrate concentration $\left\{\hat{S}_1(t), \hat{S}_2(t)\right\}$, given by the observer (15), converges to the set of indistinguishable trajectories of the plant $\{S_1(t), S_2(t)\}$, i.e. $\left\{\hat{S}_1(t), \hat{S}_2(t)\right\} \to \{S_1(t), S_2(t)\}$ as $t \to \infty$, and after a finite time they become equal, i.e. $\left\{\hat{S}_1(t), \hat{S}_2(t)\right\} = \{S_1(t), S_2(t)\}$ for $t \geq \tau$. However, this does not mean that after a finite time $\hat{S}_1(t) = S_1(t)$ or $\hat{S}_2(t) = S_2(t)$, at least for all the time.

4.2 Estimation of the Unknown Input

Besides the estimation of the state variables it is also important to estimate the possible unknown inputs causing the measurements. Given a possible state trajectory, it is possible to estimate a possible input $S_{in}(t)$. The following observer is able to calculate, from the estimated state variable, the corresponding unknown input in finite time, if the derivative of $S_{in}(t)$ is uniformly bounded.

Proposition 3. The system

$$BUIO: \begin{cases} \dot{\hat{\mathfrak{S}}}_j(t) = -l_{j1}\phi_1(e_{\mathfrak{S}j}) - \beta\hat{\mu}X + \\ \qquad +D\left(\hat{S}_{inj}(t) - \hat{S}_j\right), \ \hat{\mathfrak{S}}_j(t_0) = \hat{\mathfrak{S}}_{j0} \\ \dot{\hat{S}}_{inj}(t) = -l_{j2}D\phi_2(e_{\mathfrak{S}j}), \ \hat{S}_{inj}(t_0) = \hat{S}_{inj0} \end{cases} \tag{16}$$

for $j = 1, 2$, and where

$$e_{\mathfrak{S}j} = \mathfrak{S}_j - \hat{S}_j$$
$$\phi_1(e_{\mathfrak{S}j}) = \nu_{1j}\sqrt{|e_{\mathfrak{S}j}|}sign(e_{\mathfrak{S}j}) + \nu_{2j}e_{\mathfrak{S}j},$$
$$\phi_2(e_{\mathfrak{S}j}) = \frac{\nu_{1j}^2}{2}sign(e_{\mathfrak{S}j}) + \frac{3}{2}\nu_{1j}\nu_{2j}\sqrt{|e_{\mathfrak{S}j}|}sign(e_{\mathfrak{S}j}) + \\ + \nu_{2j}^2 e_{\mathfrak{S}j}, \ \nu_{1j} > 0, \ \nu_{2j} \geq 0,$$

where $\hat{S}_j$ and $\hat{\mu}$ are obtained from (15), and for sufficiently high gains $l_{j1} > 0$, $l_{j2} > 0$ provides in finite time an estimate of S_{in} corresponding the the each possible (indistinguishable) states S_1 and S_2, estimated by the observer (15).

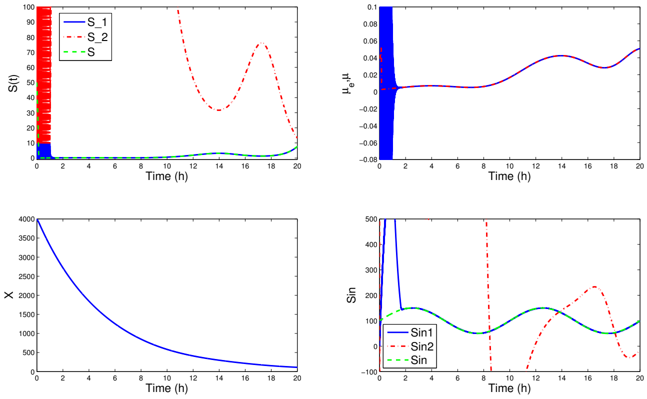

Fig. 3. State trajectories of the plant with $(X_0, S_0, D, S_{in}) = (4000, 50, 0.2\sigma(t), 100 + 50\sin(0.2\pi t))$, and the corresponding estimations of the bivalued observer without measurement noise. Upper Left: The plant's trajectory (green, dashed line) and the two different estimation values of $S(t)$ (red dash-dot line and blue solid line). Upper right: The behavior of the plant's specific growth rate $\mu(S(t))$ (red dash-dot line) and its estimation (blue solid line). Lower Left: The (measured) trajectory of biomass concentration $X(t)$. Lower right: The plant's (unmeasured) input $S_{in}(t)$ (green, dashed line) and the two different estimation values of $S_{in}(t)$ (red dash-dot line and blue solid line).

PROOF. It is similar to the previous one, and it is not given here because lack of space. ■

In Fig. 3 the behavior of these observers is presented considering that there is no measurement noise. In Fig. 4 the same results under measurement noise are given. Note that for each time instant the state and unknown input observers provide two estimation values for S and S_{in}. They coincide with the indistinguishable values for these variables after the finite convergence time (approx. 2 hours in the simulations). As discussed before it is theoretically impossible to decide which is the true trajectory in the reactor. However, using physical considerations it is possible to decide at some time instants which is the correct one. For example, in the Upper Left corner of Fig. 4 the estimation S_1 is very large during the initial period, and this could help to decide that it is not possible in the reactor. Or in the Lower Right corner of Fig. 4 the estimation S_{in2} of the unknown input is very large during the initial period, and it becomes negative around time 9 hours. Since this is impossible for a real reactor, one can know (a posteriori) that the true trajectory of the plant is given by the estimation $(S_1(t), S_{in1}(t))$.

5. CONCLUSIONS

We have studied a simple but very important class of (bio)reactors, with unknown inputs, that is neither observable nor detectable in any of the classical senses. Thus it is impossible to construct a (classical or univalued) observer for it. We were able to completely characterize the observability properties of this realistic system, and we conclude

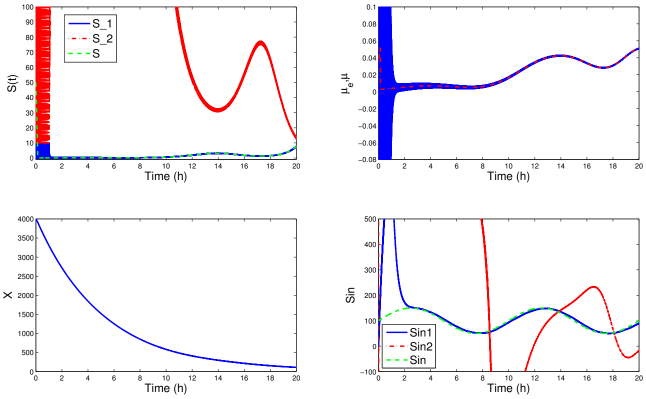

Fig. 4. State trajectories of the plant with $(X_0, S_0, D, S_{in}) = (4000, 50, 0.2\sigma(t), 100 + 50\sin(0.2\pi t))$, and the corresponding estimations of the bivalued observer with additive measurement noise simulated by $0.05\sin(1200\pi t)$. Upper Left: The plant's trajectory (green, dashed line) and the two different estimation values of $S(t)$ (red dash-dot line and blue solid line). Upper right: The behavior of the plant's specific growth rate $\mu(S(t))$ (red dash-dot line) and its estimation (blue solid line). Lower Left: The (measured) trajectory of biomass concentration $X(t)$. Lower right: The plant's (unmeasured) input $S_{in}(t)$ (green, dashed line) and the two different estimation values of $S_{in}(t)$ (red dash-dot line and blue solid line).

that for every input/output (measured) pair there are exactly two internal states and two unknown inputs, that are compatible with the behavior of the system. We are then able to construct an observer that provides a finite-time estimation of both possible (and indistinguishable) states and unknown inputs, what provides a complete solution to the observation problem for the system.

This idea can be extended to more general nonlinear systems, and this is part of future work. An interesting question is if it is possible to use the bivalued observer to control the reactor. This issue will be addressed in a future work.

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