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POST-OPTIMALITY ANALYSIS OF STEADY-STATE LINEAR TARGET CALCULATION IN MODEL PREDICTIVE CONTROL

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Abstract: Model predictive control (MPC) is an advanced process control strategy that is usually separated into two levels; steady-state target calculation and dynamic optimization. The existence of uncertainty in model parameters of target calculation can significantly affect the overall performance of the controller. Methods have been proposed to deal with model uncertainty using robust optimization. In this study, a new approach using post-optimality analysis is proposed to study the effect of uncertainty or variation in model parameters on the optimal solution of linear target calculation. This approach can compute the stability limits, for simultaneous variation in objective function coefficients or process limitations, before the optimal target or basis are changed.

Keywords: MPC, linear target calculation, and post-optimality analysis.

1. INTRODUCTION

Model predictive control or receding horizon control is a class of advanced process control strategies. It has been received wide acceptance in industries since it was proposed by Culter and Ramaker (1980). Badgwell and Qin (2003) conducted a survey of industrial model predictive control technology and discussed history, formulation, application, limitation and future perspectives of MPC. Also, Lee and Morari (1999) made a detailed review of MPC. One main advantage of MPC is its ability to handle process constraints for multi-variable system (Palazoglu and Romagnoli, 2005).

In many modern plants, the control hierarchy consists of multiple levels including MPC. MPC can be divided into a steady-state target calculation and a dynamic calculation. Target calculation uses output feedback information to determine where the process should go at the steady state, while the dynamic optimization determines the best way to drive the process to such targets. Thus, the purpose of the steady-state target calculation in MPC is to recalculate the optimal target from the local optimizer, because process changes and disturbances can change the optimal operations faster than local optimizers operate.

As with many other optimization problems, steady-state target calculation suffers from the existence of uncertainty in model parameters. This uncertainty can lead to an unstable optimal solution and poor control performance. Currently, control stability is one of the main challenges to many industrial MPC algorithms (Badgwell and Qin, 2003). Mayne et al. (2000) conducted a detailed review on stability and optimality of constrained MPC. Rao and Rawlings (1999) proposed an algorithm that utilizes exact penalties to treat the constrained system in a unified fashion and yields

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a unique steady-state target. Model uncertainty in linear target calculation was studied by Badgwell et al. (2000). They showed how robust LP prevents oscillations in the inputs and outputs caused by the model mismatch. Rawlings and Wang (2004a, 2004b) proposed a new robust MPC method that uses *min-max* optimization problem to guarantee stability and offset-free set point tracking in the presence of model uncertainty.

In this work, we focus on studying post-optimality analysis of a linear steady-state target calculation. The main objective is to study the effect of the uncertainty or changes in economic parameters on the optimal solution of target calculation. In addition, we aim to improve the closed-loop stability of MPC by assessing the required accuracy of parameter estimates.

2. STEADY-STATE TARGET CALCULATION AND UNCERTAINTY

Many MPC products perform steady-state target calculation and dynamic optimization (or receding horizon regulation) separately as shown in Fig 1. At each control cycle, the local target calculation computes the steady-state input, state, and output targets to be as close as possible to those targets supplied by the unit optimizer without violating the model constraints. These results, from the target calculation, determine where the system should go at steady state and they are passed to the dynamic optimizer to compute optimal movements toward these new targets.

Mathematically, many target calculations can be formulated as a single linear optimization problem that minimizes the difference between the economic targets ($\mathbf{u}_o$ and $\mathbf{y}_o$), from unit optimizer, and future steady-state inputs ($\mathbf{u}_s$) and outputs ($\mathbf{y}_s$). In this study, we use a linear economic objective function to present the target calculation problem as follows:

$$P = \mathbf{d}^T\mathbf{u}_s + \mathbf{e}^T\mathbf{y}_s \qquad (1)$$

and linear input and output inequality constraints:

$$\mathbf{u}^l \leq \mathbf{u}_s \leq \mathbf{u}^h \qquad (2)$$

$$\mathbf{y}^l \leq \mathbf{y}_s \leq \mathbf{y}^h \qquad (3)$$

and include the linear model or equality relation between steady-state inputs and outputs in the existence of the model bias ($\boldsymbol{\beta}$):

$$\Delta\mathbf{y} = \mathbf{G}\Delta\mathbf{u} + \boldsymbol{\beta} \qquad (4)$$

where $\Delta\mathbf{u} = \mathbf{u}_s - \mathbf{u}_o$, $\Delta\mathbf{y} = \mathbf{y}_s - \mathbf{y}_o$, $\mathbf{G} \in \Re^{m\times n}$ is a model gain matrix, n is a number of inputs, m

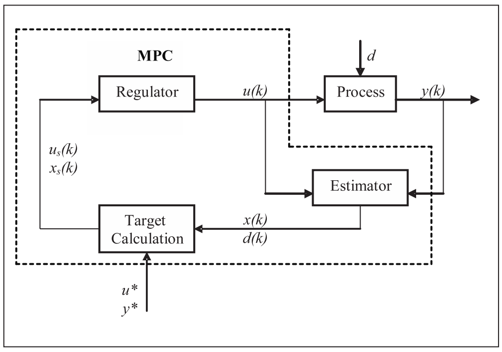

Fig. 1. Typical architecture of MPC controller.

is a number of outputs, $\mathbf{d} \in \Re^n$ and $\mathbf{e} \in \Re^m$ are economic objective coefficients, l and h represent minimum and maximum bounds, respectively.

In this optimization problem, the decision variables are the steady-state inputs ($\mathbf{u}_s$) and outputs ($\mathbf{y}_s$) and it can be expressed in a general LP form:

$$\begin{aligned} &min && \mathbf{d}^T\mathbf{u}_s + \mathbf{e}^T\mathbf{y}_s \\ &s.t. && -\mathbf{G}\mathbf{u}_s + \mathbf{y}_s = -\mathbf{G}\mathbf{u}_o + \mathbf{y}_o + \boldsymbol{\beta} \qquad (5) \\ &&& \mathbf{A}_u\ \mathbf{u}_s \leq \mathbf{b}_u \\ &&& \mathbf{A}_y\ \mathbf{y}_s \leq \mathbf{b}_y \end{aligned}$$

where $\mathbf{A}_u \in \Re^{2n\times n}$ and $\mathbf{b}_u \in \Re^{2n}$ such that $\mathbf{A}_u = [I \quad -I]^T$ and $\mathbf{b}_u = [\mathbf{u}^h \quad -\mathbf{u}^l]^T$. Similar notation and relations are used for the outputs. The main feature of this linear optimization problem, is that the optimal solution lies at the vertex of $n+m$ constraint boundaries and any variation in the vectors $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{b}$ and matrix $\boldsymbol{G}$ can lead to new optimal targets at each control execution or make the current target infeasible. Thus, unstable optimal solution may cause cycling in the closed-loop system and lead to poor overall performance of the MPC (Badgwell et al., 2000). In this work, we study the effect of simultaneous variation in vectors $\boldsymbol{d}$ and $\boldsymbol{e}$ or in vector $\boldsymbol{b}$ on the optimal solution or basis (active constraints) using a proposed method of post-optimality analysis in linear problems.

3. POST-OPTIMALITY ANALYSIS OF LP

Consider this LP problem:

$$max\ \{\mathbf{c}^T\mathbf{x}\ :\ \ \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\} \qquad (6)$$

where $\mathbf{x} = [\mathbf{u}_s\ \mathbf{y}_s]^T$ is the decision variable vector, $\mathbf{c} = [-\mathbf{d} \quad -\mathbf{e}]^T$, $\mathbf{b} = [\mathbf{b}_u\ \ \mathbf{b}_y]^T$, and $\mathbf{A} = [\mathbf{A}_u\ \ \mathbf{A}_y]^T$. Existence of uncertainty in problem (6), can impact the optimality of the solution. Therefore, consideration of uncertainty in optimization problems becomes of great importance to the researchers. There are two main approaches to deal with the uncertainty in model parameters; optimization under uncertainty (Sahinidis, 2004) and post-optimality analysis which is our interest here. Post-optimality analysis studies the effect of uncertainty or variation in model parameters on the obtained optimal solution or basis. It has received little attention in the MPC literature; however, it can provide valuable information about a particular control application and can answer many crucial questions such as:

- How the optimal target is affected by individual or simultaneous change(s) in problem parameters?
- What are the perturbation limits that do not change the optimality of the solution?
- What are the sensitive constraints that need further monitoring?
- What are the sensitive parameters that need accurate estimation?

All of these questions can be answered by computing the stability limits of the parameters.

The terms "post-optimality" , "sensitivity", and "stability" analysis, are generally used to study the behavior of an optimal solution with respect to changes in problem's parameters. Post-optimality analysis is a general term for understanding the effect of perturbations in problem parameters on the optimal solution; whereas, sensitivity and stability analysis are more specific depending on the purpose of analysis.

3.1 Sensitivity and stability analysis

For problem (6), sensitivity analysis investigates how the optimal primal solution, $\mathbf{x}^*$, the dual solution, $\boldsymbol{\lambda}^*$, and the optimal objective value, P^* change with small perturbations in problem parameters. Examples are $\frac{\partial \mathbf{x}^*}{\partial \mathbf{b}}$ and $\frac{\partial P^*}{\partial \mathbf{c}}$ (Gal, 1984). Thus, the term "sensitivity information" is often used to mean parameter derivatives where there is no change in the optimal basis (Fiacco, 1983).
Stability analysis is defined as a study of how much we can perturb the parameters of the problem without changing the optimal solution or the optimal basis (Leontev et al., 1995). For example, stability analysis of linear problems usually refers to determining certain bounds, $\boldsymbol{\varepsilon}$, $\boldsymbol{\Lambda}$, and/or $\boldsymbol{\beta}$ in the problem:

$$max \ \ \{(\mathbf{c}+\boldsymbol{\varepsilon})^T\mathbf{x} \ : \ (\mathbf{A}+\boldsymbol{\Lambda})\mathbf{x} \leq (\mathbf{b}+\boldsymbol{\beta}), \ \mathbf{x} \geq \mathbf{0}\}$$

such that the optimal basis of the original problem remains unchanged (Kozeratskaya et al., 1984).

3.2 Proposed approach for stability analysis of LP

During the last few decades, many stability approaches have been proposed, for variation in vectors $\mathbf{b}$, $\mathbf{c}$, and matrix $\mathbf{A}$ of LP (Gal and Greenberg, 1997). To date there is no single approach that dominates. In contrast to other approaches, the tolerance approach (Wendell, 1985; Wondolowski, 1991; Filippi, 2005) leads to easy-to-use results and considers simultaneous and independent variation in the model parameters. The proposed approach is a modified tolerance approach depends on the active constraints rather than optimal basis in analyzing the stability limits. It computes a better approximation of the stability region for simultaneous variation in vector $\mathbf{b}$ or $\mathbf{c}$. In addition, it is more computationally efficent for large LP problems.

To demonstrate this approach, consider this maximization problem:

$$Max \ \ \{\mathbf{c}^T\mathbf{x} \ : \ \ \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq 0, \ \mathbf{x} \in R^n \ \} \qquad (7)$$

that has a unique optimal solution, $\mathbf{x}^*$, such that $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\boldsymbol{\lambda}$. To define the entire stability region (cone) for variations in objective function coefficients (i.e. vector $\mathbf{c}$), duality information or Lagrange multipliers, $\boldsymbol{\lambda}$, are used:

$$\nabla_b \ \mathbf{P}^* = \boldsymbol{\lambda} = \mathbf{c}^T\mathbf{A}_A^{-1} \qquad (8)$$

where $\mathbf{A}_A \subset \mathbf{A}$ contains the coefficients of the active constraints. The Lagrange multiplier, λ_i, represents the increase in optimal value, for maximization problem, when the associated coefficient in vector $\mathbf{b}$ is increased by one unit. By introducing perturbations vector, $\triangle\mathbf{c}^T$,

$$\boldsymbol{\lambda}^{'} = (\mathbf{c}^T + \triangle\mathbf{c}^T)\mathbf{A}_A^{-1} \qquad (9)$$

Using the non-negativity condition on the optimal solution, there is no change in optimal solution if $\boldsymbol{\lambda}^{'} > 0$; and by substituting and rearranging, we get:

$$\boldsymbol{\lambda} + \triangle\mathbf{c}^T\mathbf{A}_A^{-1} > 0$$
$$-\triangle\mathbf{c}^T\mathbf{A}_A^{-1} < \boldsymbol{\lambda} \qquad (10)$$

This inequality relation presents the stability cone with respect to $\triangle\mathbf{c}$. The solution remains optimal for any perturbation satisfy these inequality equations. To obtain easy-to-use information, a hyperbox is built inside the stability cone using:

$$\tau_j = \frac{\lambda_j}{\sum_{i=1}^{n} |A_{A_{ij}}^{-1}|} \qquad (11)$$

where $i = 1, .., n$. To find the largest hyperbox, let

$$\mathbf{R} = sign(A_{A_{ij}}^{-1}) \times \boldsymbol{\tau} \qquad (12)$$

the stability region for each coefficient in vector $\mathbf{c}$ under simultaneous perturbations is obtained by:

$$(\max_j R_{ij} : R_{ij} < 0) < \triangle c_i < (\min_j R_{ij} : R_{ij} > 0) (13)$$

$$R_i^u < \triangle c_i < R_i^l$$

The above relation is the first stage of this analysis and it leads to the upper and the lower limit for each c_i. In addition, it is useful to understand the relation between vectors $\mathbf{c}$ and $\boldsymbol{\tau}$. Based on this relation, we can sort the matrix $\mathbf{A}_A^{-1}$ into three matrices as follow:

- $\mathbf{S} \in R^{n\times n}$: contains coefficients of $\mathbf{A}_A^{-1}$ used in the analysis of first step
- $\mathbf{P} \in R^{n\times n}$: contains unused positive coefficients of $\mathbf{A}_A^{-1}$, $\mathbf{P} = 0.5[(\mathbf{A}_A^{-1} - \mathbf{S}) + |\mathbf{A}_A^{-1} - \mathbf{S}|]$
- $\mathbf{N} \in R^{n\times n}$: contains unused negative coefficients of $\mathbf{A}_A^{-1}$, $\mathbf{N} = 0.5[(\mathbf{A}_A^{-1} - \mathbf{S}) - |\mathbf{A}_A^{-1} - \mathbf{S}|]$

letting

$\mathbf{l} = [R_1^l, R_2^l, ..., R_n^l]$, $\mathbf{u}$=[$R_1^u, R_2^u, ..., R_n^u$], $\boldsymbol{\beta} = \mathbf{Pu}$, and $\boldsymbol{\alpha} = \mathbf{Nl}$,

The obtained stability limit can be expanded using:

$$\tau_j^{new} = \frac{\lambda_j - \beta_j - \alpha_j}{\sum_{i=1}^{n} |S_{ij}|} \tag{14}$$

Then to compute an enlarged stability region, τ^{new}, Eq. (12) and Eq. (13) are used where $\boldsymbol{\tau}^{new} \geq \boldsymbol{\tau}$. The computational algorithm of this approach is shown in the appendix.

For perturbations in vector $\mathbf{b}$, the dual problem is used:

$$min \ \{\boldsymbol{\lambda}^T\mathbf{b} \ : \ \boldsymbol{\lambda}^T\mathbf{A} \geq \mathbf{c}^T, \ \boldsymbol{\lambda} \geq 0\}$$

and a similar analysis is employed by using

$$-\triangle\mathbf{b}^T(\mathbf{A}_A^T)^{-1} < \boldsymbol{\eta} \tag{15}$$

instead of Eq. (10), where the vector $\boldsymbol{\eta}$ consists of the basic optimal solution, $\mathbf{x}^B$ and non-zero slack variables, $\mathbf{s}$, such that $\boldsymbol{\eta} = [\mathbf{x}^B \ \mathbf{s}]$.

3.3 Stability analysis of linear target calculation

The previous section describes a computational method for stability limits under simultaneous variation in vector $\mathbf{c}$ or $\mathbf{b}$. In linear target calculations, stability analysis of the objective coefficients, biases, and process limitations can improve the robustness of the optimal target and avoid cycling among optimal targets.

In design of the objective function for target calculation, the objective coefficients are usually selected based on economic information or certain priorities for the process variables. Proper choices for these coefficients can be made using stability analysis to enhance the controller robustness. For example, coefficients with narrow stability limits are considered as sensitive coefficients which need more consideration than other coefficients.

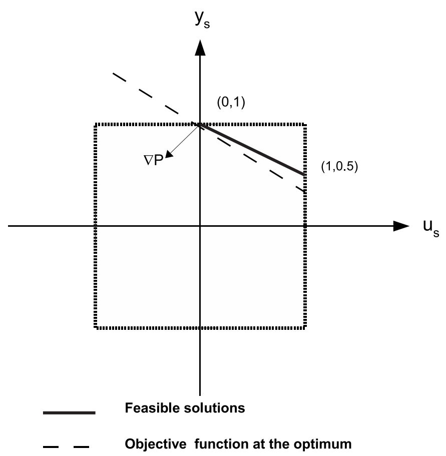

Fig. 2. Feasible region of Example 1.

In closed-loop control, bias update that is used to compensate for mismatch between the model and the process, can lead to an infeasible optimal target or cycling, which affects the controller performance (as in Example 1). The proposed stability analysis can determine the bias limits where the obtained target is feasible and robust. Thus, biases within the obtained stability limits guarantee that the optimal target is feasible in the target calculation. Moreover, performing stability analysis for process limits (i.e. vector $\mathbf{b}$) under simultaneous variation may explore the flexibility and sensitivity of the parameters.

4. EXAMPLES

Example 1: Consider this simple single-input and single-output first order system, with no time delay and a time constant of 10 for both true plant and model as follows:

$$g_p = \frac{-2.0}{10s+1}, \quad g_m = \frac{-0.5}{10s+1} \tag{16}$$

The designed linear target calculation of this system is given by:

$$\begin{aligned} \text{min} \quad & 2u_s + 3y_s \\ \text{s.t.} \quad & 0.5 \ \mathrm{u}_s + y_s = -0.5u_o + y_o + \beta \\ & -1 \leq u_s \leq 1 \\ & -1 \leq y_s \leq 1 \end{aligned}$$

where $u_o = 0$ and $y_o = 1$. Fig. 2 shows the feasible solutions when $\beta = 0$. The nominal optimal solution is (0,1) but a small variation in vector $\mathbf{c}$ to $\mathbf{c}$=[1.667, 3.333] leads to new optimal solution like (1,0.5) and changes the active set of constraints. Therefore, determining the stability limits for model parameters is important before using them in the controller. Using the proposed method, the stability limits, before the optimal solution

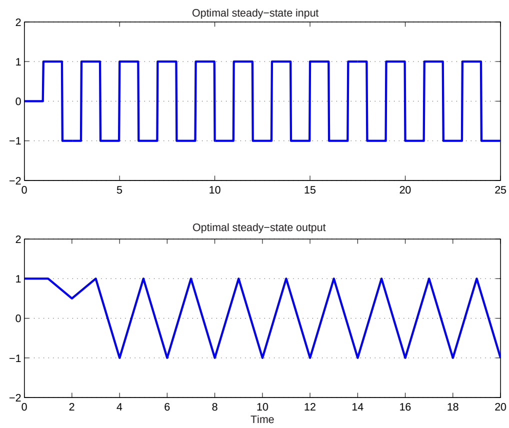

Fig. 3. Cycling in the steady-state target calculation of Example 1.

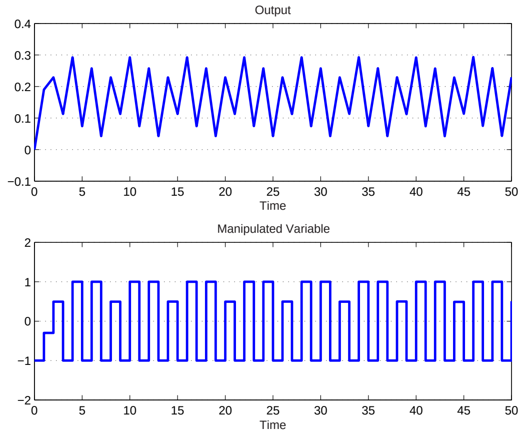

Fig. 4. MPC controller for Example 1, $(y_{set} = 0.2,\ p = 20, m = 10)$

loses its optimality, for simultaneous variation in objective function coefficients are:

$$-0.333 < \Delta c_1 < \infty$$
$$-\infty < \Delta c_2 < 0.333$$

These stability limits for simultaneous variations, help in selecting parameters values in controller design especially for large control problems. For example, the variation in the current objective coefficients should be within the above limits, and any variation at or beyond these limits may lead to situations where the optimal target starts cycling between two solutions (0,1) and (1,0.5) and the MPC loses its stability. Now, consider the controller with only steady-state target calculation, in the closed-loop system with no external disturbances and long sample time between each calculation to reach steady state condition. Due to mismatch between the model and the real process, this closed-loop target calculation cycles between two targets (1,1) and (-1,-1) as shown in Fig 3. Now, consider using MPC controller, dynamic optimization is added, with $y_{set} = 0.2$. Fig 4 shows how cycling in target calculation leads to poor control performance. Thus, in each target calculation, there are limits for the model mismatch (bias). Beyond these limits optimal target is either cycling or infeasible and both problems lead to poor performance of the MPC. For Example 1, the computed bias limit is:

$$-0.5 \leq \Delta\beta \leq 0.5$$

This issue become more important in designing multi-variable controllers where there are many equality constraints (gain equations) that need to be satisfied. Determination of these limits before the solution become infeasible can assess the required accuracy of model estimate and the flexibility of the model (i.e., variation or modification in vector **b**, as in Example 2).

Example 2: Consider a target calculation problem with two inputs and two outputs:

$$\min \quad -0.2u_1 - 0.1u_2 + y_1 + 1.5y_2$$

s.t.

$$\begin{aligned} 2u_1 + 3u_2 - y_1 &= \beta_1 \\ -u_1 - 2u_2 - y_2 &= \beta_2 \\ -1 \leq u_1 &\leq 1 \\ -1 \leq u_2 &\leq 1 \\ -0.5 \leq y_1 &\leq 0.5 \\ -0.5 \leq y_2 &\leq 0.5 \end{aligned}$$

At $\boldsymbol{\beta} = \mathbf{0}$, the optimal solution is $\mathbf{x}^* = [-1.0\ 0.75\ 0.25\ -0.5]^T$ and $\boldsymbol{\lambda}^* = [1.00\ 1.45\ 0.35\ 0.05]^T$ for equality and active inequality constraints, respectively. To maintain this solution as an optimal target, simultaneous variation in objective coefficient should be within:

$$\begin{aligned} -0.2278 &< \Delta c_1 < \infty \\ -2.375 &< \Delta c_2 < 0.0167 \\ -0.227 &< \Delta c_3 < 0.0167 \\ -0.0167 &< \Delta c_4 < \infty \end{aligned}$$

In addition, for simultaneous changes in process limits and the bias, the optimal basis remains unchanged for changes within these stability limits:

$$\begin{aligned} -0.0455 &< \Delta\beta_1 < 0.0556 \\ -0.0455 &< \Delta\beta_2 < 0.0556 \\ -1.9545 &< \Delta u_1^h \\ -0.0455 &< \Delta u_1^l < 0.0556 \\ -0.1768 &< \Delta u_2^h \\ -1.6717 &< \Delta u_2^l \\ -0.0455 &< \Delta y_1^h \\ -0.500 &< \Delta y_1^l \\ -0.9444 &< \Delta y_2^h \\ -0.0556 &< \Delta y_2^l < 0.0455 \end{aligned}$$

These limits help in determining when the duality information is still valid and any variations beyond these limits, may change the active constraints and the obtained optimal target in the steady-state target calculation.

5. CONCLUSION

Uncertainty or variation in model parameters can affect the stability of the controller and may lead to cycling. In this study, a new approach is proposed to perform post-optimality analysis for linear steady-state target calculation of MPC. It studies the effect of uncertainty or variation on the optimal target and basis and helps to assess the accuracy of parameters estimate. The proposed approach can compute the allowable limits for simultaneous variation in objective function coefficients and process limitations of target calculation model before the optimal target lose its optimality or feasibility. In addition, two simple examples are used to explain the effect of variation and to present the obtained stability limits before the controller starts cycling or has infeasible target.

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Appendix

Algorithm of the proposed approach for perturbations in vector c

Inputs: active rows of matrix **A**, and the vector of Lagrange multipliers, $\boldsymbol{\lambda}$.
Outputs: stability limits for each coefficient, $R_i^u < \triangle c_i < R_i^l$.

for j=1:m
compute $\tau_j = \frac{\lambda_j}{\sum_{i=1}^n |A_{A_{ij}}^{-1}|}$
end
find $\mathbf{R} = sign(A_{A_{ij}}^{-1}) \times \boldsymbol{\tau}$
let $\mathbf{S} = \mathbf{0}$ *where* $\mathbf{S} \in \mathbf{R}^{n\times n}$
for i=1:n
find $R_i^u = (\max_j R_{ij} : R_{ij} < 0)$
$R_i^l = (\min_j R_{ij} : R_{ij} > 0)$
let $S_{ij} = A_{A_{ij}}^{-1}$ at $R_i^l = R_{ij}$ or $R_i^u = R_{ij}$
end
compute $\mathbf{P} = 0.5[(\mathbf{A}_A^{-1} - \mathbf{S}) + |\mathbf{A}_A^{-1} - \mathbf{S}|]$
$\mathbf{N} = 0.5[(\mathbf{A}_A^{-1} - \mathbf{S}) - |\mathbf{A}_A^{-1} - \mathbf{S}|]$
let $\mathbf{l} = [R_1^l, R_2^l, ..., R_n^l]$, $\mathbf{u}$=[$R_1^u, R_2^u, ..., R_n^u$], $\boldsymbol{\beta} = \mathbf{Pu}$, and $\boldsymbol{\alpha} = \mathbf{Nl}$
for j=1:m
compute $\tau_j^{new} = \frac{\lambda_j - \beta_j - \alpha_j}{\sum_{i=1}^n |S_{ij}|}$
end
find $\mathbf{R}$= sign($A_{A_{ij}}^{-1}$) $\times \boldsymbol{\tau}^{new}$
for $i = 1 : n$
find $R_i^u = (\max_j R_{ij} : R_{ij} < 0)$
$R_i^l = (\min_j R_{ij} : R_{ij} > 0)$
end