8th International IFAC Symposium on
Dynamics and Control of Process Systems
Preprints Vol.2, June 6-8, 2007, Cancún, Mexico

INVENTORY REGULATION AND SYNCHRONIZATION OF DYNAMIC SUPPLY CHAINS BY NONLINEAR BOUNDED PI CONTROL

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Abstract:
A nonlinear bounded PI control for inventory level regulation, and for production and incoming rate synchronization in linear dynamic supply chains is proposed. Control boundedness is required to satisfy physical and operational limitations. The control varies and synchronizes the production and incoming rates while regulating the inventory levels. The dynamic models allow reckoning multi-product and multi-purpose systems by considering production ratios. For regulation purposes, PI techniques are introduced via nominal references. A stability analysis based on linearization is performed. Simulations of a multi-product petrochemical company show the controller performance.

Keywords:
Supply chain, linear dynamics, PI control, bounded control, synchronization.

1. INTRODUCTION

Business network and strategic alliances have generated major changes in the relations between suppliers, producers, manufacturers, distributors and costumers. Meanwhile, dynamic market trends require companies to provide low cost and effective production in a competitive way. Therefore, most supply chains pursue for minimizing the inventories of raw material and finished products, as well as quick distribution networks. This allows to considerer the synchronization of production and distribution for instantaneous consumption. Supply chain synchronicity occurs when the consumer business world is linked together by technology making each of the constitutive parts: consumers, suppliers, producers, associates, and distributors synchronous with the whole. In others words, when the consumer thinks of a need, there is a synchronized retailer or distributor there to deliver it (Fujimoto, 2002), (Steidtmann, 2004). In (Koudal, 2003) the demand and supply are integrated for the automotive value chain, yielding flexibility and fast consumer respond. In textile industry , TAL Appareal Group applied electronic and communications platforms to evolve into a flexible manufacturer, growing from a single local textile mill

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to a global multinational company (Koudal and Wei-teh Long, 2005).

A supply chain model must capture relevant activities associated with the flow and transformation of goods from the raw material stage to finished products and end user delivery. Several approaches in the modeling of supply chains have been considered, see (Daganzo, 2002) and (Shapiro, 2001) for extensive overviews. In (Perea-Lopez *et al.*, 2001) a discrete dynamic model based on balances of inventories and orders, and intended for planning and scheduling, is presented. By using traffic dynamics (Helbing, 2003) proposed a continuous time dynamic model to represent the behavior of the inventories and the production rates.

Aiming to simultaneously regulate the inventories and the production and incoming rates, a nonlinear bounded PI control is here proposed. This controller also ensures synchronization of the supply chain with respect to the demand at each one of its constitutive entities. Differential dynamic models with production ratios are introduced to consider multi-product and multi-purpose systems. A stability analysis and some tuning rules are provided.

The paper is organized as follows. The model of the supply chain is presented in Section 2. In Section 3 a bounded nonlinear PI control for inventory regulation and supply chain synchronization is proposed. In Section 4 stability analysis and tuning conditions are provided. A simulation study of the controller is shown in Section 5. Section 6 presents some conclusions.

2. MODEL OF THE LINEAR DYNAMIC SUPPLY CHAIN

Consider a supply chain formed by n-nodes with information and material flows as depicted in Figure 1. The information flows represent the orders of material and goods. However and without loss of generality along this article the information flows are not considered.

Each node i (for $i = 1, \ldots, n$) is represented by its inventory level N_i and its production or incoming rate λ_i, depending whether it is a producer or a warehouse, supplier or distribution node.

Combining modeling strategies proposed in (Helbing, 2003) and (Perea-Lopez *et al.*, 2001), a supply chain can be represented by linear differential equations. Furthermore by introducing production ratios multi-product and multi-purpose systems can be considered. The production or product ratio $F_{i,j}$, reflects the quantity of product i required to produce a unit of product j, and it allows to change the recipes on the producer nodes, such that different products may be yield.

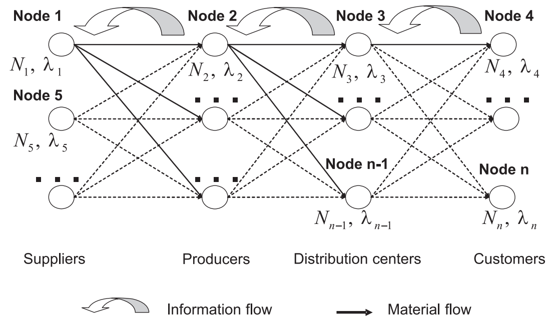

Fig. 1. Material and information flows in a supply chain.

2.1 Dynamic model for the constitutive nodes

The inventory N_i is important for both producer and non producer nodes. The inventory dynamics is given by a balance of the incoming λ_i and the delivery (outgoing) $\lambda_{d\_p,i}$ rates, $\lambda_{d\_p,i}$ represents the total demand of products for the node i, thus

$$\frac{dN_i}{dt} = \lambda_i - \lambda_{d\_p,i}; \qquad \lambda_{d\_p,i} = \sum_{j=1}^{r} F_{i,j}\lambda_j \quad (1)$$

which takes into account the demand of all the r nodes requiring products or material from node i, with individual demanding rates λ_j and production or product ratio $F_{i,j}$.

The production or incoming rate constitutes the control action to vary the inventory dynamics of producer and non producer nodes respectively.

For a non producer node its incoming rate corresponds to the product or materials that are received from its suppliers, such that it does not possess a dynamics of its own. Meanwhile the production rate of a producer node varies accordingly to production policies. A change in the production rate involves several activities that require an adaptation time named T_i. If W_i denotes the control action that varies the production rate λ_i, then its dynamics is given by

$$\frac{d\lambda_i}{dt} = \frac{1}{T_i}(W_i - \lambda_i) \quad (2)$$

Remark 1. In the model (1), and (2), it is assumed that upstream nodes deliver the demanded product or material as soon as it is required by the demanding nodes, therefore delivering delays are neglected and storage of raw materials at each node are not considered. This hypothesis establishes one of the major advantages and at the same time a major challenge for supply chain synchronization. Since raw material storages are not present savings in costs and time are achieved, but instantaneous delivering is required.

Remark 2. For multi-product approach an independent stock for each product is considered at its producer and distributor nodes, each independent product stock being modeled by (1).

3. INVENTORY REGULATION AND SUPPLY CHAIN SYNCHRONIZATION CONTROL

To capture the characteristic and limitations of the supply chain some conditions are introduced.

Condition 3. The inventory levels N_i, and the production and incoming rates λ_i are bounded in order to consider the physical and operational limitations present in supply chains, thus

$$N_{i,min} \leq N_i \leq N_{i,max} \tag{3}$$

$$0 \leq \lambda_i \leq \lambda_{i,max} \tag{4}$$

where $N_{i,min}, N_{i,max}$ are the minimum and maximum inventory level respectively, and $\lambda_{i,max}$ is the maximum production or incoming rate.

Condition 4. Since the production rate λ_i in (2) is bounded by (4), then the control action must be bounded accordingly. That is

$$0 \leq W_i \leq \lambda_{i,max} \tag{5}$$

The control action W_i for a producer node i is proposed as

$$W_i = \lambda_{i,max} \left(2 - \frac{1}{1 + e^{-\alpha_i (N_i - N_{c,i})}} - \frac{1}{1 + e^{-\alpha_i N_{c,i}}} \right) \tag{6}$$

Meanwhile for a non producer node its inventory dynamics (1) is modified by its incoming rate λ_i, such that it is proposed to vary as

$$\lambda_i = \lambda_{i,max} \left(2 - \frac{1}{1 + e^{-\alpha_i (N_i - N_{c,i})}} - \frac{1}{1 + e^{-\alpha_i N_{c,i}}} \right) \tag{7}$$

where $\lambda_{i,max}$ is the maximum production or incoming rate in the node, α_i is a parameter that regulates the convergence rate of N_i. The larger α_i, the faster the convergence rate. However too large values of α_i can induce instability.

$N_{c,i}$ acts as a nominal reference and forces N_i to a desired constant value $N_{d,i}$, and is given by

$$N_{c,i} = N_{d,i} - K_{P,i}(N_i - N_{d,i}) - K_{I,i} \int (N_i - N_{d,i}) dt \tag{8}$$

where $K_{P,i}$ and $K_{I,i}$ are the positive proportional and integral control gains, respectively. The integral action renders a steady error equal to zero around the equilibrium point, while the proportional action regulates the convergence rate.

Remark 5. The controller (6) and (7) use exponential functions to render a bounded control action, while allowing a smooth and fast convergence. The term in between parenthesis in (6) and (7) is bounded in $[0, 1]$, after multiplying this term by $\lambda_{i,max}$ the physical and operational limitations on the production or incoming rate are recovered.

4. STABILITY ANALYSIS

The controller (6) and (7) induce nonlinearities in the closed loop system, so that, linearization techniques are considered for the stability analysis.

Theorem 6. The equilibrium point (λ_i^*, N_i^*) of the closed loop formed by a producer node (1), (2) and (6) is given by $\lambda_i^* = \lambda_{d\_p,i}$ for the production rate, and $N_i^* = N_{d,i}$ for the inventory level.

Proof: From (1) and (2) it follows that the equilibrium conditions are

$$0 = \lambda_i^* - \lambda_{d\_p,i} \tag{9}$$

$$0 = \frac{1}{T_i} (W_i - \lambda_i^*) \tag{10}$$

thus $\lambda_i^* = \lambda_{d\_p,i}$, and simultaneously $W_i = W_i(N_i^*) = \lambda_i^*$ such that $W_i(N_i^*)$ must be constant. Then by substitution of (6) and (8), it follows that because $\lambda_{i,max}$ and α_i, are constants, $W_i(N_i^*)$ is constant if and only if $N_i^* = N_{d,i}$. ■

Remark 7. The closed loop of a non producer is given by (1) and (7), such that its closed loop equilibrium point implies $\lambda_i^* = \lambda_{d\_p,i}$ and $N_i^* = N_{d,i}$.

Remark 8. The equilibrium point of both producer and non producer nodes implies that $\lambda_i^* = \lambda_{d\_p,i}$, therefore λ_i synchronizes with the total demanding rate $\lambda_{d\_p,i}$, such that instantaneous consumption and supply chain synchronization are achieved. Simultaneously the inventory fulfills $N_i^* = N_{d,i}$, thus inventory regulation is obtained.

Theorem 9. The closed loop system formed by a producer node (1) and (2) with the controller (6) and (8) is locally asymptotically stable and converge to the equilibrium point $\lambda_i^* = \lambda_{d\_p,i}$, $N_i^* = N_{d,i}$, if the gains α_i, $K_{P,i}$ and $K_{I,i}$ satisfy

$$\alpha_i \geq 1 \tag{11}$$

$$\zeta_1 \leq K_{P,i} < \zeta_2 \tag{12}$$

$$0 < K_{I,i} \leq \frac{1}{4} \frac{K_{P,i}}{T_i} \tag{13}$$

where

$$\zeta_1 = \left| \frac{(1+e^{-(\alpha_i N_{d,i})})^2(\alpha_i \lambda_{i,max} T_i - 1)}{\alpha_i \lambda_{i,max} T_i (e^{(-\alpha_i N_{d,i})} - 1)^2} \right|$$

$$\zeta_2 = \left| \frac{(1+e^{-(\alpha_i N_{d,i})})^2}{(e^{(-\alpha_i N_{d,i})} - 1)^2} \right|$$

Furthermore, the above conditions ensure an over-damped closed loop system, avoiding large overshoots and keeping conditions (3) and (4).

Proof: In the equilibrium point (λ_i^*, N_i^*), the closed loop can be linearized around small deviations δN_i and $\delta \lambda_i$ as

$$\dot{x} = Ax \tag{14}$$

where $x = (\delta N_i, \delta \lambda_i)^T$, with

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{T_i} W'(N_i^*) & -\frac{1}{T_i} \end{pmatrix}$$

whose eigenvalues $s_{1,2}$ are given by

$$s_{1,2} = \frac{-\frac{1}{T_i} \pm \sqrt{\frac{1}{T_i^2} + \frac{4}{T_i} W_i'(N_i^*)}}{2} \tag{15}$$

Then the closed loop is asymptotically stable and overdamped (i.e. both eigenvalues are negative and purely real), if the derivative of the control function $W_i'(N_i^*)$ fulfills

$$-\frac{1}{4T_i} \leq W'_i(N_i^*) < 0 \tag{16}$$

which imposes conditions on $\alpha_i, K_{p,i}, K_{I,i}$. Replacing (8) in (6) yields

$$W'_i(N_i^*) = \alpha_i \lambda_{i,max} \left(-\frac{1}{4}(1 + K_{P,i} + tK_{I,i}) + \frac{(K_{P,i} + tK_{I,i}) e^{-(\alpha_i N_{d,i})}}{(1+e^{-(\alpha_i N_{d,i})})^2} \right) \tag{17}$$

where t represents the integration time. When $N_i^* = N_{d,i}$ there is not integral action, thus it can be taken $t = 0$, and from (17) and by algebraic manipulation it follows that sufficient conditions on $K_{P,i}$, for (16) being satisfied, are given by (12). By defining the regulation error $e_i = (N_i - N_{d,i})$, replacing it in (8) and taking first derivative with respect to time it is obtained

$$\frac{dN_{c,i}}{dt} = -K_{p,i}\frac{de_i}{dt} - K_{I,i} e_i \tag{18}$$

In the equilibrium point N_i becomes constant, i.e. $N_i \rightarrow N_i^*$, thus (18) equals to zero and by Laplace transform it is obtained the pole

$$s = -\frac{K_{I,i}}{K_{P,i}} \tag{19}$$

Considering that the pole in (19) must verify condition (16) to limit the dynamics of the closed loop, then $K_{I,i}$ must satisfy the condition (13). Finally, since only $K_{P,i}$ through the condition (12) depends on α_i, it can be to some extend freely chosen. Thus for convenience and to obtain fast convergence it is taken that $\alpha_i \geq 1$. ■

Theorem 10. The closed loop system formed by a non producer node (1) with the controller (7) is locally asymptotically stable and converge to the equilibrium point $\lambda_i^* = \lambda_{d\_p,i}$, $N_i^* = N_{d,i}$, if the gains α_i, $K_{P,i}$ and $K_{I,i}$ satisfy the conditions

$$\alpha_i \geq 1 \tag{20}$$

$$K_{P,i} < \left| \frac{\left(1+e^{-\alpha_i N_{d,i}}\right)^2}{\left(e^{-\alpha_i N_{d,i}} - 1\right)^2} \right| \tag{21}$$

$$K_{I,i} > 0 \tag{22}$$

Furthermore, the above conditions ensure an over-damped closed loop system.

Proof: It follows as for Theorem 9. ■

5. SIMULATION STUDY

The controller is tested by simulations on a multi-product petrochemical company, which produces different grades of polyethylene products, see Figure 2. The numbers on the left upper side of the nodes identifies the numbering used through the simulations and graphics of the results.

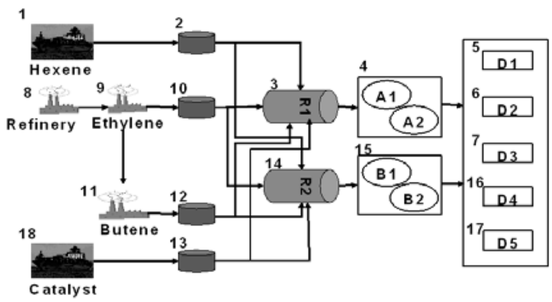

Fig. 2. Supply chain for a multi-product polyethylene petrochemical plant.

Hexene and catalyst are imported, whereas ethylene is obtained from a local refinery. The production of ethylene and butene is carried out by independent production plants. There exist intermediate storages for the hexene, ethylene, butene and catalyst feedstocks. Only five demand sources are taken into consideration, from D1 to D5.

The reactors R1 and R2 produce different polymeric products depending on the fed material (production ratio) and operation conditions. Each reactor produces two polymers: R1 produces A1 and A2, and R2 produces B1 and B2 in a cyclic way, according to a given schedule; R1, R2 and their storages are of multi-product kind, thus have different stocks per product. For supply chain synchronization, it is considered that the demanded product is supplied to the costumer only during the production time of the corresponding product.

The simulated period is 10 hrs. Reactors R1 and R2 change from producing product A1 to A2 and B2 to B1 at $t = 5$ hrs. respectively. The production ratios and demands for the four products are listed in Table 1. The rest of the production ratios are independent of the kind of product, such that $F_{8,9} = 1.5$ and $F_{9,11} = 0.8$, and those for storage purposes are all equal to 1. The plant capacity per reactor is 34.24 [MT/hr].

For reactor R1 and product A1 a desired inventory level of $N_{d,3} = 400$ is considered, while for product A2, $N_{d,3} = 395$. For the distributor of products A1 and A2, $N_{d,4} = 440$ for A1 and $N_{d,4} = 420$ for A2. Similarly for reactor R2 and its distributor $N_{d,14} = 380$ and $N_{d,15} = 1000$ for B1, $N_{d,14} = 420$ and $N_{d,15} = 1300$ for B2.

	A1	A2		B1	B2
$F_{2,3}$	0.25	0.4	$F_{2,14}$	0.6	0.4
$F_{10,3}$	0.15	0.2	$F_{10,14}$	0.15	0.1
$F_{12,3}$	0.5	0.3	$F_{12,14}$	0.1	0.3
$F_{13,3}$	0.2	0.1	$F_{13,14}$	0.15	0.2
D_1	5	8	$D4$	9	14
D_2	3	6	$D5$	13	8
D_3	12	11			

Table 1. Demands in [MT/hr] and production ratios for products A1, A2, B1 and B2.

The storage capacity of the plant is of 2000 [MT] for nodes 2, 10, and 12; 10000 [MT] for nodes 4, 13 and 15; and for nodes 3, 8, 9, 11 and 14 of 500 [MT]. According to a monthly schedule with daily resolution, inventory levels on nodes 1 and 13 must be of 3000 [MT] and 2500 [MT], respectively. The desired inventory levels for the rest of the nodes are listed in Table 2. Note that the planning or scheduling instance must take into account the physical and operational limitations listed in table 2 to provide feasible and attainable inventories.

The initial values at $t = 0$ hrs and the operational limitations are listed in Table 2. For the multi-product reactor R1 its initial values are for product A1 $N_3(0) = 405$, while for product A2, $N_3(0) = 390$. For the distributor the initial values are $N_4(0) = 435$ for A1 and $N_4(0) = 425$ for A2. Similar for R2 and its distributor the initial values are $N_{14}(0) = 385$ and $N_{15}(0) = 990$ for B1, $N_{14}(0) = 415$ and $N_{15}(0) = 1310$ for B2.

Note that the initial values for the inventories are near to the desired ones as to generate illustrative curves with small oscillations and fast convergence. Nevertheless the controller can deal with large differences on the initial inventories and productions rates with respect to the desired ones.

The bounds for the control gains where calculated according to the theorems 9 and 10, such that

Node	1	2	3	4
$N_i(0)$ [MT]	3000	992		
$\lambda_i(0)$ [MT/hr]			24	
T_i [hr]			0.5	
$\lambda_{i,max}$ [MT/hr]		120	34.24	40
$N_{d,i}$ [MT]		1000		
Node	8	9	10	11
$N_i(0)$ [MT]	377	377	360	356
$\lambda_i(0)$ [MT/hr]	36	39		33
T_i [hr]	0.1	0.1		0.2
$\lambda_{i,max}$ [MT/hr]	50	60	50	55
$N_{d,i}$ [MT]	370	380	365	360
Node	12	13	14	15
$N_i(0)$ [MT]	360	2500		
$\lambda_i(0)$ [MT/hr]			22.8	
T_i [hr]			0.1	
$\lambda_{i,max}$ [MT/hr]	120	60	34.24	60
$N_{d,i}$ [MT]	365	2500		

Table 2. Initial values, maximum production and incoming rates, and desired inventories.

the gain values (Table 3) where chosen inside the corresponding bounds.

Node	1	2	3	4	8	9
α_i		1	2	10	1	2
$K_{P,i}$		0.1	0.9	1	0.5	0.3
$K_{I,i}$		0.07	0.08	0.07	0.1	0.04
Node	10	11	12	13	14	15
α_i	1	1	1		5	1
$K_{P,i}$	0.1	0.2	0.1		0.6	0.1
$K_{I,i}$	0.07	0.04	0.07		0.04	0.07

Table 3. Control gain values

Figure 3 presents the inventory levels for producer N_3, N_9 and non producer N_4, N_{10} nodes. Note that all inventories converge to their desired values with smooth response. The inventory N_9 shows higher oscillations than the others during transient ($t < 1$) because it is the node most to the left of the shown ones, such that it is affected by the dynamic changes of all the related downstream nodes. This is the phenomena that origins the bullwhip effect in large supply chains. Notice that the inventories N_3, N_4 implies individual stock levels for the products A1 and A2.

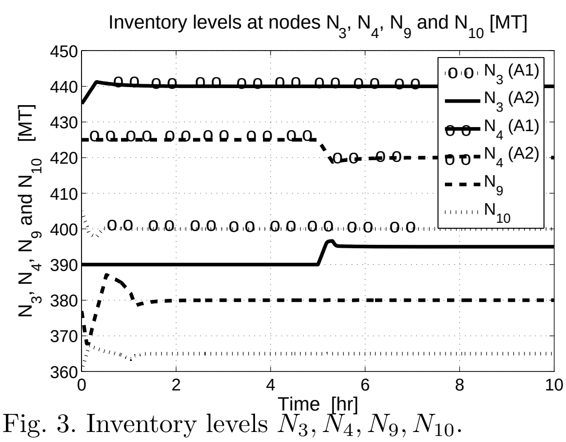

Fig. 3. Inventory levels N_3, N_4, N_9, N_{10}.

From Figures 4 and 5 notice that all rates touch their boundaries at transient and when changes in

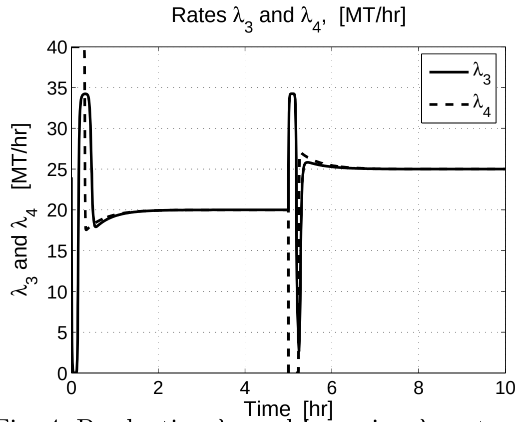

Fig. 4. Production λ_3 and incoming λ_4 rates.

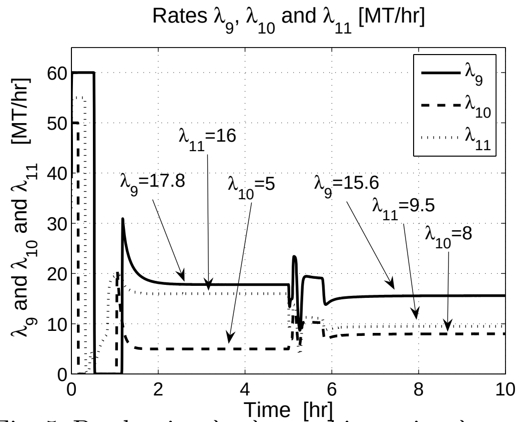

Fig. 5. Production λ_9, λ_{11} and incoming λ_{10} rates.

production from A1 to A2 are required (t=5 hrs), therefore the physical and operational bounds (4) are held. As a result of changing the production from A1 to A2, λ_3 changes its value, while the inventories of the products A1 and A2 converge to their desired values, see Figure 3. The producer λ_9 and $\lambda_1 1$, and non producer λ_{10} rates are shown in Figure 5, notice that λ_9 synchronizes to its demanding rates, with $F_{9,10} = 1$ and $F_{9,11} = 0.8$. Also notice that λ_3 and λ_4 synchronizes between them accordingly to the proportion $F_{3,4} = 1$, meanwhile the incoming rate λ_4 synchronizes to the total demand of product A1 of 20 [MT/hr] and A2 of 25 [MT/hr], such that instantaneous consumption synchronization is achieved.

Figure 6 shows a comparison study for different control gains, notice that although the PI is filtered by the bounded function, see eqs. (6, 7, 8), the behavior of the PI actions is preserved, such as bigger overshoot but faster convergence when increasing the P action, and smaller stationary error by increasing the I action.

6. CONCLUSIONS

A nonlinear control for inventory regulation and supply chain synchronization by manipulating the production or incoming rate has been developed. The controller is bounded, such that production

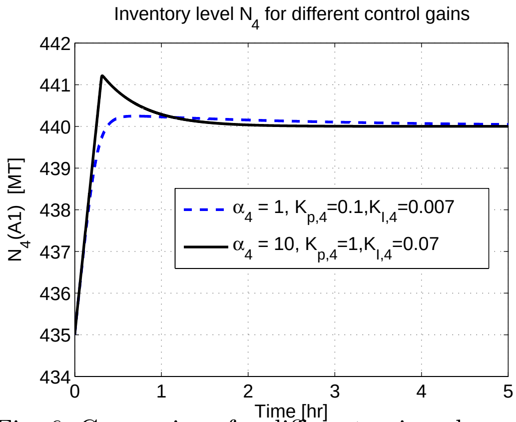

Fig. 6. Comparison for different gain values.

capacity, physical and operational limitations of the supply chain are held. A formal stability analysis and conditions for the control gains have been presented, such that asymptotic stability and convergence to the equilibrium point are guaranteed. PI techniques are considered for robustness and convergence of the closed loop system. Simulation results show robustness against changes on production rates and operational conditions such as changes of products or desired inventories.

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