8th International IFAC Symposium on
Dynamics and Control of Process Systems
Preprints Vol.3, June 6-8, 2007, Cancún, Mexico

ON THE EFFECT OF UN-IDENTIFIABILITY ON CONTROL

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Abstract: Mathematical models often depend on unknown parameters that must be identified. Un-identifiable models have parameters that cannot be identified from input-output data. In this work, it is shown that un-identifiability can affect the closed-loop performance of systems. This conclusion holds even for minimal systems. It is shown that a change of coordinates can be used to transform any linear, time-invariant un-identifiable system into one that is identifiable up to a change in initial conditions. For such systems, it is possible to construct controller/observer pairs that do not depend on any un-identifiable parameters.

Keywords: Parameter Identification, Control, Closed Loop System, Parameterization, Observability

1. INTRODUCTION

Model-based control is very popular as both a research subject and in application (Nijmeijer and van der Schaft, 1990). Almost all mathematical models of physical systems rely on parameters which may be unknown, to generate an input-output behaviour (Walter and Pronzato, 1995). Some models contain more parameters than can be estimated from the input-output behaviour of the system. Un-identifiable systems (Bellman and Astrom, 1970; Ljung and Glad, 1994) are models where some parameters cannot be estimated because of a defect in the model structure. That is, the parameter estimation problem is not well posed regardless of the type and quantity of input-output data collected. This is in contrast with un-estimable systems (Jaquez and Grief, 1985), where each parameter is identifiable, but the estimation problem is not well posed due to noise or lack of measured data.

In this work the identifiability of a model (or lack thereof) is shown to be an is an important consideration for control system design. Specifically, lack of identifiability may cause classical control techniques to produce poor closed-loop behavior. This is true even for systems that are both controllable and observable.

2. PROBLEM DEFINITION

In this work the structure of a mathematical models will be treated as mappings $M : p \mapsto M(p)$ mapping from a parameter space $\mathscr{P}$ to the set of observed input-output mappings. The parameter space $\mathscr{P}$ is open and dense in some subset of $\mathbb{R}^s$. Roughly speaking a model M takes parameter values $p, r \in \mathscr{P}$ and produces observed input output behaviours $M(r)$ and $M(p)$. Input-output behaviors will be treated as mappings as mappings, $M(\cdot) : \mathscr{U} \to \mathscr{Y}$, from an input space $\mathscr{U}$ containing (sufficiently) differentiable functions $u : \mathscr{U} \to \mathbb{R}$ on a closed interval $\mathscr{T} = [0, t_f] \in \mathbb{R}$ to an output space $\mathscr{Y}$ containing (sufficiently)

[1] The author's work is partially supported by Canada's Natrual Science and Engineering Research Council

differentiable functions $y : \mathscr{Y} \to \mathbb{R}$ on $\mathscr{T}$. The results of our work can easily be extended to multi-input multi-output systems. For our work, the property of differentiability implies that the restriction of $\mathscr{T}$ to any open subset $\mathscr{T}_o \subset \mathscr{T}$ will be differentiable.

A model structure M is identifiable if and only if for any two candidate parameter values r and p the following holds.

$$M(p) = M(r) \Leftrightarrow p = r \tag{1}$$

Condition 1 implies that M is a bijection and each observed input-output behaviour corresponds to a single set of parameter values. Note that $M(p)$ itself does not have to be a bijection.

The goal of our work is to show that lack of identifiability (i.e., un-identifiability) is relevant to control design. That is, although a parameter or combination of parameters can have no effect on input-output behaviour, they can have an effect on the closed loop performance of a system. This somewhat counter-intuitive result can be illustrated using linear time invariant (LTI) dynamical systems of the following form.

$$\begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)u(t) \\ y(t) &= C(p)x(t) \end{aligned} \tag{$\mathcal{S}$}$$

where $u \in \mathscr{U}$, $y \in \mathscr{Y}$, $x : \mathscr{T} \to \mathbb{R}^n$ is sufficiently differentiable, $p \in \mathscr{P}$ and $A(p)$, $B(p)$, $C(p)$ are real-valued matrices of appropriate dimensions. Also, $A(p)$ will be assumed invertible on a dense subset of $\mathscr{P}$.

3. A MOTIVATING EXAMPLE

The focus of this work is the effects of identifiability on control. As a result, it is advantageous to pick a system with a simple structure whose properties can be easily verified. Consider the following LTI system.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 1 & p_1 \\ p_2 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [p_2, 0]x(t) \end{aligned} \tag{$\mathcal{X}$}$$

whose transfer function is

$$G(s,p) = \frac{p_1 p_2}{s^2 - 3s + (2 - p_1 p_2)} \tag{2}$$

System $\mathcal{X}$ is not identifiable because the input-output behavior of the system (as described by $G(s)$) depends only on the product $p_1 p_2$ which is assumed to be non-zero. This assumption is equivalent to assuming that the input-output behavior of the system is not identically zero as a function of u. Furthermore, as a result of this assumption $\mathscr{P} \subset \mathbb{R}^2$ is given by

$$\mathscr{P} = \mathbb{R}^2 \setminus \{(a,b) \in \mathbb{R}^2 \mid a \neq 0, b \neq 0\}$$

which is open and dense as required.

System $\mathcal{X}$ is observable because it has an observability Grammian given by

$$\begin{bmatrix} C(p) \\ C(p)A(p) \end{bmatrix} = \begin{bmatrix} p_2 & 0 \\ p_2 & p_1 p_2 \end{bmatrix}$$

whose determinant is $p_2^2 p_1 \neq 0$.

System $\mathcal{X}$ is also controllable because it has a controllability Grammian given by

$$[B(p), A(p)B(p)] = \begin{bmatrix} 0 & p_1 \\ 1 & 2 \end{bmatrix}$$

whose determinant is $-p_1 \neq 0$.

System $\mathcal{X}$ can be feed-back stabilized. Indeed, one may design an observer/controller pair with arbitrarily fast convergence. Specifically, for any two eigenvalue pairs $\{\tau_1, \tau_2\}$ and $\{\lambda_1, \lambda_2\}$, the feedback law

$$u(t) = -B(p)K(p)\hat{x}(t)$$

with

$$K(p) = \left[\frac{1}{p_1}\left(1 - \tau_2 - \tau_1 + p_1 p_2 + \tau_2\tau_1\right), 3 - \tau_2 - \tau_1\right]$$

and the observer gain

$$L(p) = \begin{bmatrix} \dfrac{-1}{p_2}\left(\lambda_1 + \lambda_2 - 3\right) \\ \dfrac{4 - 2\lambda_1 - 2\lambda_2 + p_1 p_2 + \lambda_1\lambda_2}{p_1 p_2} \end{bmatrix}$$

with $\hat{x}(t)$ the estimated state whose dynamics correspond to the observer law

$$\dot{\hat{x}} = A(p)\hat{x}(t) + B(p)u(t) + L(p)(y(t) - C(p)\hat{x}(t)$$

will produce an observer error $e_o(t) = x(t) - \hat{x}(t)$ whose dynamics are given by

$$\dot{e}_o(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and whose control dynamics are described by the equation

$$A(p) - B(p)K(p) = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}$$

Note however, that the feedback matrix $K(p)$ is a function of the parameter p_1 which must be known independently of p_2, and that the observer gain matrix $L(p)$ is a function of p_2 which must be known independently of p_1.

The full dynamics of System $\mathcal{X}$ are given by the following system.

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} &= \mathcal{A} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \\ y(t) &= \mathcal{C}[x(t), \hat{x}(t)]^T \end{aligned}$$

with

$$\mathcal{A}(p) = \begin{bmatrix} A(p) - B(p)K(p) & 0 \\ A(p) - B(p)K(p) - L(p)C(p) & L(p)C(p) \end{bmatrix}$$

and

$$\mathcal{C}(p) = [1, 0, 0, 0]$$

Consider the following two parameter values for System $\mathcal{X}$.

$$p = [10, 2] \tag{3}$$
$$r = [-20, -1] \tag{4}$$

Note that $p_1 p_2 = r_1 r_2$ and so

$$G(s,p) = G(s,r) \tag{5}$$

and the two parameter values induce an identical input-output behaviour. The key observation in this work is that Condition 5 *does not* imply that

$$\mathcal{A}(p) = \mathcal{A}(r)$$

and more importantly, does not guarantee that the *closed-loop* output matches. That is, generally, if $y_r(t)$ is the closed loop trajectory for an initial condition x_0 and parameter value $r \in \mathscr{P}$ and $y_p(t)$ is the closed loop trajectory for (the same) initial condition x_0 and parameter value $p \in \mathscr{P}$, $p \neq r$, then $y_r(t) \neq y_p(t)$ because

$$\mathcal{C}(p)\left(Is + \mathcal{A}(p)\right)^{-1} \neq \mathcal{C}(r)\left(Is + \mathcal{A}(r)\right)^{-1}$$

where I is the identity matrix of the same dimension as $\mathcal{A}$.

This issue will now be illustrated using System $\mathcal{X}$ with parameter values $p, r \in \mathscr{P}$ given in Equations 3 and 4. Let us specify

$$\tau_1 = -5 \quad \tau_2 = -7 \quad \lambda_1 = -3 \quad \lambda_2 = -2 \tag{6}$$

then

$$\mathcal{A}(p) = \begin{bmatrix} 1 & 10 & 0 & 0 \\ -\frac{24}{5} & -13 & 0 & 0 \\ 8 & 0 & -7 & 10 \\ 4 & 0 & -\frac{44}{5} & -13 \end{bmatrix}$$

and

$$\mathcal{A}(r) = \begin{bmatrix} 1 & 20 & 0 & 0 \\ -\frac{12}{5} & -13 & 0 & 0 \\ 8 & 0 & -7 & 20 \\ 2 & 0 & -\frac{22}{5} & -13 \end{bmatrix}$$

As a result,

$$\mathcal{C}(p)\left(Is + \mathcal{A}(p)\right)^{-1} = \left[\frac{2(13+s)}{s^2+12s+35}, \frac{20}{s^2+12s+35}, 0, 0\right]$$

and

$$\mathcal{C}(r)\left(Is + \mathcal{A}(r)\right)^{-1} = \left[\frac{13+s}{s^2+12s+35}, \frac{20}{s^2+12s+35}, 0, 0\right]$$

which implies that $y_r(t) \neq y_p(t)$ for any initial condition

$$x(0) = [a, b] \in \mathbb{R}^2$$

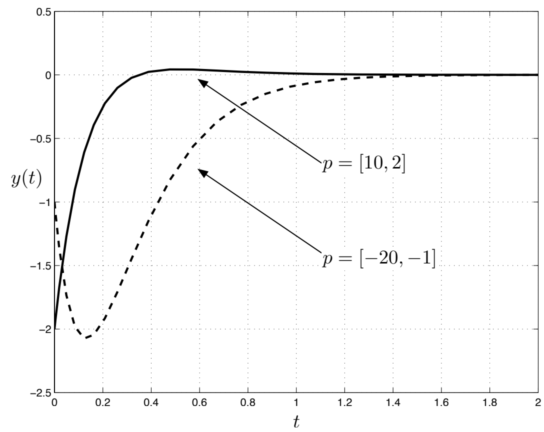

Fig. 1. Output time-traces for System $\mathcal{X}$ for controller/observer pairs constructed using two different parameter values.

where $a \neq 0$. Figure 1 shows the output of System $\mathcal{X}$ as a function of time with the initial condition $x(0) = [-1, 1]$ and $\hat{x}(0) = [0, 0]$ for the two different parameter estimates. Note how for the apparent quality of the controller is completely different for the two parameter values. As a practical matter, the control law used in this example is not useful because it may produce a poor closed loop response. This is true regardless of the number and type of experiments used to identify the system parameters because p_1 and p_2 are not, by themselves, identifiable. Un-identifiability can also cause the dynamics of the states and their estimates to be dependent on the particular parameter estimate used. Figure 2 shows the time-trajectory of the states $x(t)$ from the initial condition $x(0) = [-1, 1]$ and $\hat{x}(0) = [0, 0]$ for the two different parameter values. Note how the state trajectories for one set of parameter estimates is completely different than the other.

In this section it is shown that, even for a system that are minimal, closed-loop behaviour may depend on parameters that cannot be identified from input-output behaviour. As will be shown, this unexpected result is caused by the fact that the output depends not only on the input, but also on initial conditions. It is worth noting that for System $\mathcal{X}$ the initial conditions would not generally be known because x_2 is not observed and x_1, is observed as the product $p_2 x_1(t)$ which involves the un-identifiable parameter p_2.

4. CONTROL USING A CANONICAL REPRESENTATION

In this section an approach is developed for designing observer/controller pairs for System $\mathcal{X}$

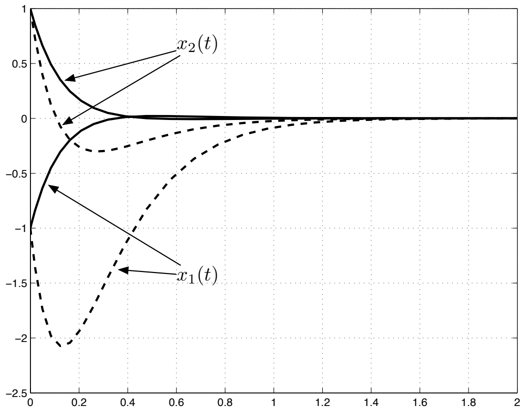

Fig. 2. Output time-traces for System $\mathscr{X}$ for controller/observer pairs constructed using two different parameter values: $p = [10, 2]$ (solid) and $p = [-20, -1]$ (dashed).

so that the closed loop response of the system (including control and observer dynamics) is independent of un-identifiable parameters. The proposed approach does not rely on a specific formulation for controller/observer design. Rather, it is shown that System $\mathscr{X}$ can always be transformed into one for which the classical control design techniques apply. The approach, therefore, depends on expressing System $\mathscr{X}$ using appropriate coordinates.

Using the input-output transfer function in Equation 2 one can construct a minimal realization (based on the standard observable realization) given by

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ (p_1p_2 - 2) & 3 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (\mathscr{Z})$$
$$y(t) = [p_1p_2, 0]z(t)$$

System $\mathscr{Z}$ is un-identifiable in the sense that p_1 and p_2 cannot be known independently. However, p_1 and p_2 always appear as the product p_1p_2 and, as a result, any control or observer gain matrices designed for this system require only knowledge of the product p_1p_2 which can be determined from input-output behaviour. For example, by applying the same controller/obsever scheme to System $\mathscr{Z}$ as was used for System $\mathscr{X}$ the following controller and observer gain matrices (respectively) are obtained.

$$K(p) = \left[\frac{-2 + p_1p_2 + \tau_1\tau_2}{p1p2}, \frac{3 - \tau_1 - \tau_2}{p1p2}\right]$$
$$L(p) = \begin{bmatrix} 3 - \lambda_1 - \lambda_2 \\ 7 - 3\lambda_1 + p_1p_2 - 3\lambda_2 + \lambda_1\lambda_2 \end{bmatrix}$$

Note that $K(p)$ and $L(p)$ are both functions only of the product p_1p_2 regardless of the value of p_1, p_2, τ_1, τ_2, λ_1 or λ_2. As a result $K(p)$ and $L(p)$ are invariant under all parameter pairs (p_1, p_2) that produce the observed product p_1p_2.

5. ANALYSIS OF UN-IDENTIFIABILITY

The transformation $x \mapsto z$ is linear and has a matrix representation

$$T(p) = \begin{bmatrix} p_2 & 0 \\ p_2 & p_1p_2 \end{bmatrix}$$

so that at any time $t \in \mathscr{T}$, $z(t) = T(p)x(t)$. Note that T is a function only of p_1 which cannot be determined from input-output behaviour. This issue is not critical for controller design because $T(p)$ is not used explicitly in the formulation of either the controller or the observer. However, the coordinate change defined by $T(p)$ changes the nature of the un-identifiability in the system. In particular, at $t = 0$ the following condition holds.

$$z_0 = T(p)x_0 = \begin{bmatrix} p_2x_1(0) \\ p_2x_1(0) + p_1p_2x_2(0) \end{bmatrix}$$

The change of coordinates given by $T(p)$ had the effect of transforming our original initial state x_0 to z_0. In the z coordinates, the dynamics of the system depend only on the input-output behaviour $G(p, s)$ and so z_0 can be calculated. Indeed, System $\mathscr{Z}$ is constructed using the standard observable realization so that $y(t) = z_1(t)$ and $y'(t) = z_2(t)$. The initial states $x_0 = T^{-1}(p)z_0$, however, cannot be re-constructed because p_1 is unknown. Thus the coordinate transformation had the effect of constructing a system whose dynamics can be used to generate a consistent control strategy. However, this comes at the cost of having an initial state, z_0, that cannot be used to reconstruct x_0, thereby losing all knowledge of the initial states. The un-identifiability of the system, in other words, has been transferred entirely to the initial states of the system.

This approach of "expressing" the un-identifiable part of a system as part of the initial conditions can be applied to any LTI state-space system. This idea is formulated in the following proposition.

Proposition 1. Consider System $\mathcal{S}$ and let

$$G(s, p) = C(p)(Is - A(p))^{-1}B(p)$$

be a strictly rational function of s. There exists a second system given by

$$\dot{z}(t) = \tilde{A}(p)z(t) + \tilde{B}(p)u(t) \qquad (\tilde{S})$$
$$y(t) = \tilde{C}(p)z(t)$$

that is equivalent to System $\mathcal{S}$ in the sense that there exists a full rank transformation $T(p)$ on a dense subset of $\mathscr{P}$ so that

$$
\begin{aligned}
z(t) &= T(p)x(t) \\
\tilde{A}(p) &= T(p)A(p)T^{-1}(p) \\
\tilde{B}(p) &= T(p)B(p) \\
\tilde{C}(p) &= C(p)T^{-1}(p)
\end{aligned}
$$

and Systems $\mathcal{S}$ and $\tilde{S}$ have identical input-output behaviour described by

$$
\begin{aligned}
G(p,s) &= C(p)(Is - A(p))^{-1}B(p) \\
&= \tilde{C}(p)(Is - \tilde{A}(p))^{-1}\tilde{B}(p)
\end{aligned}
$$

such that $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are functions only of identifiable parameter combinations.

proof It has been shown by (Bellman and Astrom, 1970) that the combinations of system parameters that can be identified from input output behaviour are the independent coefficients in the transfer function $G(p,s)$. Furthermore, there exists a (standard) observable realization for $G(p,s)$ given by the triplet $\left(\tilde{A}(p), \tilde{B}(p), \tilde{C}(p)\right)$ whose entries depend only on the independent coefficients of $G(p,s)$. Letting System $\tilde{S}$ be the standard observable realization of $G(p,s)$ completes the proof. The restriction of the proof to a dense subset of $\mathscr{P}$ is due to the fact that $A(p)$ is not invertible for all parameter values. □

Note that our approach amounts to scaling the states by a matrix $T(p)$ of possibly unknown quantities. For System $\mathcal{X}$, $T(p)$ is a function of p_2 and p_1p_2. This implies that the original System $\mathcal{X}$ is not observable in any practical sense. (Ben-Zvi *et al.*, 2004) has previously observed that un-identifiability implies that the states of a system can be arbitrarily scaled. This notion of un-identifability as lack of observability due to parameter uncertainty can be used to pose the un-identifiability problem as a sensor placement problem. For example, if the second state x_2 of System $\mathcal{X}$ was observed directly then given two parameter estimates $p, r \in \mathscr{P}$, $p \neq r$ the following conditions holds.

$$y'(0) = z_2(0) = p_2 x_{p,1}(0) + p_1 p_2 x_2(0) \qquad (7)$$

$$y'(0) = z_2(0) = r_2 x_{r,1}(0) + r_1 r_2 x_2(0) \qquad (8)$$

where $x_{p,1}(0)$ and $x_{r,1}(0)$ are the unknown initial conditions of x_1. Using Equations 7 and 8 and consequently

$$x_{p,1}(0) = \frac{r_2}{p_2} x_{r,1}(0)$$

and one cannot determine the value of p_2 which can be used to scale the state x_1. Alternatively, by observing x_1 directly, one obtains the relation

$$y(0) = p_2 x_1(0) = r_2 x_1(0)$$

and p_2 can be calculated for $x_1(0) \neq 0$. This conclusion can be verified by noting that observing x_1 directly amounts to setting $p_2 = 1$ in System $\mathcal{X}$. It is important to note that System $\mathcal{X}$ was chosen for its simplicity, and in the general nonlinear case it may be very difficult to determine which state should be directly observed (Ben-Zvi *et al.*, 2004).

6. CONCLUSION

Mathematical models used for control are often dependent on unknown parameters. A model is identifiable if and only if every parameter can be uniquely estimated from some set of input-output data. In this work it is shown that lack of identifiability affects the closed loop performance of dynamical systems. That is, two parameter estimates that yield identical input-output behavior may yield a controller/observer pair with completely different *closed-loop* input output behaviour even for identical initial conditions. This conclusion holds even for systems that are both observable and identifiable.

A coordinate representation was constructed for an LTI system so that lack of identifiability did not affect controller/observer design. This was done at the cost of making the initial conditions of the original system unrecoverable in the general case. This is because the coordinate transformation matrix $T(p)$ was generally a function of unknown parameters.

Using a coordinate transformation, the problem of un-identifiability was recast as a sensor placement problem. This problem could be solved in order to determine what state information is necessary to identify all system parameters.

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