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On the convergence of boundary control strategies designed using ODE approximations of diffusive PDE systems

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This paper considers the convergence upon grid refinement of control strategies derived from ODE aproximations of diffusive boundary-controlled linear PDE systems. It focuses specifically on the Dirichlet boundary control of the heat equation as a canonical model for more general diffusive PDE systems. It treats two classes of problems: the *controllability problem* (that is, the determination of a control distribution to move a system exactly from a specified initial state to a specified terminal state in finite time) and the *state feedback control problem* (that is, the determination of an optimal feedback rule $\mathbf{u} = K\mathbf{x}$ which minimizes some quadratic cost function J measuring both the state of the system and the control input), in the latter problem focusing specifically, for simplicity, on the infinite-horizon (that is, constant-gain) case. Both classes of problems require special attention beyond the usual considerations commonly known for the control of low-order ODE systems. Specifically, convergence of the control strategies upon refinement of the ODE approximation used in the controller calculation is not guaranteed. Note that the present study considers sine, finite difference, and Chebyshev discretizations, all of which provide consistent results, indicating that the results reported are not a spurious artificat of any particular numerical discretization.

1 The 1D heat equation

Consider first the PDE system

$$\frac{\partial \Phi}{\partial t} = \mathcal{A}\Phi \quad \text{on} \quad 0 < y < 1, \quad t > 0, \tag{1a}$$

with $\mathcal{A} = \partial^2/\partial y^2$ and $\Phi = \Phi(y,t)$ with inhomogeneous boundary conditions

$$\Phi(y=0,t) = v^0(t) \quad \text{and} \quad \Phi(y=1,t) = v^1(t). \tag{1b}$$

To simplify the analysis, we may **lift** the boundary conditions by defining $\phi = \phi(y,t)$ such that

$$\Phi(y,t) = \phi(y,t) + f(y)v^0(t) + g(y)v^1(t) \tag{2a}$$

$$\text{where} \quad f(y) = 1 - y \quad \text{and} \quad g(y) = y. \tag{2b}$$

Note that $f(0) = 1$ and $f(1) = 0$, whereas $g(1) = 1$ and $g(0) = 0$, and that $\mathcal{A}f = \mathcal{A}g = 0$. Defining

$$b^0(y) = -f(y), \quad b^1(y) = -g(y), \quad u^0(t) = dv^0(t)/dt, \quad \text{and} \quad u^1(t) = dv^1(t)/dt, \tag{2c}$$

it follows that

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi + b^0u^0 + b^1u^1 \quad \text{on} \quad 0 < y < 1, \quad t > 0, \tag{3}$$

with $\mathcal{A} = \partial^2/\partial y^2$ and $\phi = \phi(y,t)$ with homogeneous boundary conditions $\phi(y=0,t) = \phi(y=1,t) = 0$. Given (2), the systems defined by (1) and (3) are equivalent. The analysis below is performed using the more "convenient" (in terms of the application of control theory) form in ϕ, as given in (3), whereas many of the subsequent plots are made using the more "intuitive" (in terms of the physical application) form in Φ, as given in (1).

1.1 Approximate controllability

The control problem considered in §1.1 is to find $u^0(t)$ and $u^1(t)$ on $t \in [0,T]$ to bring the system (3) from a specified initial state $\phi(y,t=0)=\phi^0$ to a specified terminal state $\phi(y,t=T)=\phi^T$. Without loss of generality, we will select $\phi^T=0$. As $\phi(y,t)$ has homogeneous boundary conditions, consider the sine series expansions

$$\phi(y,t)=\sum_{n=1}^{\infty}\hat{\phi}_n(t)\sin(k_{y_n}y) \quad \text{and} \quad \phi^0(y)=\sum_{n=1}^{\infty}\hat{\phi}_n^0\sin(k_{y_n}y), \tag{4}$$

where $k_{y_n}=\pi n$. Defining the inner product $\langle a,b\rangle = 2\int_0^1 a\cdot b\,dy$, note that

$$\langle \sin(k_{y_n}y), \sin(k_{y_{n'}}y)\rangle = \begin{cases} 1 & n=n' \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Taking the inner product of (3) with $\sin(k_{y_{n'}}y)$, applying the expansions in (4), and noting (5) leads to a system that may be written in the form

$$\frac{\partial\hat{\phi}_n}{\partial t} = -k_{y_n}^2\hat{\phi}_n + \hat{b}_n^0u^0 + \hat{b}_n^1u^1 \quad \text{on} \quad t>0 \quad \text{for} \quad n=1,2,3,\ldots \tag{6}$$

where $\hat{\phi}_n=\hat{\phi}_n(t)$, $u^0=u^0(t)$, $u^1=u^1(t)$, and

$$\hat{b}_n^0 = 2\int_0^1 (y-1)\sin(k_{y_n}y)\,dy = -2/k_{y_n}, \quad \hat{b}_n^1 = 2\int_0^1 (-y)\,\sin(k_{y_n}y)\,dy = (-1)^n 2/k_{y_n}.$$

Note that the $\hat{b}_n^0$ are all distinct and nonzero, as are the $\hat{b}_n^1$. The exact solution of (6) for each n is given by

$$\hat{\phi}_n(t) = e^{-k_{y_n}^2 t}\hat{\phi}_n^0 + \hat{b}_n^0\int_0^t u^0(t')e^{-k_{y_n}^2(t-t')}\,dt' + \hat{b}_n^1\int_0^t u^1(t')e^{-k_{y_n}^2(t-t')}\,dt'. \tag{7}$$

Now consider the finite-dimensional approximation of the PDE control problem given by truncating the sine decompositions in (4) after the N'th terms. We may proceed by assuming the control distributions $u^0(t)$ and $u^1(t)$ on $t\in[0,T]$ are expanded using cosine series in time,

$$u^0(t)=\sum_{m=1}^{M}\hat{u}_m^0\cos(\omega_m t), \quad u^1(t)=\sum_{m=1}^{M}\hat{u}_m^1\cos(\omega_m t), \tag{8}$$

where $\omega_m=\pi m/T$. Noting (7), defining

$$c_{nm}=\int_0^T \cos(\omega_m t)e^{-k_{y_n}^2(T-t)}\,dt = \frac{(-1)^m - e^{-k_{y_n}^2T}}{(\omega_m/k_{y_n})^2+k_{y_n}^2}, \tag{9}$$

taking $M=N/2$, and applying the desired result that $\hat{\phi}_n(T)=0$ for the modes $n=1,2,\ldots,N$ which have been retained in the finite-dimensional approximation leads to an $N\times N$ nonsingular linear system of equations

$$0=e^{-k_{y_n}^2T}\hat{\phi}_n^0 + \hat{b}_n^0\sum_{m=1}^{N/2}c_{nm}\hat{u}_m^0 + \hat{b}_n^1\sum_{m=1}^{N/2}c_{nm}\hat{u}_m^1 \quad \text{for} \quad n=1,2,\ldots,N, \tag{10}$$

which may be written in matrix form as

$$\begin{bmatrix} C^0 & C^1 \end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}^0 \\ \hat{\mathbf{u}}^1\end{bmatrix} = \hat{\boldsymbol{\phi}}^0, \tag{11}$$

where

$$\hat{\mathbf{u}}^0 = \begin{pmatrix} \hat{u}_1^0 \\ \vdots \\ \hat{u}_{N/2}^0 \end{pmatrix}, \quad \hat{\mathbf{u}}^1 = \begin{pmatrix} \hat{u}_1^1 \\ \vdots \\ \hat{u}_{N/2}^1 \end{pmatrix}, \quad \hat{\boldsymbol{\phi}}^0 = \begin{pmatrix} \hat{\phi}_1^0 \\ \vdots \\ \hat{\phi}_N^0 \end{pmatrix}, \quad c_{nm}^0 = -e^{k_{y_n}^2 T}\hat{b}_n^0 c_{nm}, \quad c_{nm}^1 = -e^{k_{y_n}^2 T}\hat{b}_n^1 c_{nm}.$$

Given the expansion coefficients of the initial conditions $\hat{\phi}_n^0$ for $n = 1,2,\dots,N$ for any even N, this non-singular system may be solved (using Gaussian elimination) for the expansion coefficients $\hat{u}_m^0$ and $\hat{u}_m^1$ for $m = 1,2,\dots,N/2$, thereby achieving $\hat{\phi}_n(T) = 0$ for the modes $n = 1,2,\dots,N$. Thus, any finite-dimensional approximation of the system (3) [equivalently, (6)] given by truncating the expansions in (4) after the N'th terms is controllable via appropriate selection of $u^0(t)$ and $u^1(t)$ on $t \in [0,T]$. In the limit that $N \to \infty$, the issue of the regularity of the resulting control distributions $u^0(t)$ and $u^1(t)$ on $t \in [0,T]$ is governed by the rate of decay of $|\hat{u}_m^0|$ and $|\hat{u}_m^1|$ with m [see (8)], and is a function of the coefficients c_{nm}^0 and c_{nm}^1 in (11) as well as the regularity of the initial conditions $\phi^0(y)$ considered (that is, the rate of decay of $|\hat{\phi}_n^0|$ with n). If a control distribution of the requisite regularity on $u^0(t)$ and $u^1(t)$ may be found for any initial conditions of the assumed regularity on $\phi^0(y)$, then the PDE system is said to be **controllable**; if it can not, but (as in this problem) any finite-dimensional approximation of the control problem is solvable and the neglected system modes are exponentially stable, then the PDE system is said to be **approximately controllable**. In the present case, considering arbitrary initial conditions $\phi^0(y) \in L^2(y|0 \le y \le 1)$ and seeking appropriate control distributions $u^0(t) \in L^2(t|0 \le t \le T)$ and $u^1(t) \in L^2(t|0 \le t \le T)$, the PDE system (3) is found to be approximately controllable, as discussed further below and illustrated in Figures 1-2.

Recall that (1) is equivalent to (3); thus, the above discussion also applies to (1), taking [noting (2c) and (8)] the control distributions

$$v^0(t) = \sum_{m=1}^{M} \frac{\hat{u}_m^0}{\omega_m}\sin(\omega_m t) \quad \text{and} \quad v^1(t) = \sum_{m=1}^{M} \frac{\hat{u}_m^0}{\omega_m}\sin(\omega_m t). \tag{12}$$

By (12), $u^0(0) = u^0(T) = u^1(0) = u^1(T) = 0$, and thus $\Phi(y,t=0) = \phi^0$ and $\Phi(y,t=T) = \phi^T$. Note that $v^0(t)$ and $v^1(t)$ are more regular than $u^0(t)$ and $u^1(t)$, as the sine series expansion coefficients of $v^0(t)$ and $v^1(t)$ decrease more quickly with m than do the cosine series expansion coefficients of $u^0(t)$ and $u^1(t)$.

Solving (11) in the case with $\phi^0(y)$ chosen to be a square wave (see solid curves in Figure 1), selecting a time horizon $T = 0.03$, and approximating the PDE system with various values of N in the ODE control formulation described above results in control distributions of

$$\begin{aligned}
N = 2 \quad &\Rightarrow \quad u^0(t) = u^1(t) = [3.38\cos(\omega_1 t)] \times 10^2,\\
N = 4 \quad &\Rightarrow \quad u^0(t) = u^1(t) = [3.84\cos(\omega_1 t) + 12.4\cos(\omega_2 t)] \times 10^2,\\
N = 8 \quad &\Rightarrow \quad u^0(t) = u^1(t) = [-0.908\cos(\omega_1 t) + 12.7\cos(\omega_2 t) + 45.5\cos(\omega_3 t) + 35.6\cos(\omega_4 t)] \times 10^2,\\
N = 16 \quad &\Rightarrow \quad u^0(t) = u^1(t) = [-5.46\cos(\omega_1 t) - 17.0\cos(\omega_2 t) - 4.20\cos(\omega_3 t) + 77.5\cos(\omega_4 t) +\\
&\qquad 193\cos(\omega_5 t) + 219\cos(\omega_6 t) + 125\cos(\omega_7 t) + 28.9\cos(\omega_8 t)] \times 10^2.
\end{aligned}$$

The resulting distributions of $\Phi(y,t)$ in the controlled PDE system (1) are plotted in Figures 1-2. Note in Figure 1 that, as N is increased, an increasing number of the low frequency modes of the terminal state of the PDE system, $\Phi(y,T)$, are brought close to the target distribution $\phi^T = 0$. Unfortunately, as shown above, the magnitude of the cosine coefficients generally *increase* with m. In other words, as N is increased in the control formulation, the control distributions do not converge to smooth functions of time, as illustrated in Figure 2. We thus say that the present PDE system is only approximately controllable. Reducing the time horizon T results in control distributions that are even larger in magnitude (due to the fact that the diffusion in the system works to the controller's advantage when larger time horizons are used), and further excites the modes that are not targetted by the ODE controller formulation.

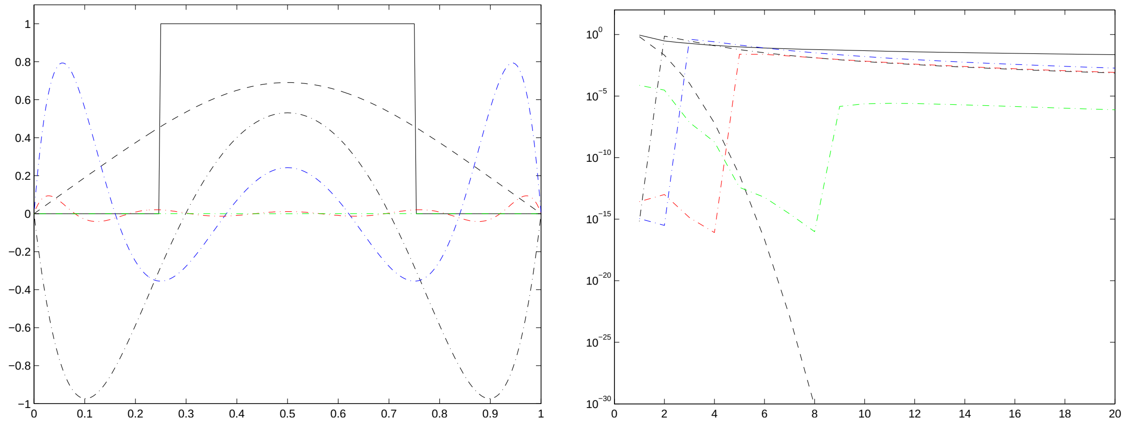

Figure 1: Distributions of Φ in the controlled PDE system (1) with $T = 0.03$. (a) The initial state (solid), the terminal state without control applied (dashed), and the terminal states with control applied using $N = 2$ (black dot-dashed), $N = 4$ (blue dot-dashed), $N = 8$ (red dot-dashed), and $N = 16$ (green dot-dashed) in the ODE controller calculation. Note that, as N is increased, an increasing number of the low frequency modes of the terminal state of the PDE system, $\Phi(y, T)$, are brought close to zero; the corresponding evolutions of the controlled PDE system in space-time are depicted in Figure 2. (b) The magnitudes of the leading odd sine series coefficients of the corresponding states, using the same line styles as used in (a). Note that, in the controlled cases, the desired number of modes are driven to values that are several orders of magnitude smaller than the uncontrolled case, but not to zero due to numerical errors in the computation. Also note that the control applied excites higher modes of the temperature distribution not accounted for in the low-dimensional ODE controller calculation.

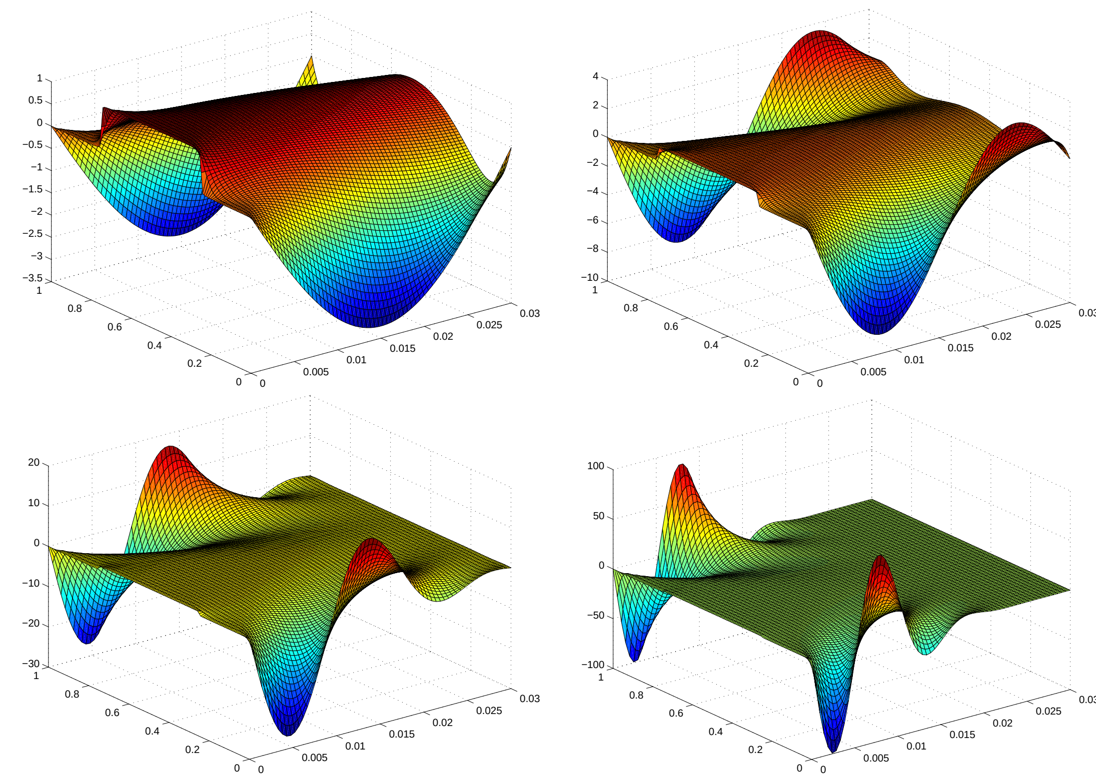

Figure 2: The evolution of the controlled PDE system in the $T = 0.03$ case in space-time with control applied using a value of N in the controller calculation of $N = 2$ (a), $N = 4$ (b), $N = 8$ (c), and $N = 16$ (d).

Defining

$$
\hat{\boldsymbol{\phi}} = \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_N \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -k_{y,1}^2 & & 0 \\ & \ddots & \\ 0 & & -k_{y,N}^2 \end{pmatrix},
$$
$$
\hat{\mathbf{b}}^0 = \begin{pmatrix} \hat{b}_1^0 \\ \vdots \\ \hat{b}_N^0 \end{pmatrix}, \quad \hat{\mathbf{b}}^1 = \begin{pmatrix} \hat{b}_1^1 \\ \vdots \\ \hat{b}_N^1 \end{pmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{\mathbf{b}}^0 & \hat{\mathbf{b}}^1 \end{bmatrix}, \quad \mathbf{u} = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \tag{13}
$$

we may write the relation in (6), truncated at the N'th term, in the **modal-coordinate** state-space form

$$
\frac{d\hat{\boldsymbol{\phi}}}{dt} = \Lambda\hat{\boldsymbol{\phi}} + \hat{B}\mathbf{u}. \tag{14}
$$

Note that the system matrix Λ is diagonal in this representation. This numerical approximation of the PDE system (3) is clearly controllable, as

$$
\text{rank}\{\begin{bmatrix} \hat{B} & \Lambda\hat{B} & \dots & \Lambda^{N-1}\hat{B} \end{bmatrix}\} = N. \tag{15}
$$

Now define a matrix S with components $s_{jn} = \sin(k_{y_n} y_j)$ where $y_j = j/(N+1)$ for $j = 1, 2, \dots, N$, noting that the n'th column $\mathbf{s}^n$ of the matrix $S = \begin{bmatrix} \mathbf{s}^1 & \mathbf{s}^2 & \dots & \mathbf{s}^N \end{bmatrix}$ is a discretization of the mode $\sin(k_{y_n} y)$ on the gridpoints y_1 to y_N. We also define $\boldsymbol{\phi} = S\hat{\boldsymbol{\phi}}$, which simply amounts to enforement of (4), with the sums truncated after the N'th term, on the N gridpoints y_1 to y_N. Premultiplying (14) by S thus leads to the equivalent state-space form

$$
\frac{d\boldsymbol{\phi}}{dt} = A\boldsymbol{\phi} + B\mathbf{u}, \tag{16}
$$

where we have defined $A = S\Lambda S^{-1}$, $\mathbf{b}^0 = S\hat{\mathbf{b}}^0$, $\mathbf{b}^1 = S\hat{\mathbf{b}}^1$, and $B = S\hat{B} = \begin{bmatrix} \mathbf{b}^0 & \mathbf{b}^1 \end{bmatrix}$. Note that the system matrix A is full in this representation. By (15) and the equivalence of (14) and (16), it follows that (16) is controllable.

Note that A is some numerical approximation of $\mathcal{A} = \partial^2/\partial y^2$, and that the columns of B are some discretizations of $(y-1)$ and $(-y)$. It is thus quite tempting to conclude that AB should be zero in any "good" numerical discretization of (3), and thus

$$
\text{rank}\{\begin{bmatrix} B & AB & \dots & A^{N-1}B \end{bmatrix}\} \tag{17}
$$

should be two, from which it would follow that (16) would in fact *not* be controllable. This line of reasoning, however, is not correct. Indeed, for the N modes retained in the truncated form of the expansions (4), the present numerical approximation is, in a sense, "exact", as it follows from the definition $A = S\Lambda S^{-1}$ that $A\mathbf{s}^n = -k_{y_n}^2 \mathbf{s}^n$ where the vector $\mathbf{s}^n$ is a discretization of the sine wave $\sin(k_{y_n} y)$. However, it is important to note that A is built from the discretization of the modes $\sin(k_{y_n} y)$, which effectively incorporate the homogeneous boundary conditions[1] on $\phi(y,t)$; these boundary conditions are *not* shared by $b^0(y)$ and $b^1(y)$. Thus, as illustrated in Figure 3, $AB \neq 0$, and the conclusion that (17) should be two is in fact incorrect. Indeed, we may write

$$
\begin{bmatrix} B & AB & \dots & A^{N-1}B \end{bmatrix} = \begin{bmatrix} S\hat{B} & (S\Lambda S^{-1})S\hat{B} & \dots & (S\Lambda S^{-1})^{N-1}S\hat{B} \end{bmatrix} = S\begin{bmatrix} \hat{B} & \Lambda\hat{B} & \dots & \Lambda^{N-1}\hat{B} \end{bmatrix},
$$

from which it follows immediately, by (15) and the invertability of S, that (17) is in fact N.

[1] Note that (3) is not satisfied at the discretization points $y_0 = 0$ and $y_{N+1} = 1$ on the boundary of the domain $y \in [0,1]$. In fact, the only way that the homogeneous boundary conditions on ϕ inherent to the PDE form (3) may be enforced when writing a discrete approximation of (3) in the general state-space form (16) is by somehow incorporating the effect of these boundary conditions in A.

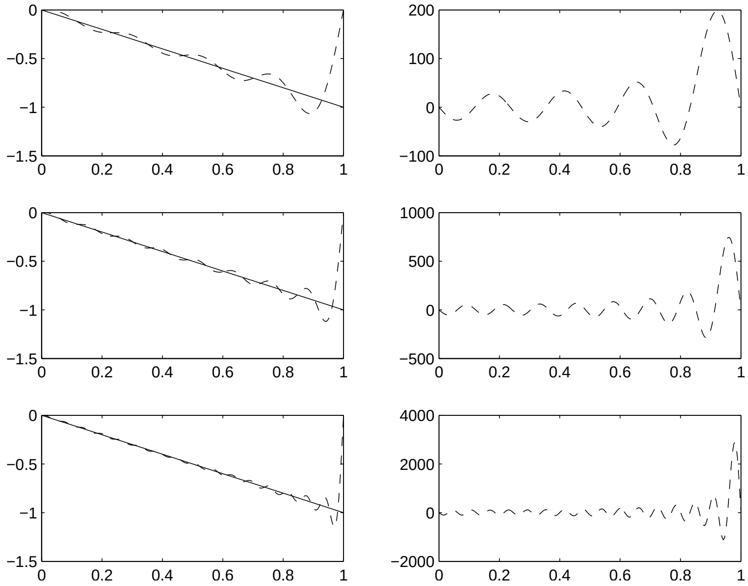

Figure 3: Demonstration of Gibbs phenomenon when performing a sine reconstruction of a **sawtooth function** (that is, a periodic extension of a linear function): (left) $b^1(y)$ and three N-term sine reconstructions of $b^1(y)$ using (top) $N = 8$, (middle) $N = 16$, and (bottom) $N = 32$, and (right) the second derivative of these N-term sine reconstructions of $b^1(y)$.

In general, the problem of **controllability** (that is, the question of whether or not a control distribution can be found to move a system *exactly* from a specified initial state to a specified terminal state in finite time) is a demanding problem that often far exceeds the actual needs of the controller effectiveness in real applications. It is often quite sufficient for the control algorithm to bring the system in some sense *close* to the desired target in the specified time, and for the perturbations of the system to be stable thereafter. Thus, the problem of **stabilizability** (that is, the question of whether or not the unstable modes of the system are controllable) is often the more relevant question to address[2]. Indeed, assuming infinite-precision arithmetic, we can move as many individual modes of the PDE system as we like exactly to the desired target in finite time via solution of the nonsingular system of equations in (11). Further, the higher modes of a diffusive system are certainly very stable. We thus say that the PDE control problem considered here is **approximately controllable**—a characterization which is, in a sense, somewhere between being stabilizable and being controllable.

Note in Figure 2 that, as the discretization is refined (that is, as N is increased) in the ODE control formulation, the control distribution generated to move the ODE approximation of the system described above exactly to the specified target becomes increasingly irregular. As $N \to \infty$, the control distribution does not converge to a smooth function of time; in other words, it fails the test of **convergence upon grid refinement** which is essential in connecting the ODE control problem solved numerically to the PDE control problem which it purports to approximate. Also, as N is increased, the matrix in the linear system of equations determining the control distribution, (11), becomes increasingly ill-conditioned; thus, its solution using finite-precision arithmetic becomes increasingly prone to numerical error. This is reflected by the green curve of Figure 2 for the first few modes reported, which would be closer to zero if (11) were solved using higher-precision arithmetic.

[2]Note that the present problem is not only stabilizable, it is in fact stable with no control inputs applied whatsoever. However, an argument may still be made that this problem is a useful one to study, as several related PDE systems (e.g., the forced heat equation, Burgers' equation, the Orr-Sommerfeld/Squire equation, the Navier-Stokes equation, etc.), are dominated by a diffusion component at the higher wavenumbers, and thus the consideration of how to handle these wavenumbers correctly in a control-oriented context is of significant interest.

1.2 Alternative formulations

Before moving on to the problem of feedback, it is worthwhile to mention that there are several alternative ways to formulate and discretize the control problem considered above. We present three such alternative formulations below. The terminal states reported in Figure 1, which were computed “semi-analytically” via a sine decomposition of ϕ as described above, were also determined numerically via the alternative formulations presented below, all of which confirmed the results reported in Figure 1. The fact that sine, finite-difference, and Chebyshev discretizations all accurately reproduced Figure 1 in this study verify that the result reported is not a spurious artifact of the particular numerical discretization used.

1.2.1 A generalized formulation based on sine-series expansions

Following the notation of §1, we have

$$\frac{\partial \Phi}{\partial t} = \mathcal{A}\Phi, \quad \Phi(y=0,t) = v^0(t), \quad \Phi(y=1,t) = v^1(t),$$

with $\mathcal{A} = \partial^2/\partial y^2$. Defining

$$\Phi(y,t) = \phi(y,t) + f(y)v^0(t) + g(y)v^1(t),$$

where now $f(y)$ and $g(y)$ are some (as-yet, unspecified) continuous functions with

$$f(0) = 1, \quad f(1) = 0, \quad g(1) = 1, \quad \text{and} \quad g(0) = 0, \tag{18}$$

and defining

$$b^0(y) = -f(y), \quad b^1(y) = -g(y), \quad dv^0(t)/dt = -av^0(t) + u^0(t), \quad \text{and} \quad dv^1(t)/dt = -av^1(t) + u^1(t)$$

for some $a > 0$, it follows [cf. (3)] that

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi + (a - \mathcal{A}b^0)v^0 + (a - \mathcal{A}b^1)v^1 + b^0u^0 + b^1u^1 \quad \text{on} \quad 0 < y < 1, \quad t > 0, \tag{19}$$

with $\mathcal{A} = \partial^2/\partial y^2$ and $\phi = \phi(y,t)$ with homogeneous boundary conditions $\phi(y=0,t) = \phi(y=1,t) = 0$. Again, consider the sine series expansion

$$\phi(y,t) = \sum_{n=1}^{\infty} \hat{\phi}_n(t)\sin(k_{y_n} y) \quad \text{where} \quad k_{y_n} = \pi n. \tag{20}$$

Defining the inner product $\langle a,b\rangle = 2\int_0^1 a \cdot b\, dy$, taking the inner product of (19) with $\sin(k_{y_{n'}} y)$ gives

$$\frac{\partial \hat{\phi}_n}{\partial t} = -k_{y_n}^2\hat{\phi}_n + (\widehat{a - \mathcal{A}b^0})_n v^0 + (\widehat{a - \mathcal{A}b^1})_n v^1 + \hat{b}_n^0 u^0 + \hat{b}_n^1 u^1 \quad \text{on} \quad t > 0 \quad \text{for} \quad n = 1,2,3,\ldots \tag{21}$$

where $\hat{\phi}_n = \hat{\phi}_n(t)$, $u^0 = u^0(t)$, $u^1 = u^1(t)$, and

$$\hat{b}_n^0 = 2\int_0^1 b^0(y)\sin(k_{y_n} y)\, dy, \qquad \hat{b}_n^1 = 2\int_0^1 b^1(y)\sin(k_{y_n} y)\, dy, \tag{22a}$$

$$(\widehat{a - \mathcal{A}b^0})_n = 2\int_0^1 [a - \mathcal{A}b^0(y)]\sin(k_{y_n} y)\, dy, \quad (\widehat{a - \mathcal{A}b^1})_n = 2\int_0^1 [a - \mathcal{A}b^1(y)]\sin(k_{y_n} y)\, dy. \tag{22b}$$

Equation (21) is still exact. We now approximate the system numerically by retaining only the first N modes of (21). Writing

$$\hat{\mathbf{x}} = \begin{bmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_N \end{pmatrix} \\ \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \begin{pmatrix} -k_{y_1} & & 0 \\ & \ddots & \\ & & -k_{y_N}^2 \end{pmatrix} & \begin{pmatrix} (\widehat{a-\mathcal{A}b^0})_1 & (\widehat{a-\mathcal{A}b^1})_1 \\ \vdots & \vdots \\ (\widehat{a-\mathcal{A}b^0})_N & (\widehat{a-\mathcal{A}b^1})_N \end{pmatrix} \\ 0 & \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \begin{pmatrix} \hat{b}_1^0 & \hat{b}_1^1 \\ \vdots & \vdots \\ \hat{b}_N^0 & \hat{b}_N^1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}, \quad \mathbf{u} = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix},$$

we have

$$\frac{d\hat{\mathbf{x}}}{dt} = \hat{A}\hat{\mathbf{x}} + \hat{B}\mathbf{u}. \tag{23}$$

Defining $C = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \dots & \hat{A}^{N+1}\hat{B} \end{bmatrix}$, it follows immediately, as before, that

$$\operatorname{rank}(C) = N+2 \quad \text{if} \quad \underbrace{a=0}_{(i)}, \quad \text{and} \quad \underbrace{(\widehat{\mathcal{A}b^0})_n = 0, \quad (\widehat{\mathcal{A}b^1})_n = 0 \quad \forall n}_{(ii)}.$$

That is, if conditions (i) and (ii) hold, the system given in (23) is controllable. If conditions (i) and/or (ii) are relaxed, this rank condition would not necessarily change; however, for particular (that is, bad) choices for a, f, and g, it is possible that the rank condition might fail. One might be tempted to bypass the continuous description of $b^0(y) = -f(y)$ and $b^1(y) = -g(y)$, instead selecting only their discretizations on the interior gridpoints y_1 to y_N to be in the nullspace of some discretization of $\mathcal{A}$ together with the specified inhomogeneous boundary conditions (18). It is important to identify that continuous functions $b^0(y)$ and $b^1(y)$ that discretize in such a manner do *not* satisfy $(\widehat{\mathcal{A}b^0})_n = 0$ and $(\widehat{\mathcal{A}b^1})_n = 0$ for all n [see (22b)], and thus it follows from the definition of C given above that the conclusion that $\operatorname{rank}(C)$ should reduce to two whenever such lifting functions are used is again incorrect.

1.2.2 A second-order central finite difference discretization

As an alternative to the formulations based on sine-series expansions illustrated above, we may instead discretize (1) directly with the state-space form

$$\frac{d\mathbf{\Phi}}{dt} = A\mathbf{\Phi} + B\mathbf{v},$$

using standard second-order central finite difference methods, taking Φ_j as the discretization of $\Phi(y)$ on the interior gridpoint $y_j = j/(N+1)$ for $j = 1, 2, \dots, N$, in which case

$$A = \frac{1}{(\Delta y)^2} \begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}, \quad \mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{N-1} \\ \Phi_N \end{pmatrix}, \quad B = \frac{1}{(\Delta y)^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix},$$

where $\Delta y = 1/(N+1)$. Note that a so-called **boundary bordering** method is used above to account for the effect of the inhomogeneous boundary conditions (that is, for $v_0 = \Phi_0$ and $v_1 = \Phi_{N+1}$) as a rhs forcing vector $\mathbf{v}$.

In this case, the controllability matrix is

$$C = \begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix} = \frac{1}{(\Delta y)^2} \begin{pmatrix} 1 & 0 & \frac{-2}{(\Delta y)^2} & 0 & \frac{5}{(\Delta y)^4} & 0 & \dots \\ 0 & 0 & \frac{1}{(\Delta y)^2} & 0 & \frac{-4}{(\Delta y)^4} & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{(\Delta y)^4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(\Delta y)^4} & \dots \\ 0 & 0 & 0 & \frac{1}{(\Delta y)^2} & 0 & \frac{-4}{(\Delta y)^4} & \dots \\ 0 & 1 & 0 & \frac{-2}{(\Delta y)^2} & 0 & \frac{5}{(\Delta y)^4} & \dots \end{pmatrix}.$$

Rearranging C by premultiplying by a permutation matrix P (that is, without changing its eigenvalues) reveals that

$$PC = \frac{1}{(\Delta y)^2} \begin{pmatrix} 1 & 0 & \frac{-2}{(\Delta y)^2} & 0 & \frac{5}{(\Delta y)^4} & 0 & \dots \\ 0 & 1 & 0 & \frac{-2}{(\Delta y)^2} & 0 & \frac{5}{(\Delta y)^4} & \dots \\ 0 & 0 & \frac{1}{(\Delta y)^2} & 0 & \frac{-4}{(\Delta y)^4} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{(\Delta y)^2} & 0 & \frac{-4}{(\Delta y)^4} & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{(\Delta y)^4} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(\Delta y)^4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \quad \text{where} \quad P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is thus seen again that, for any finite-dimensional discretization of this problem, the controllability matrix C has full rank. However, as $N \to \infty$, and thus $\Delta y \to 0$, the controllability matrix C becomes increasingly ill-conditioned; in fact, the ratio of the first singular value to the third grows without bound as N is increased.

1.2.3 A Chebyshev discretization

Similarly, we may discretize (1) with the state-space form

$$\frac{d\mathbf{\Phi}}{dt} = A\mathbf{\Phi} + B\mathbf{v},$$

using Chebyshev methods, now taking Φ_j as the discretization of $\Phi(y)$ on the interior gridpoint $y_j = \cos(j\pi/(N+1))$ for $j = 1, 2, \dots, N$. In this case, A is the section of the Chebyshev collection second derivative matrix corresponding to the interior gridpoints, and, following a boundary bordering method analogous to that described above, B is built from the first and last columns of the Chebyshev collection second derivative matrix, thereby corresponding to the effect of Φ at the boundary gridpoints (that is, for $v_0 = \Phi_0$ and $v_1 = \Phi_{N+1}$). Following this approach for various values of N, it is again found that the controllability matrix has full rank, but becomes increasingly ill-conditioned as N is increased.

1.3 Infinite-horizon (i.e., constant-gain) feedback control

1.3.1 Formulation penalizing ϕ and $\mathbf{u} = d\mathbf{v}/dt$

Now consider the generation of an infinite-horizon, constant-gain, optimal feedback control rule for the PDE system (3) defined such that

$$u^0(t) = \int_0^1 k^0(y)\phi(y,t)dy, \quad u^1(t) = \int_0^1 k^1(y)\phi(y,t)dy, \tag{24}$$

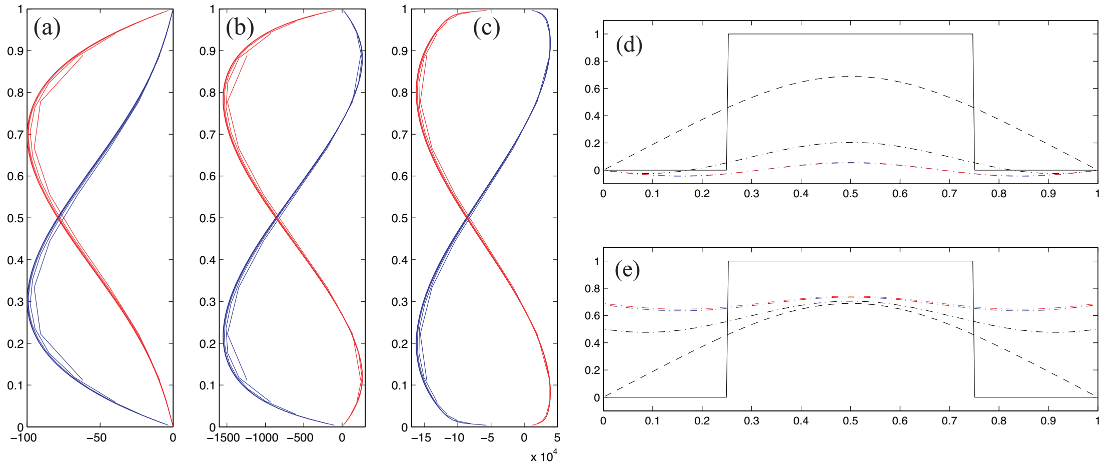

Figure 4: The elements of the first (ilustrated in blue in subfigures a-c) and second (ilustrated in red in subfigures a-c) rows of the infinite-horizon LQR feedback matrix K in (26), plotted sideways as a function of the y location to which the gains correspond, as N is increased from $N = 8$ to $N = 256$ by powers of two, taking (a) $\ell^2 = 10^{-4}$, (b) $\ell^2 = 10^{-6}$, and (c) $\ell^2 = 10^{-10}$. In each of these three cases, note that the two columns of K_0 converge to continuous functions, denoted $k^0(y)$ and $k^1(y)$, as N is increased. (d) Effectiveness of these feedback gains on the state ϕ at time $t = 0.03$ for the control cases with $\ell^2 = 10^{-4}$ (black, dot-dashed), $\ell^2 = 10^{-6}$ (red, dot-dashed), and $\ell^2 = 10^{-10}$ (blue, dot-dashed), as compared with the case with no control (dashed), and the initial conditions of the system (solid). Note that the red curve lies almost exactly on top of the blue curve. (e) The same as in (d) above, but plotting Φ instead of ϕ; cf. Figure 1 for the case in which the control $u(t)$ on the interval $t \in [0,T]$ is computed directly.

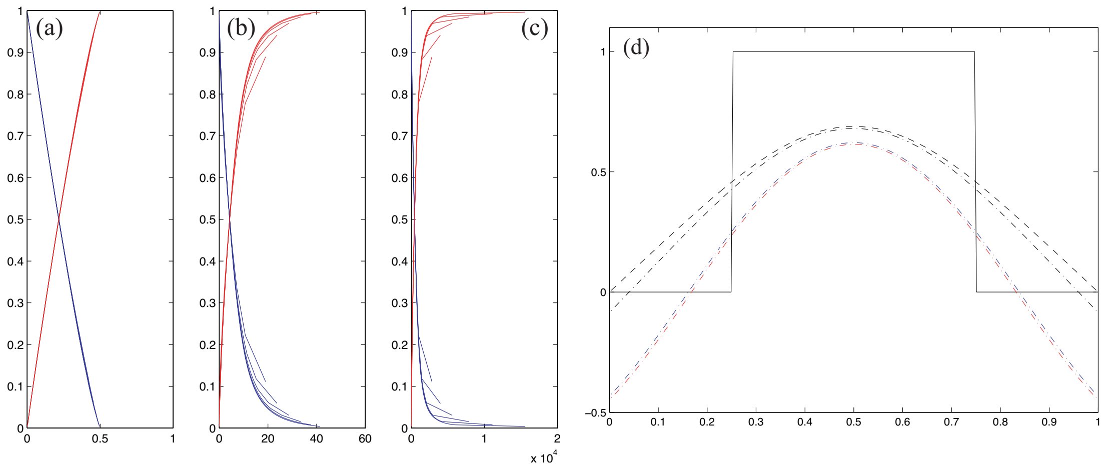

Figure 5: The elements of the first (blue) and second (red) rows of the infinite-horizon LQR feedback matrix K in (30), plotted sideways as a function of the y location to which the gains correspond, as N is increased from $N = 8$ to $N = 256$ by powers of two, taking (a) $\ell^2 = 1$, (b) $\ell^2 = 10^{-2}$, and (c) $\ell^2 = 10^{-6}$. Again, note that the two columns of K converge to continuous functions as N is increased. (d) Effectiveness of these feedback gains on the state Φ at time $t = 0.03$ for the control cases with $\ell^2 = 1$ (black, dot-dashed), $\ell^2 = 10^{-2}$ (red, dot-dashed), and $\ell^2 = 10^{-6}$ (blue, dot-dashed), as compared with the case with no control (dashed), and the initial conditions of the system (solid). Again, note that the red curve lies almost exactly on top of the blue curve.

where the **gain functions** (a.k.a. **functional gains**) $k^0(y)$ and $k^1(y)$ are to be chosen to minimize the cost function

$$\mathcal{J} = \int_0^\infty \Big(\int_0^1 \{\phi(y,t)\}^2\, dy + \ell^2[\{u^0(t)\}^2 + \{u^1(t)\}^2] \Big)\, dt. \tag{25}$$

Discretizing (3) with the state-space form (16) as presented in §1.1, we may approximate this control problem with the state-space feedback control rule

$$\mathbf{u}(t) = \frac{1}{N} K \boldsymbol{\phi}(t), \tag{26}$$

where K is chosen to minimize a cost function approximating (25) such that

$$J = \int_0^\infty \Big[\boldsymbol{\phi}^T Q \boldsymbol{\phi} + \mathbf{u}^T R \mathbf{u}\Big]\, dt \quad \text{where} \quad Q = I/N, \quad R = \ell^2 I. \tag{27}$$

This discretized optimal control problem may be solved by the standard LQR approach via solution of the associated continuous-time algebraic Riccati equation. When the problem is scaled appropriately, as illustrated above, the two rows of K are found to converge upon grid refinement to two functions $k^0(y)$ and $k^1(y)$, as indicated in (24), which are continuous and smooth, as illustrated in Figure 4a-c. The effectiveness of this feedback is illustrated in Figure 4d.

Note very little change in the controller effectiveness between the $\ell^2 = 10^{-6}$ and $\ell^2 = 10^{-10}$ cases, as illustrated in Figure 4d, even though the feedback gains differ by two orders of magnitude, as illustrated in Figures 4b and c. In these cases, due to the large feedback gains used, the state ϕ is essentially constrained to evolve on a manifold which is orthogonal to the gain functions $k^0(y)$ and $k^1(y)$. There are effectively two factors "limiting" or "regularizing" the control effort applied. The first factor is the standarad penalty on the control effort, $[u_0^2(t) + u_1^2(t)]$, incorporated into the cost function [see (25)]. Note, however, that as the coefficient ℓ^2 on this term is reduced towards zero, there is a limit to the controller's effectiveness, indicating that there is something in the formulation preventing the control from returning the system state to zero faster than that depicted by the red and blue dot-dashed curves at $t = 0.03$ in Figure 4d. This is due to a second important factor that comes into play: that is, that the present system is severely **underactuated**, meaning that it has significantly fewer independent actuators (in this case, 2) than it has degrees of freedom (in this case, once the problem is discretized, N). Effectively, control of the states on the interior of the system is achieved by exciting the states near the boundary of the system, which are, in turn, penalized by the cost function defined in (25) even for vanishing values of the factor ℓ^2.

Note that the control problem formulated above only penalizes the discretization of the homogeneous part of the state, ϕ, and the time derivative of the actual boundary conditions on the physical system, $\mathbf{u} = d\mathbf{v}/dt$. Thus, the control objective in this formulation in terms of the original PDE [see (1)], stated in words, is to make the state approximately constant (but not necessarily zero) across the domain, while keeping the square of the time rate of change of the boundary values (but not necessarily the square of the boundary values themselves) small. When plotting the solution of the system in the physical variable Φ, as illustrated in Figure 4e, it is seen that, for small ℓ^2, the controller indeed achieves these objectives fairly well. An alternative control formulation targetting Φ and $\mathbf{v}$ directly is presented in the following section.

1.3.2 A generalized formulation capable of penalizing Φ and $\mathbf{v}$ directly

Now consider the generation of an alternative optimal feedback control rule for the PDE system (1) defined such that

$$v_0(t) = \int_0^1 k^0(y)\Phi(y,t)dy, \quad v_1(t) = \int_0^1 k^1(y)\Phi(y,t)dy, \tag{28}$$

where $k^0(y)$ and $k^1(y)$ are to be chosen to minimize the cost function

$$\mathcal{J} = \int_0^\infty \Big(\int_0^1 \Phi^2(y,t)\, dy + \ell^2 [v_0^2(t) + v_1^2(t)] \Big)\, dt. \tag{29}$$

In this case, for simplicity, we discretize (1) with the state-space form

$$\frac{d\mathbf{\Phi}}{dt} = A\mathbf{\Phi} + B\mathbf{v},$$

where A and B are constructed using standard second-order central finite difference methods, as described in §1.2.2. We then design a state-space feedback control rule

$$\mathbf{v}(t) = \frac{1}{N} K\mathbf{\Phi}(t), \tag{30}$$

where K is chosen to minimize the cost function

$$J = \int_0^\infty \Big[\mathbf{\Phi}^T Q \mathbf{\Phi} + \mathbf{v}^T R \mathbf{v} \Big]\, dt \quad \text{where} \quad Q = I/N, \quad R = \ell^2 I.$$

This discretized problem may again be solved via the standard approach via solution of the associated continuous-time algebraic Riccati equation. When the problem is scaled appropriately, as illustrated above, the two rows of K are again found to converge upon grid refinement to two functions $k^0(y)$ and $k^1(y)$, as indicated in (28), which are continuous and smooth, as illustrated in Figure 5a-c. The effectiveness of this feedback is illustrated in Figure 5d. Similar trends are seen as in §1.3.1, though in this case it is the weighted sum of the integral of the square of Φ and the square of $\mathbf{v}$ that are of minimized via the control input.

2 The 2D heat equation

In order for the ODE control problem solved numerically to be directly applicable to the PDE control problem which it purports to approximate, it is essential that it pass the test of convergence upon grid refinement. As introduced in the previous section, the standard "controllability problem" applied to the boundary control of the 1D heat equation fails this important test (see §1.1), whereas the (perhaps, more relevant) feedback problems posed previously pass this test (see §1.3). Unfortunately, not all feedback formulations for the boundary control of PDE systems pass the test of convergence upon grid refinement. To examine this issue further, we first extend the PDE control problem considered above to two dimensions. Towards this end, consider now the PDE system [cf. (1)]

$$\frac{\partial \Phi}{\partial t} = \mathcal{A}\Phi \quad \text{on} \quad 0 \le x < L_x, \quad 0 < y < 1, \quad t > 0, \tag{31a}$$

with $\mathcal{A} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\Phi = \Phi(x,y,t)$ with inhomogeneous boundary conditions in y such that

$$\Phi(x, y=0, t) = v^0(x,t) \quad \text{and} \quad \Phi(x, y=1, t) = v^1(x,t) \tag{31b}$$

and periodic boundary conditions in x such that

$$\Phi(x = L_x, y, t) = \Phi(x = 0, y, t). \tag{31c}$$

To simplify the analysis, we decompose $v^0(x,t)$ and $v^1(x,t)$ with infinite Fourier series in x such that

$$v^0(x,t) = \sum_{m=-\infty}^{\infty} \hat{v}_m^0(t) e^{ik_{xm}x}, \quad v^1(x,t) = \sum_{m=-\infty}^{\infty} \hat{v}_m^1(t) e^{ik_{xm}x} \quad \text{with} \quad k_{xm} = 2\pi m/L_x, \tag{32a}$$

and then lift the boundary conditions by defining $\phi = \phi(x,y,t)$ such that [cf. (2)]

$$\Phi(x,y,t) = \phi(x,y,t) + \sum_{m=-\infty}^{\infty} \left[\hat{f}_m(y)\hat{v}_m^0(t) + \hat{g}_m(y)\hat{v}_m^1(t) \right] e^{ik_{xm}x} \tag{32b}$$

$$\text{where} \quad \hat{f}_m(y) = \begin{cases} 1-y & m=0, \\ \hat{f}_m^+ e^{k_{xm}y} + \hat{f}_m^- e^{-k_{xm}y} & m \neq 0, \end{cases} \qquad \hat{g}_m(y) = \begin{cases} y & m=0, \\ \hat{g}_m^+ e^{k_{xm}y} + \hat{g}_m^- e^{-k_{xm}y} & m \neq 0, \end{cases} \tag{32c}$$

with $\hat{f}_m^+ = -e^{-k_{xm}}/(e^{k_{xm}} - e^{-k_{xm}})$, $\hat{f}_m^- = e^{k_{xm}}/(e^{k_{xm}} - e^{-k_{xm}})$, and $\hat{g}_m^+ = 1/(e^{k_{xm}} - e^{-k_{xm}})$, $\hat{g}_m^- = -\hat{g}_m^+$. Note that $\hat{f}_m(0) = 1$ and $\hat{f}_m(1) = 0$, whereas $\hat{g}_m(1) = 1$ and $\hat{g}_m(0) = 0$, and that $\mathcal{A}\hat{f}_m(y)e^{ik_{xm}x} = \mathcal{A}\hat{g}_m(y)e^{ik_{xm}x} = 0$ for all m. Defining

$$u^0(x,t) = \frac{dv^0(x,t)}{dt}, \quad u^1(x,t) = \frac{dv^1(x,t)}{dt} \quad \Leftrightarrow \quad \hat{u}_m^0(t) = \frac{d\hat{v}_m^0(t)}{dt}, \quad \hat{u}_m^1(t) = \frac{d\hat{v}_m^1(t)}{dt}, \tag{32d}$$

it follows [cf. (3)] that

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi - \sum_{m=-\infty}^{\infty} \left[f_m(y)\hat{u}_m^0(t) + g_m(y)\hat{u}_m^1(t) \right] e^{ik_{xm}x} \quad \text{on} \quad 0 \le x < L_x, \quad 0 < y < 1, \quad t > 0, \tag{33a}$$

with $\mathcal{A} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\phi = \phi(x,y,t)$ with homogeneous boundary conditions in y such that

$$\phi(x, y=0, t) = \phi(x, y=1, t) = 0 \tag{33b}$$

and periodic boundary conditions in x such that

$$\phi(x = L_x, y, t) = \phi(x = 0, y, t). \tag{33c}$$

Given (32), the systems defined by (31) and (33) are equivalent.

2.1 Approximate controllability

As $\phi(x,y,t)$ has periodic boundary conditions in x and homogeneous boundary conditions in y, consider the Fourier/sine series expansions [cf. (4)]

$$\phi(x,y,t) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{\phi}_{m,n}(t) e^{ik_{xm}x} \sin(k_{y_n} y) \quad \text{and} \quad \phi^0(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{\phi}_{m,n}^0 e^{ik_{xm}x} \sin(k_{y_n} y), \tag{34}$$

where $k_{y_n} = \pi n$. Defining the inner product $\langle a, b \rangle = \frac{2}{L_x} \int_0^{L_x} \int_0^1 a \cdot \overline{b} \, dy \, dx$, note [cf. (5)] that

$$\langle e^{ik_{xm}x} \sin(k_{y_n} y), e^{ik_{xm'}x} \sin(k_{y_{n'}} y) \rangle = \begin{cases} 1 & \text{for } m = m' \text{ and } n = n', \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Taking the inner product of (33) with $e^{ik_{xm'}x} \sin(k_{y_{n'}} y)$, applying the expansions in (34), and noting (35) leads to a system that may be written in the form [cf. (6)]

$$\frac{\partial \hat{\phi}_{m,n}}{\partial t} = (-k_{xm}^2 - k_{y_n}^2)\,\hat{\phi}_{m,n} + \hat{b}_{m,n}^0 \hat{u}_m^0 + \hat{b}_{m,n}^1 \hat{u}_m^1 \quad \text{on} \quad t > 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \text{and} \quad m = 0, \pm 1, \pm 2 \ldots \tag{36}$$

where

$$\hat{b}_{m,n}^0 = -2 \int_0^1 f_m(y) \sin(k_{y_n} y) \, dy = -2k_{y_n}/(k_{xm}^2 + k_{y_n}^2),$$
$$\hat{b}_{m,n}^1 = -2 \int_0^1 g_m(y) \sin(k_{y_n} y) \, dy = (-1)^n 2k_{y_n}/(k_{xm}^2 + k_{y_n}^2).$$

Note that the subset of the equations given in (36) for each particular value of m is completely decoupled from those equations for other values of m, with the subset corresponding to $m = 0$ identical to (6).

We now proceed with discretization, again in an analogous manner as to that done in §1.1. Defining [cf. (13)]

$$\hat{\boldsymbol{\phi}}_m = \begin{pmatrix} \hat{\phi}_{m,1} \\ \vdots \\ \hat{\phi}_{m,N} \end{pmatrix}, \quad \Lambda_m = \begin{pmatrix} -k_{x_m}^2 - k_{y_1}^2 & & 0 \\ & \ddots & \\ 0 & & -k_{x_m}^2 - k_{y_N}^2 \end{pmatrix},$$
$$\hat{\mathbf{b}}_m^0 = \begin{pmatrix} \hat{b}_{m,1}^0 \\ \vdots \\ \hat{b}_{m,N}^0 \end{pmatrix}, \quad \hat{\mathbf{b}}_m^1 = \begin{pmatrix} \hat{b}_{m,1}^1 \\ \vdots \\ \hat{b}_{m,N}^1 \end{pmatrix}, \quad \hat{B}_m = \begin{bmatrix} \hat{\mathbf{b}}_m^0 & \hat{\mathbf{b}}_m^1 \end{bmatrix}, \quad \mathbf{u} = \begin{pmatrix} \hat{u}_m^0 \\ \hat{u}_m^1 \end{pmatrix}, \tag{37}$$

we may write the relation in (36), truncated to include N sine modes in the y direction and $2M+1$ Fourier modes in the x direction, in the modal-coordinate state-space form [cf. (14)]

$$\frac{d\hat{\boldsymbol{\phi}}_m}{dt} = \Lambda_m \hat{\boldsymbol{\phi}}_m + \hat{B}_m \hat{\mathbf{u}}_m \quad \text{for} \quad m = 0, \pm 1, \pm 2, \ldots, \pm M. \tag{38}$$

Note that $\hat{\boldsymbol{\phi}}_m$ is completely decoupled from one value of m to the next, and that the system matrix Λ_m is diagonal for each value of m in this representation.

Note also that, for each m, the $\hat{b}_{m,n}^0$ are all distinct and nonzero, as are the $\hat{b}_{m,n}^1$. Thus, by the same argument as that given in §1.1 above, it follows that (36), and therefore (33) and (31), are approximately controllable. That is, truncating the expansions of the state in x and y [see (34)] to any desired order, assuming infinite-precision arithmetic is used, a control distribution may be found to bring the discrete approximation of the system [see (38)] back to rest exactly from arbitrary initial conditions. However, as the numerical discretization is refined, this control distribution becomes highly irregular; that is, the control formulation fails the test of convergence upon grid refinement.

Again, however, the problem of controllability (that is, the question of whether or not a control distribution can be found to move a system exactly from a specified initial state to a specified terminal state in finite time) is a demanding problem that often far exceeds the actual needs of the controller effectiveness in real applications. Thus, in the following section, we focus more carefully on the feedback problem.

In order to better understand the PDE considered, we also introduce a useful “half-transformed” (that is, physical in y but Fourier in x) representation $\check{\boldsymbol{\phi}}_m = S\hat{\boldsymbol{\phi}}_m$ where, as before, the matrix S has components $s_{jn} = \sin(k_{y_n} y_j)$ where $y_j = j/(N+1)$ for $j = 1, 2, \ldots, N$. The elements of $\check{\boldsymbol{\phi}}_m$ represent the values of $\check{\phi}_m(y)$ on the corresponding y gridpoints. Multiplying (38) by S, it follows that the dynamics are still decoupled for each m, that is [cf. (16)],

$$\frac{d\check{\boldsymbol{\phi}}_m}{dt} = \check{A}_m \check{\boldsymbol{\phi}}_m + \check{B}_m \hat{\mathbf{u}}_m \quad \text{for} \quad m = 0, \pm 1, \pm 2, \ldots, \pm M, \tag{39}$$

where $\check{A}_m = S\Lambda_m S^{-1}$ and $\check{B}_m = S\hat{B}_m$. Note that $\check{\boldsymbol{\phi}}_m$ is completely decoupled from one value of m to the next, and that the $N \times N$ system matrix $\check{A}_m$ is full for each value of m in this representation.

Further transforming the $\check{\boldsymbol{\phi}}_m$ completely back to **physical space** (that is, physical in both y and x) leads to a single (that is, coupled) system for the discretization of $\phi(x,y)$ on the entire 2D domain. The $[(2M+1)N] \times [(2M+1)N]$ system matrix in this representation is full. For “reasonable” values of N and M [say, $N = M = O(100)$] designed to resolve a diffusive PDE system with any substantial degree of complexity (for instance, if the system considered is extended to include a nonlinear term; see footnote 2 on page 6), the feedback control formulations considered in the following section become intractable without first applying some sort of open-loop model reduction, which poses certain disadvantages (specifically, a loss of any guarantees of closed-loop stability, robustness, and performance). Thus, we primarily leverage the decoupled (that is, Fourier-in-x) formulations listed above in the discussion that follows.

2.2 Infinite-horizon feedback control

Now consider the generation of an infinite-horizon, constant-gain, optimal feedback control rule for the PDE system (33) defined such that [cf. (24)]

$$u^0(x,t) = \frac{1}{L_x}\int_0^1\int_0^{L_x} k^0(x',y)\,\phi(x-x',y,t)\,dx\,dy, \quad u^1(x,t) = \frac{1}{L_x}\int_0^1\int_0^{L_x} k^1(x',y)\,\phi(x-x',y,t)\,dx\,dy, \tag{40}$$

with

$$u^0(x,t) = \sum_{m=-\infty}^{\infty} \hat{u}_m^0(t)e^{ik_{x_m}x} \quad \text{and} \quad u^1(x,t) = \sum_{m=-\infty}^{\infty} \hat{u}_m^1(t)e^{ik_{x_m}x},$$

where the **feedback convolution kernels**[3] $k^0(x',y)$ and $k^1(x',y)$ are to be chosen to minimize the cost function

$$\mathcal{J} = \int_0^\infty \frac{1}{L_x}\int_0^{L_x}\Big(\int_0^1 \{\phi(x,y,t)\}^2\,dy + \ell^2[\{u^0(x,t)\}^2 + \{u^1(x,t)\}^2]\Big)\,dx\,dt. \tag{41}$$

The feedback convolution kernels $k^0(x',y)$ and $k^1(x',y)$, which are used to determine $u^0(x,t)$ and $u^1(x,t)$ as shown above, are said to be **shift invariant** when, as in the present case, they do not explicitly depend on x (that is, they only depend on x'). It is easily argued that the shift invariance of the feedback convolution kernels associated with the solution of an optimal control problem such as that formulated above is a direct consequence of the shift invariance of the underlying PDE [see (31)], cost function [see (41)], and feedback control rule [see (40)] upon which this optimal control problem is based.

As mentioned above, determination of such energetically-optimal feedback convolution kernels (by solution of the associated Riccati equations) is computationally intractable in typical well-resolved discretizations when performed in physical space unless open-loop model reduction is used, as the system matrix is both very large and full in such discretizations. Thus, as in [1], we leverage the equivalent, decoupled, Fourier-in-x system formulations listed above in the derivation that follows in order to block decouple a single, large, unmanageble control problem into an equivalent set of many small, decoupled, manageable control problems.

Note first that a convolution integral in physical space (for example, $u(x) = \frac{1}{L_x}\int_0^{L_x} k(x')\phi(x-x')\,dx'$ for all x), corresponds to a product at each wavenumber in transform space (that is, $\hat{u}_m = \hat{k}_m\hat{\phi}_m$). This can be seen by expanding $u(x)$, $k(x')$, and $\phi(x-x')$ in the former with infinite Fourier series, which leads to

$$\begin{aligned}\sum_{m=-\infty}^{\infty}[\hat{u}_m]e^{\imath k_m x} &= \frac{1}{L_x}\int_0^{L_x}\Big(\sum_{p=-\infty}^{\infty}\hat{k}_p e^{\imath k_p x'}\Big)\Big(\sum_{m=-\infty}^{\infty}\hat{\phi}_m e^{\imath k_m(x-x')}\Big)\\ &= \sum_{m=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\hat{k}_p\hat{\phi}_m\underbrace{\Big(\frac{1}{L_x}\int_0^{L_x} e^{\imath(k_p-k_m)x'}dx'\Big)}_{=\delta_{pm}}e^{\imath k_m x} = \sum_{m=-\infty}^{\infty}[\hat{k}_m\hat{\phi}_m]e^{\imath k_m x} \quad \forall x.\end{aligned}$$

Applying this result to the feedback control rules in (40) leads to

$$\hat{u}_m^0(t) = \int_0^1 \check{k}_m^0(y)\,\check{\phi}_m(y,t)\,dy, \quad \hat{u}_m^1(t) = \int_0^1 \check{k}_m^1(y)\,\check{\phi}_m(y,t)\,dy. \tag{42}$$

Similarly, by Parseval's theorem, the cost function (41) may be rewritten

$$\mathcal{J} = \int_0^\infty \sum_{m=-\infty}^{\infty}\Big(\int_0^1\{\check{\phi}_m(y,t)\}^2dy + \ell^2[\{\hat{u}_m^0(t)\}^2 + \{\hat{u}_m^1(t)\}^2]\Big)\,dt. \tag{43}$$

[3] Note that "feedback convolution kernels", as discussed here, are distinct from "functional gains", as discussed in 1.3.1, as the latter are not associated with convolution integrals, whereas the former are.

Applying the discretization presented in §2.1 by restricting our attention to a finite range of both m and n, the feedback control rules (42) may be approximated [cf. (26)] as

$$\hat{\mathbf{u}}_m(t) = \frac{1}{N}\check{K}_m\check{\boldsymbol{\phi}}_m(t), \tag{44}$$

where the columns of $\check{K}_m$ are discretizations of $\check{k}^0_m(y)$ and $\check{k}^1_m(y)$ on the gridpoints y_1 to y_N, and are to be chosen to minimize a cost function approximating (43) such that [cf. (27)]

$$J = \sum_{m=-M}^{M} J_m \quad \text{where} \quad J_m = \int_0^\infty \left(\check{\boldsymbol{\phi}}_m^H Q\check{\boldsymbol{\phi}}_m + \hat{\mathbf{u}}_m^H R\hat{\mathbf{u}}_m\right) dt, \quad \text{where} \quad Q = I/N, \quad R = \ell^2 I. \tag{45}$$

Examining (39), (44), and (45) together, it is seen that the problem of minimizing each J_m by the appropriate selection of the $\check{K}_m$ is completely decoupled at each m, and thus each of these fairly small LQR problem may be solved independently. Once each of these LQR problems is solved using standard techniques, the net result minimizes J. Finally, the resulting columns of the the $\check{K}_m$ can be assembled and inverse transformed to physical space to obtain a numerical approximation of the feedback convolution kernels $k^0(x',y)$ and $k^0(x',y)$ sought [see (40)].

3 Summary of the key result

The key result of the present investigation is now summarized. After a series of related warm-up problems, the problem of the feedback control of the 2D heat equation on a periodic strip via actuation of the Dirichlet boundary conditions on the temperature was considered carefully. In this study, $\Phi(x,y,t)$ denotes the temperature within the strip, $v^0(x,t)$ and $v^1(x,t)$ denote the boundary values of this temperature on the upper and lower edges of the strip, $u^0(x,t) = \partial v^0(x,t)/\partial t$ and $u^1(x,t) = \partial v^1(x,t)/\partial t$ denote the time derivatives of these boundary values, and $\phi(x,y,t)$ denotes a convenient transformation of $\Phi(x,y,t)$ with homogeneous boundary conditions on the upper and lower edges of the strip.

The problem set up and solved in §2.2, penalizing the squares of $\phi(x,y,t)$, $u^0(x,t)$, and $u^1(x,t)$ in the cost function in a manner analogous to that presented in §1.3.1 for the 1D case, happens to converge upon grid refinement to smooth 2D feedback convolution kernels $k^0(x',y)$ and $k^1(x',y)$, in a manner similar to that depicted in Figure 4 for the 1D case.

However, a very slightly modified formulation, penalizing the squares of $\Phi(x,y,t)$, $v^0(x,t)$, and $v^1(x,t)$ in the cost function in a manner analogous to that presented in §1.3.2 for the 1D case, spectacularly fails to converge upon grid refinement to smooth 2D convolution kernels, in sharp contrast to the result depicted in Figure 5 in the 1D case. Rather, when this formulation is used, the feedback gain functions get larger and larger in magnitude as the wavenumber k_{xm} is increased. Thus, when inverse transformed, the resulting feedback convolution kernels are dominated by oscillations at the highest frequencies retained in the discretization, in a manner similar to that depicted for the controllability problem considered in Figure 2.

Further, this failure to obtain convergence upon grid refinement is not a consequence of the discretization method implemented in the y direction, as similar results are obtained for all of the numerical discretizations in y used in this study (sine, Chebyshev, and finite difference). Rather, it is delicate issue related to the regularity of the PDE control problem posed (even in this fairly "simple" canonical problem). Notably, *it is not sufficient simply to penalize the mean squares of Φ, v^0, and v^1 for the system described in* (31) *in order to obtain a feedback control solution that passes the important test of convergence upon grid refinement.*

The problem described here is not new; in fact, for those working in this area, it has been a longstanding challenge in the boundary control of diffusive PDE systems that not all "reasonable" feedback control formulations appear to converge upon grid refinement to something meaningful, indicating a gap in the available regularity theory for feedback control problems for diffusive, spatially-invariant PDE systems.

A new mathematical theory to establish a sharp sufficient condition which guarantees that a given feedback control formulation will in fact converge upon grid refinement in the manner described above is currently under development by the authors, and will be discussed at the conference.

References

[1] BEWLEY, T.R., & LIU, S. 1998 Optimal and robust control and estimation of linear paths to transition. *J. Fluid Mech.* **365**, 305.