# ON THE OPERABILITY OF HIGH-ORDER MULTIVARIABLE NON-SQUARE SYSTEMS

**Fernando V. Lima and Christos Georgakis**

*Department of Chemical and Biological Engineering &*
*Systems Research Institute, Tufts University*

Abstract: Non-square process control systems, with fewer inputs than the controlled outputs, are quite common in industrial chemical processes. In these systems, it is impossible to control all the outputs at specific set-points and thus, many of them are controlled within intervals. The objective of this paper is to extend the non-square Operability methodology proposed by Lima and Georgakis (2006) to high-order systems to be used in the design of non-square constrained controllers. This is achieved by analyzing the general problem with *n* outputs, *m* inputs and 1 disturbance. The developed methodology is applied to non-square systems obtained from industrial problems.


Keywords: Operability, Ranges, Output Variables, Polygons, MIMO, Predictive Control

## 1. INTRODUCTION

In general for multivariable and highly interactive chemical processes, MPC controllers aim to control non-square systems in which there are more controlled outputs than manipulated inputs. Based on the input constraints, generally specified *a priori* due to the physical limitations of the process, an important task is to define the output ranges, or constraints within which one wants to control the process. Because the improper selection of these constraints can make the controller infeasible, hard constraints are either converted to soft constraints that can be violated under certain conditions (Hovd, 2004; Rawlings, 2000) or the intervals between the upper and lower values are widened, resulting in less tight control. Looser control of the outputs causes the operating point of the process to stray further away from the true economic optimum, which is often at the boundary of the acceptable region of operation.

The Operability methodology developed by Vinson and Georgakis (1998, 2000) serves an important role in selecting proper output constraints, so that they are as tight as possible and do not render the controller infeasible. This enables one to verify the achievability of control objectives before implementing the MPC controller. This framework was introduced as a measure to assess the input-output open-loop operability of a multivariable square chemical process at the steady-state, a necessary condition for overall process operability. It provides a quantitative result for multivariable systems and a graphical representation for systems with 3 dimensions or less, permitting modifications to the design that may improve process operability before the selection of a control structure. This methodology takes into account the limited range available for the control inputs during the design phase. Essentially, the Operability framework can quantify the ability of a process to change from one steady-state to another and reject expected disturbances, utilizing the finite control action available. This concept is important because once the design is fixed no control methodology can overcome limitations on operability (Vinson, 2000; Georgakis *et al.*, 2003).

Based on the non-square Operability concepts introduced by Lima and Georgakis (2006), we present an iterative algorithm for the extension of this framework to high-order non-square systems. This is done by addressing the general problem with $n$ outputs, $m$ inputs and 1 disturbance.

## 2. OPERABILITY OF MULTIVARIABLE NON-SQUARE SYSTEMS

In order to quantify the steady-state operability of non-square linear systems, the process outputs are classified into two categories: *set-point controlled*: variables that are controlled at exact set-point values (production rates and product qualities); *set-interval controlled*: variables that are controlled within specified ranges (pressure, temperature and level). In the latter case we refer to the operability as *interval operability*. The set-point and range variables are selected according to the process control objectives. Moreover, when assessing the interval operability of a process one aims to fix critical outputs at their set-points, allowing the others to vary within their maximum and minimum limits. Process outputs must have at least one feasible operating point within the desired interval. To clarify the idea of the Operability concept, it is first necessary to define some useful sets. The Available Input Set (*AIS*) is the set of values that the process input variables can take based on the design and constraints of the process. For an $n \times m \times q$ *(outputs x inputs x disturbances)* non-square system:

$$AIS = \left\{ \mathbf{u} \mid u_i^{\min} \le u_i \le u_i^{\max};\ 1 \le i \le m \right\}$$

Moreover, the Desired Output Set (*DOS*) is given by the ranges of the outputs that are desired to be achieved and might be represented by:

$$DOS = \left\{ \mathbf{y} \mid y_i^{\min} \le y_i \le y_i^{\max};\ 1 \le i \le n \right\}$$

Finally, the Expected Disturbance Set (*EDS*) represents the expected steady-state values of the disturbances:

$$EDS = \left\{ \mathbf{d} \mid d_i^{\min} \le d_i \le d_i^{\max};\ 1 \le i \le q \right\}$$

We will here limit our attention to a single disturbance and thus $q$ = 1. Based on the steady-state model of the process, expressed by the process gain matrix ($G$) and the disturbance gain matrix ($G_d$), the Achievable Output Set for a specific disturbance value ($AOS(d)$) is defined by the ranges of the outputs that can be achieved using the inputs inside the *AIS*:

$$AOS(\mathbf{d}) = \left\{ \mathbf{y} \mid \mathbf{y} = \mathbf{G}\mathbf{u} + \mathbf{G}_d\mathbf{d};\ \mathbf{u} \in AIS \right\}$$

Thus, the servo $AOS(d = 0)$ is a subset of an $m$-dimensional manifold in $\mathbb{R}^n$. Varying the process disturbance, which can take values within the *EDS* (1-dimensional region), the $AOS(d = 0)$ is shifted in the $\mathbb{R}^n$ space along a direction determined by $G_d$ and by an amount affected by the maximum and minimum disturbance values. The union of all shifted locations for all the possible disturbance values yields the set *AOS*:

$$AOS = \bigcup_{d \in EDS} AOS(d), \text{ which is a subset of } \mathbb{R}^n$$

In order to calculate the feasible output ranges, we will use the definition of the Achievable Output Interval Set (*AOIS*) given by Lima and Georgakis (2006). The *AOIS* was defined as the smallest possible interval constraints for the outputs that can be achieved with the available range of the manipulated variables when the disturbances remain within their expected values. The idea of this calculation is to enlarge the sizes of an initial estimate of the *AOIS,* keeping a pre-specified aspect ratio between the output variables constant until the *AOIS* grows in size enough to touch or intersect the extreme sets associated with the minimum and maximum disturbance values of *AOS*. The desired degree of tightness in the control of each of the outputs will affect the aspect ratio of the corresponding side of the initial estimate of the *AOIS*. For example, an aspect ratio of 1:10 between two outputs assures that one will be controlled 10 times more tightly, approximating set-point control for the corresponding variable.

In the previous publication (Lima and Georgakis, 2006), two simple cases involving sub-systems of the Steam Methane Reformer (SMR) hydrogen-production process from Air Products and Chemicals (Vinson, 2000) were presented. This process has 4 manipulated variables (MV's), 1 disturbance variable (DV) and 9 controlled variables (CV's). The sub-systems considered there were 2 x 1 (*outputs x inputs*) and 3 x 2 examples, where all the outputs were controlled within an interval using the same relative weight. Here, examples are presented where some of the output variables are controlled more tightly than others. This is followed by an examination of cases where some of the outputs are controlled at the set-point and the rest within ranges. Finally, examples involving systems with dimensionality higher than

3-D are shown. For all cases, the property $n \geq m+1$ is observed. Thus, the *AIS*, *DOS* and *EDS* are subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^1$ respectively.

## 3. MOTIVATING SIMPLE CASES

The concepts and definitions of Interval Operability (Lima and Georgakis, 2006) are presented in this section using a 2 x 1 system obtained from the SMR process model. In this example we assume the same weight for all the outputs and the origin as the nominal steady-state point ($y_0$). Consider the following system:

$$y = G u_1 + G_d d_1 \quad \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} u_1 + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} d_1 \qquad (1)$$

Where:

$AIS = \{u_1 \mid -1 \leq u_1 \leq 1\}$, $DOS = \{\mathbf{y} \in \Re^2 \mid \|\mathbf{y}\|_\infty \leq 1\}$ and

$EDS = \{d_1 \mid -1 \leq d_1 \leq 1\}$.

Rearranging equation (1) we can calculate the interrelationship between $y_1$ and $y_2$ for different disturbance values:

$$y_2 = a_{21} \frac{y_1 - c_1 d_1}{a_{11}} + c_2 d_1 \qquad (2)$$

Consider the steady-state gain matrices:

$$G = [1.41, 0.66]^T; \quad G_d = [-0.6,\ 0.4]^T;$$

These matrices were modified from the originals for demonstration purposes. Thus, the base case servo *AOS* ($d_1$ = 0) is given by a straight line ($y_2 = a_{21}y_1/a_{11}$). Changes in the disturbance value ($-1 \leq d_1 \leq 1$) shift *AOS* ($d_1$ = 0) along the $G_d$ direction generating the *AOS*. This is seen in Figure 1, where we have also sketched the *DOS*, or desired set of output constraints, and the *AOIS* calculated for this case.

If the two outputs are controlled within some desired constraints that are wider than or equal to the *AOIS* sizes, the process will be interval operable within the available input ranges and in the presence of the expected disturbances (Lima and Georgakis, 2006). In other words, the system will be interval operable if *DOS* covers *AOIS* completely. Therefore, the Interval Operability Index (*IOI*) is defined as:

$$IOI = \frac{\mu(DOS \cap AOIS)}{\mu(AOIS)} \qquad (3)$$

Here $\mu$ represents a measure of the size of the set, e.g. area for 2-D and volume for 3-D examples, and hyper-volume for sets with higher dimensionality. This index quantifies how much of the region of the operable range of the outputs can be achieved using the desired set of output constraints. This index has a value between 0 and 1. A process is considered interval operable if the index is equal to 1. In this calculation, mathematical operations involving intersections of polytopes have to be performed to evaluate intersections like $DOS \cap AOIS$. This is done by using the Geometric Bounding Toolbox (GBT) in MATLAB (Mathworks[TM], Inc) developed by Veres *et al.* (1996).

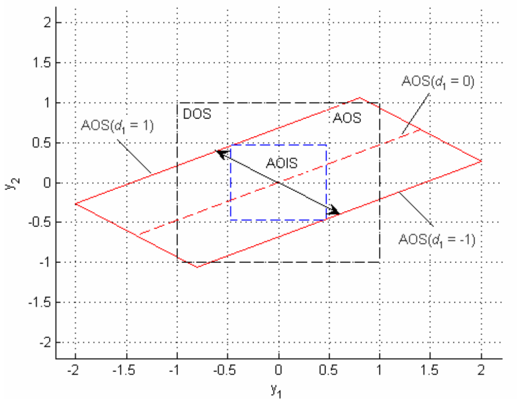


Figure 1: *AOS*, *DOS* and *AOIS*

Because the *DOS* is large enough to cover the *AOIS*, this system is interval operable. In fact, the *DOS* could still be significantly reduced to achieve tighter control of the outputs, as long as the *AOIS* remains a subset of the *DOS*.

Changing the relative output variable weights influences the aspect ratio of *AOIS* because the weights represent the tightness within which each output will be controlled around its operating point. The following relationship between the two parameters is observed for two output variables:

$$r_{12} = \frac{\Delta y_1}{\Delta y_2} = \frac{w_2}{w_1} \qquad (4)$$

Where $w_i$ and $\Delta y_i$ are the weight and the *AOIS* range, respectively, associated with output *i*. Moreover, $r_{ij}$ is the *AOIS* aspect ratio associated with the ranges between outputs *i* and *j*. Figure 2 shows an example where $r_{12}$ = 1:10 for the system above.

As the relative weight between the outputs increases, the problem approaches a set-point control problem for the variable with the greatest weight. This occurs because a set-point control problem is equivalent to a very tight interval control problem. Figure 3 illustrates this fact where the aspect ratio considered is 1000:1. The calculation of the output constraints for the interval controlled variable can then be performed based on the set-point value of the set-point controlled variable.

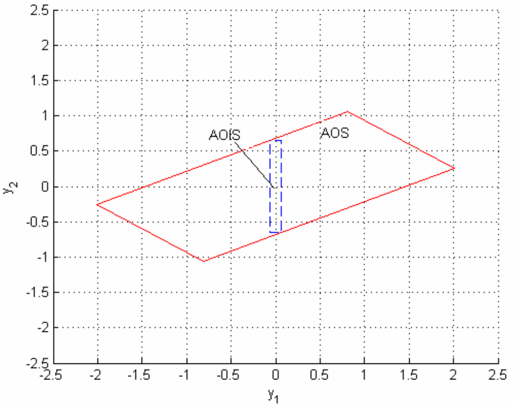


Figure 2: *AOIS* calculated using $r_{12}$= 1:10

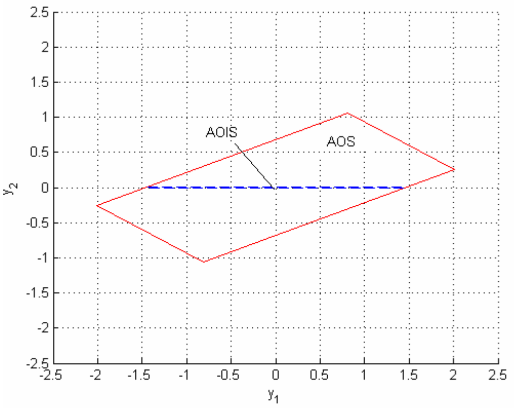


Figure 3: *AOIS* calculated using $r_{12}$= 1000:1

Now, if one wants to control the process around a steady state different from the origin using the same input window available and at the same expected disturbance range, the *AOS* will remain the same but the coordinates of the *AOIS* will be shifted, characterizing an asymmetric problem. Figure 4 shows an example of the nominal steady-state of the process ($y_0$) being shifted from the origin (0, 0) to (0.5, 0.5) considering $r_{12}$ = 4:1. Observe that the *AOIS* is the smallest rectangle that touches the lines associated with the minimum and maximum disturbance values of *AOS* for the specified aspect ratio. Notice that the *AOIS* does not need to be completely bounded within the *AOS* because we are addressing ($y_1$, $y_2$) interval operability.

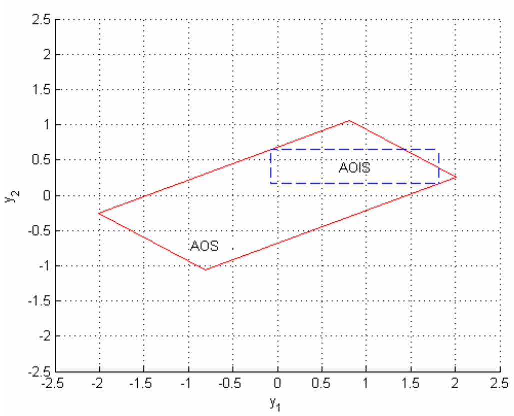


Figure 4: *AOIS* calculated for $y_0$ = (0.5, 0.5) and $r_{12}$= 4:1

## 4. ITERATIVE METHODOLOGY FOR HIGH-ORDER SYSTEMS

For problems with dimensionality higher than 3-D we are unable to geometrically represent the sets; consequently, the problem is addressed by algebraic calculations. This is performed using computational geometry tools (GBT), such as the calculation of convex hulls and intersections.

The transformation of the *AIS* and *EDS* necessary to calculate the corresponding *AOS* is easily done in any dimension using the linear process model at the disposal of the MPC controller. This is a direct mapping task. Concerning the *AOIS* calculation, the iterative approach is:

1) Define the relative weights $w_1$, $w_2$, ... $w_n$ that quantify the tightness with which each output will be controlled;

2) Specify small enough values for two scalar parameters $\alpha_0$ and $\Delta\alpha_0$. Initially, let $n$ = 0, $\alpha_n = \alpha_0$ and $\Delta\alpha_n = \Delta\alpha_0$. The initial approximation of the *AOIS* is a high dimensional parallelepiped around the steady-state value of the outputs ($y_0$) and is defined as a function of the scalar α representing its size by the following set of inequalities (5):

$$AOIS(\alpha) = \{\mathbf{y} \mid \mathbf{b_1} \le \mathbf{y} - \mathbf{y}_0 \le \mathbf{b_2}\} \quad (5)$$

$$\mathbf{b_1} = \left[\frac{-\alpha}{w_1}\ \frac{-\alpha}{w_2} \cdots \frac{-\alpha}{w_n}\right]^T;\ \mathbf{b_2} = \left[\frac{\alpha}{w_1}\ \frac{\alpha}{w_2} \cdots \frac{\alpha}{w_n}\right]^T;$$
$$\mathbf{y}_0 = [y_{01}\ y_{02} \cdots y_{0n}]^T;\ \mathbf{y} = [y_1\ y_2 \cdots y_n]^T;$$

Where $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{y}_0$ and $\mathbf{y}$ are column vectors in $\mathbb{R}^n$.

3) Test whether there is an intersection between the $AOIS(\alpha_n)$ and the $AOS$ associated with one of the extreme disturbance values, say $d_1$ = 1. If not, increase $\alpha_n$ by $\Delta\alpha_n$ and test for the intersection again. Keep increasing $\alpha_n$ until one of the $AOIS(\alpha_n)$ vertices touches $AOS$ ($d_1 = 1$);

4) Denote the respective value of $\alpha_n$ with $\alpha_{+1}$. Retain the coordinates of the $AOIS(\alpha_{+1})$ vertex obtained and denote it with $v_{+1}$;

5) Repeat the procedure for the other extreme disturbance value (say $d_1$ = -1). Denote the obtained value of $\alpha_n$ with $\alpha_{-1}$. Retain the $AOIS(\alpha_{-1})$ vertex obtained as well and denote it with $v_{-1}$;

6) The final *AOIS* will be the smallest orthogonal parallelepiped that has $v_{+1}$ and $v_{-1}$ in one of its diagonals ($AOIS = OP(v_{+1}, v_{-1})$). This will result in the minimum calculated set of output constraints that makes the process operable for all the disturbance values inside the *EDS*;

In order to find where the $AOIS(\alpha)$, a subset of $\mathbb{R}^n$, touches one of the $AOS(d_1 = \pm 1)$, the intersection of these two sets is calculated during each program iteration using the subroutine `intersct` from GBT (Veres *et al.*, 1996). However, this sub-routine requires both sets to be full dimensional. As explained previously, the two sets $AOS(d_1 = \pm 1)$, for the two limiting values of the disturbance, are $m$-dimensional objects in $\mathbb{R}^n$. To increase the dimensionality of $AOS(d_1 = \pm 1)$ from $m$ to $n$ we translate all the points of $AOS(d_1 = \pm 1)$ by $\delta\mathbf{e}_1, \delta\mathbf{e}_2, \ldots$ and $\delta\mathbf{e}_n$. Here $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$ are the usual basis in $\mathbb{R}^n$ and $\delta$ is chosen to be small enough ($\delta \approx 10^{-4}$) to not substantially change the characteristics of the respective $AOS(d_1)$. The convex hull of all points (original and translated) calculated for each set by using the subroutine `convh` define two polytopes in $\mathbb{R}^n$. The resulting polytopes can be used in the calculation of the intersections.

If two sequential values of $\alpha$ do not yield an intersection set, the increment $\Delta\alpha$ is doubled. Then the intersection measure (e.g. hyper-volume) is calculated and the value of $\alpha$ is reduced using the Secant Method (Rice and Do, 1994) until the intersection measure is reasonably small, yielding a satisfactory approximation of the intersection vertex (v).

Finally, it is worth mentioning that the calculations using `convh` and `intersct` for high dimensional objects can be computationally expensive and unstable, especially as the problem dimensionality increases. The iterative algorithm itself can be expensive as well. Alternative solutions to deal with these problems are under development, aiming at faster and more stable approaches for online implementation.

### 4.1. *High-Order Examples*

Steam Methane Reformer: In order to demonstrate the effectiveness of the proposed methodology, a higher-order system from the SMR process is addressed:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 14.07 & -0.04 & 2.66 \\ -3.92 & 0 & -1.96 \\ -7.74 & 6.03 & 0 \\ 6.60 & -3.90 & 4.89 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} 2.96 \\ -2.18 \\ 0 \\ 5.44 \end{pmatrix} d_1 \quad (6)$$

$$DOS = \{\mathbf{y} \in \Re^4 \mid \|\mathbf{y}\|_\infty \le 1\},\ AIS = \{\mathbf{u} \in \Re^3 \mid \|\mathbf{u}\|_\infty \le 1\},$$
$$EDS = \{d_1 \mid -1 \le d_1 \le 1\}$$

A set of calculated *AOIS* dimensions (low and high limits) considering $y_0$ = (0.1, 0, -0.15, 0) and output weights ($w$) = (1, 1, 1, 1) is shown in Table 1. The average computational time of 10 repeated simulations was 17.52 seconds (Dell PC with a 3.0-GHz Intel Pentium 4 processor).

Table 1: Results for SMR example

| *CV* | *Designed Low Limit* | *Designed High Limit* |
|---|---|---|
| $y_1$ | -0.2516 | 0.3285 |
| $y_2$ | -0.2285 | 0.3516 |
| $y_3$ | -0.3785 | 0.2016 |
| $y_4$ | -0.2285 | 0.3516 |

Dryer Control Problem: Consider now a 4-D system from the Dryer Control Problem provided by DuPont, described by the system of equations and sets:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5.85 & 0.22 \\ 1.52 & 0 \\ -40.86 & 1.56 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 0 \\ -0.63 \\ -0.17 \end{pmatrix} d_1 \quad (7)$$

$$DOS = \left\{ \mathbf{y} \in \Re^4 \mid \|\mathbf{y}\|_\infty \le 1 \right\}, AIS = \left\{ \mathbf{u} \in \Re^2 \mid \|\mathbf{u}\|_\infty \le 1 \right\}$$
$$EDS = \{ d_1 \mid -1 \le d_1 \le 1 \}$$

In this application, $n > m + 1$, $y_0 = (0, 0, 0, 0)$ and $w = (2, 2, 1, 1)$. The set of designed output constraints is presented in Table 2. Notice that the $y_1$ and $y_2$ bounds are twice as tight as the others. The average computational time in this case was 13.00 seconds.

Table 2: Drier Control Problem results

| *CV* | *Designed Low Limit* | *Designed High Limit* |
|---|---|---|
| $y_1$ | -0.0857 | 0.0857 |
| $y_2$ | -0.0857 | 0.0857 |
| $y_3$ | -0.1714 | 0.1714 |
| $y_4$ | -0.1714 | 0.1714 |

## 5. CONCLUSIONS

We have presented here an extension of the non-square Operability framework to high-order systems. A 2-D example has been used to demonstrate how different weights on the tightness of the desired control of each output can be accommodated. This includes the case where some of the outputs need to be controlled at set-points. These calculations can also be done for the case where the steady-state of the process is different from the origin.

Concerning the extension to high-order systems, we have employed computational geometry tools to calculate the tightest possible operable set of output constraints (*AOIS*) in $\mathbb{R}^n$, where $n$ is the number of outputs, using the Geometric Bounding Toolbox (GBT) in MATLAB (Veres *et al.*, 1996). This is achieved by starting with a small enough estimate of the *AOIS* and enlarging it until a non-zero intersection is obtained with the *AOS* sets associated with the extreme cases of the disturbance value. Our conceptual approach is not limited to the dimensionality of the input ($u$) or the output ($y$) vector, but only considers the case of a single disturbance variable.

Finally, we have successfully handled two industrial examples provided by Air Products and Chemicals and DuPont where the output-input dimensionality is 4 x 3 and 4 x 2 respectively. For higher-order problems, the proposed algorithm can be computationally expensive due to the calculation of convex hulls and intersections. Subsequent work of our group has resolved the computational limitations imposed by the algorithm presented here and larger size process examples have been thoroughly investigated.


## ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support from PRF-ACS through grant # 45400-AC9. We also wish to acknowledge David R. Vinson from Air Products and Chemicals and Phillip D. Schnelle and Julie Smith from DuPont for providing the process models.


## REFERENCES


Georgakis, C., D. Uzturk, S. Subramanian and D. R. Vinson (2003). On the operability of continuous processes. *Control Engineering Practice,* **11**, 859-869.

Hovd, M. (2004). *A brief introduction to Model Predictive Control*. Engineering Cybernetics Department, NTNU, 1-27.

Lima, F. and C. Georgakis (2006). Operability of Multivariable Non-Square Systems. *ADCHEM Proceedings*, 989-994.

Rawlings, J. B. (2000). Tutorial Overview of Model Predictive Control. *IEEE Control Systems Magazine*, **20(3)**, 38-52.

Rice, R. G. and D. D. Do (1994). *Applied Mathematics and Modeling for Chemical Engineers*. John Wiley & Sons, New York.

Veres, S. M., A. V. Kuntsevich, I. Valyi, D. S. Wall and S. Hermsmeyer (1996). Geometric Bounding Toolbox for MATLAB™. Edgbaston, UK: The University of Birmingham.

Vinson, D. R. and C. Georgakis (1998). A new measure of process output controllability. In *DYCOPS-5, 5th IFAC Symposium on Dynamics and Control of Process Systems*; Georgakis, C., Ed.; Pergamon Press: Elmsford, NY, 700-709.

Vinson, D. R. and C. Georgakis (2000). A new measure of process output controllability. *Journal of Process Control,* **10**, 185–194.

Vinson, D. R. (2000). *A new measure of process operability for the improved steady-state design of chemical processes*. Ph.D. thesis, Lehigh University, USA.