ON-LINE OPTIMIZATION VIA OFF-LINE PARAMETRIC OPTIMIZATION! - A GUIDED TOUR TO PARAMETRIC PROGRAMMING AND CONTROL

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Abstract: In this paper model predictive control problems are reformulated as multi-parametric programs. The optimal value of control variables is obtained as an explicit function of the state variables. This reduces on-line optimization to simple function evaluations that need simple computational hardware.

Keywords: Real-time optimization, model predictive control, multi-parametric programming, multi-parametric dynamic optimization, hybrid control.

1. INTRODUCTION

1.1 On-line Optimization

Model Predictive Control (MPC) (Morari and Lee, 1999) has been widely adopted by industry to address on-line optimization problems with input and output constraints. MPC is based on the so called *receding horizon* philosophy: a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the system until new measurements are available. Then, a new sequence is determined which replaces the previous one (see Figure 1). Each sequence is evaluated by means of an optimization procedure which takes into account two objectives: optimize the tracking performance, and protect the system from possible constraint violations.

While the benefits of on-line optimization are tremendous, its application is rather restricted, considering its profit potential, primarily due to its large "on-line" computational requirements which involve a repetitive solution of an optimization problem at regular time intervals (see Figure 2). This limitation is in spite of the significant advances in the computational power of the modern computers and in the area of on-line optimization over the past many years. Thus, it is fair to state that an efficient implementation of on-line optimization tools relies on a quick and repetitive on-line computation of optimal control actions.

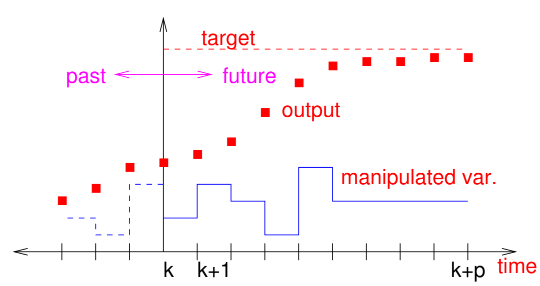

Fig. 1. Model Predictive Control

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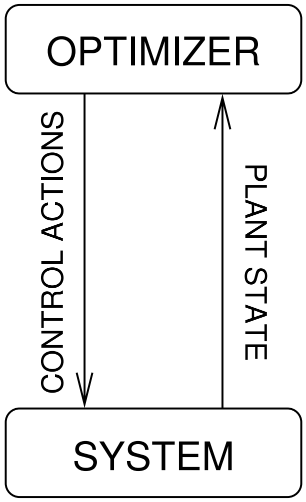


Fig. 2. On-line optimization

1.2 Multiparametric Programming

A parametric programming approach which avoids this repetitive solution is presented. In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters in the optimization problem are uncertain, parametric programming is a technique for obtaining (i) the objective function and the optimization variables as a function of these parameters and (ii) the regions in the space of the parameters where these functions are valid - see Figure 3. The main advantage of using the parametric programming techniques to address such problems is that for problems pertaining to plant operations, such as for process planning (Pistikopoulos and Dua, 1998) and scheduling, one obtains a complete map of all the optimal solutions and as the operating conditions fluctuate, one does not have to re-optimize for the new set of conditions since the optimal solution as a function of parameters (or the new set of conditions) is already available.

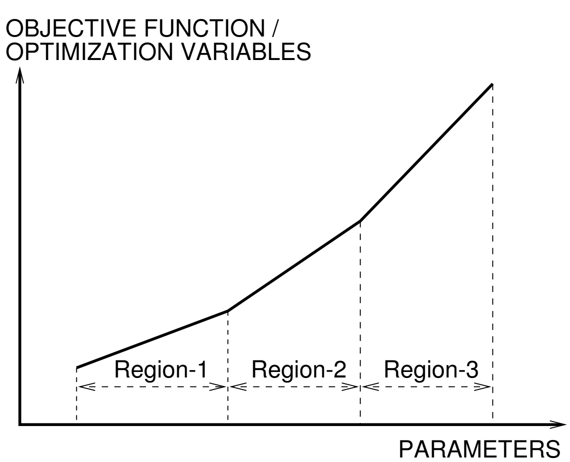


Fig. 3. Parametric Programming

By using the theory of parametric programming the control variables are obtained as a function of the state variables, and therefore on-line optimization reduces to simple function evaluations, at regular time intervals, for the given state of the plant - to compute the corresponding control actions. This results in a very small computational effort in comparison to repetitively solving an optimization problem.

The rest of the paper is structured as follows. In the next section an overview of parametric programming is provided. In section 3.1 a discrete time formulation of the on-line optimization problem is considered. While section 3.2 is concerned with continuous-time formulation, section 3.4 discusses the case when dynamics and logical decisions are simultaneously present in the problem. Concluding remarks are given in section 4.

2. PARAMETRIC PROGRAMMING AT A GLANCE

Consider the following multiparametric program:

$$\begin{aligned} & z(\theta) = \min_x \ f(x,\theta) \\ \text{s.t.} \quad & g_i(x,\theta) \leq 0, \quad \forall \ i = 1,...,p \\ & h_j(x,\theta) = 0, \quad \forall \ j = 1,...,q \\ & x \in X \subseteq \Re^n \\ & \theta \in \Theta \subseteq \Re^m \end{aligned} \tag{1}$$

Note that in an optimization framework f is the performance criterion to be minimized, $g \leq 0$ and $h = 0$ are the constraints, x is the vector of optimization variables and θ is the vector of parameters. The objective is to obtain the optimal $x(\theta)$, which when substituted into $f(x,\theta)$ provides the optimal objective function value, $z(\theta)$, as a function of θ. See Figure 4, where $x(\theta)$ is plotted as a function of θ - the region where a particular functional relationship between $x(\theta)$ and θ holds is known as a Critical Region (CR). Note that in the figure there are three CRs and $x(\theta)$ was obtained as a function of θ by solving only three optimization problems and therefore solving optimization problems for each and every value of θ has been avoided.

Depending upon whether f, g and h are linear, quadratic, nonlinear, convex, differentiable, or not, and also whether x is vector of continuous or mixed -continuous and integer- variables, the nature of the profiles and CRs shown in figure 4 changes (Dua and Pistikopoulos, 2000; Dua *et al.*, 2002; Dua and Pistikopoulos, 1999; Dua *et al.*, 2003; Sakizlis *et al.*, 2002*b*). For example, when f and g are convex and continuously differentiable and h is affine $z(\theta)$ is continuous and convex. Recently algorithms for the case when

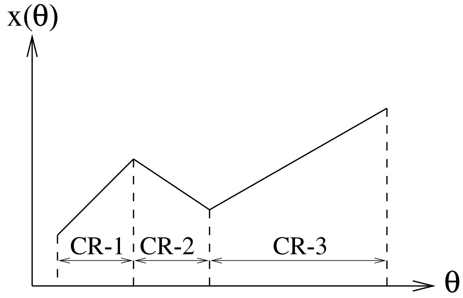


Fig. 4. Parametric Optimization

(1) involves (i) differential and algebraic equations (Sakizlis *et al.*, 2002*a*) and (ii) uncertain parameters (Sakizlis *et al.*, 2004) have been proposed.

3. PARAMETRIC CONTROL AT A GLANCE

3.1 Discrete Time Formulation

Consider the following state-space representation of a given process model (Pistikopoulos *et al.*, 2002):

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{2}$$

subject to the following constraints:

$$\begin{aligned} y_{min} \le y(t) \le y_{max} \\ u_{min} \le u(t) \le u_{max}, \end{aligned} \tag{3}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, and $y(t) \in \Re^p$ are the state, input, and output vectors respectively, subscripts min and max denote lower and upper bounds respectively and (A, B) is stabilizable. Model Predictive Control (MPC) problems for regulating to the origin can then be posed as the following optimization problems:

$$\begin{aligned} \min_U \ & J(U, x(t)) = x'_{t+N_y|t} P x_{t+N_y|t} \\ & + \sum_{k=0}^{N_y-1} x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k} \\ \text{s.t.} \ & y_{min} \le y_{t+k|t} \le y_{max},\ k = 1, \ldots, N_c \\ & u_{min} \le u_{t+k} \le u_{max},\ k = 0, 1, \ldots, N_c \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k},\ k \ge 0 \\ & y_{t+k|t} = Cx_{t+k|t},\ k \ge 0 \\ & u_{t+k} = Kx_{t+k|t},\ N_u \le k \le N_y \end{aligned} \tag{4}$$

where $U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}$, $Q = Q' \succeq 0$, $R = R' \succ 0$, $P \succeq 0$, $N_y \ge N_u$, and K is some feedback gain. The problem (4) is solved repetitively at each time t for the current measurement $x(t)$ and the vector of predicted state variables, $x_{t+1|t}, \ldots, x_{t+k|t}$ at time $t+1, \ldots, t+k$ respectively and corresponding control actions $u_t, \ldots, u_{t+k-1}$ is obtained.

In the following paragraphs, a parametric programming approach which avoids a repetitive solution of (4) is presented. First, we do some algebraic manipulations to recast (4) in a form suitable for using and developing some new parametric programming concepts. By making the following substitution in (4):

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j} \tag{5}$$

the objective $J(U, x(t))$ can be formulated as the following Quadratic Programming (QP) problem:

$$\begin{aligned} \min_U \ & \frac{1}{2} U' H U + x'(t) F U + \frac{1}{2} x'(t) Y x(t) \\ \text{s.t.} \ & GU \le W + Ex(t) \end{aligned} \tag{6}$$

where $U \triangleq [u'_t, \ldots, u'_{t+N_u-1}]' \in \Re^s$, $s \triangleq mN_u$, is the vector of optimization variables, $H = H' \succ 0$, and H, F, Y, G, W, E are obtained from Q, R and (4)–(5). The QP problem (6) can now be formulated as the following Multi-parametric Quadratic Program (mp-QP):

$$\begin{aligned} \mu(x) = \min_z \ & \frac{1}{2} z' H z \\ \text{s.t.} \ & Gz \le W + Sx(t), \end{aligned} \tag{7}$$

where $z \triangleq U + H^{-1} F' x(t), z \in \Re^s$, represents the vector of optimization variables, $S \triangleq E + GH^{-1}F'$ and x represents the vector of parameters. The main advantage of writing (4) in the form given in (7) is that z (and therefore U) can be obtained as an affine function of x for the complete feasible space of x. To derive these results, we first state the following theorem.

Theorem 1. For the problem in (7) let x_0 be a vector of parameter values and (z_0, λ_0) a KKT pair, where $\lambda_0 = \lambda(x_0)$ is a vector of nonnegative Lagrange multipliers, λ, and $z_0 = z(x_0)$ is feasible in (7). Also assume that (i) linear independence constraint qualification and (ii) strict complementary slackness conditions hold. Then,

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1} N_0 (x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix} \tag{8}$$

where,

$$M_0 = \begin{pmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_p G_q & & & -V_q \end{pmatrix}$$

$$N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_p S_p)^T$$

where G_i denotes the i^{th} row of G, S_i denotes the i^{th} row of S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i denotes the i^{th} row of W and Y is a null matrix of dimension $(s \times n)$.

The space of x where this solution, (8), remains optimal is defined as the Critical Region (CR^0) and can be obtained as follows. Let CR^R represent the set of inequalities obtained (i) by substituting $z(x)$ into the inequalities in (7) and (ii) from the positivity of the Lagrange multipliers, as follows:

$$CR^R = \{Gz(x) \le W + Sx(t), \lambda(x) \ge 0\}, \quad (9)$$

then CR^0 is obtained by removing the redundant constraints from CR^R as follows:

$$CR^0 = \Delta\{CR^R\}, \quad (10)$$

where Δ is an operator which removes the redundant constraints - for a procedure to identify the redundant constraints, see Gal (1995). Since for a given space of state-variables, X, so far we have characterized only a subset of X i.e. $CR^0 \subseteq X$, in the next step the rest of the region CR^{rest}, is obtained as follows (Dua and Pistikopoulos, 2000):

$$CR^{rest} = X - CR^0. \quad (11)$$

The above steps, (8–11) are repeated and a set of $z(x), \lambda(x)$ and corresponding CR^0s is obtained. The solution procedure terminates when no more regions can be obtained, i.e. when $CR^{rest} = \emptyset$. For the regions which have the same solution and can be unified to give a convex region, such a unification is performed and a compact representation is obtained. The continuity and convexity properties of the optimal solution are summarized in the next theorem.

Theorem 2. For the mp-QP problem, (7), the set of feasible parameters $X_f \subseteq X$ is convex, the optimal solution, $z(x) : X_f \mapsto \Re^s$ is continuous and piecewise affine, and the optimal objective function $\mu(x) : X_f \mapsto \Re$ is continuous, convex and piecewise quadratic.

Based upon the above theoretical developments, an algorithm for the solution of an mp-QP of the form given in (7) to calculate U as an affine function of x and characterize X by a set of polyhedral regions, CRs, has been developed which is summarized in Table 1.

This approach provides a significant advancement in the solution and on-line implementation of MPC problems. Since its application results in a complete set of control actions as a function of state-variables (from (8)) and the corresponding regions of validity (from (10)), which are computed off-line. Therefore during on-line optimization, no optimizer needs to be called and instead for the current state of the plant, the region, CR^0, where the value of the state variables is valid, can be identified by substituting the value of these state variables into the inequalities which define the regions. Then, the corresponding control actions can be computed by using a function evaluation of the corresponding affine function (see Figure 5).

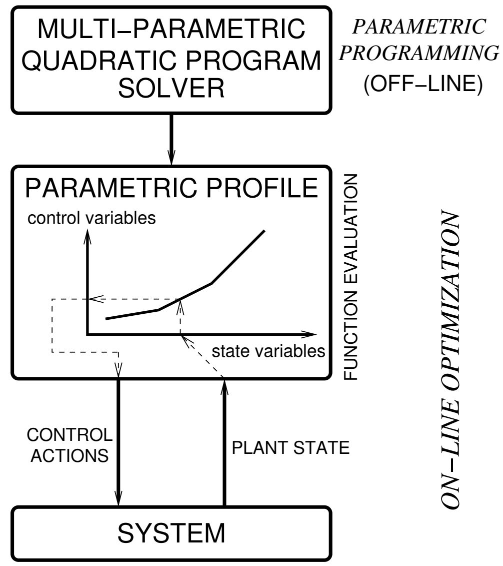


Fig. 5. On-line optimization via parametric programming

Table 1. Solution Steps of the mp-QP Algorithm

Step 1 For a given space of x solve (7) by treating x as a free variable and obtain $[x_0]$.
Step 2 In (7) fix $x = x_0$ and solve (7) to obtain $[z_0, \lambda_0]$.
Step 3 Obtain $[z(x), \lambda(x)]$ from (8).
Step 4 Define CR^R as given in (9).
Step 5 From CR^R remove redundant inequalities and define the region of optimality CR^0 as given in (10).

Step 6 Define the rest of the region, CR^{rest}, as given in (11).
Step 7 If no more regions to explore, go to the next step, otherwise go to Step 1.
Step 8 Collect all the solutions and unify a convex combination of the regions having the same solution to obtain a compact representation.

Figure 6 demonstrates how advanced controllers can be implemented on a simple hardware.

3.2 Continuous Time Formulation

Consider the following optimal control problem (Sakizlis, 2003):

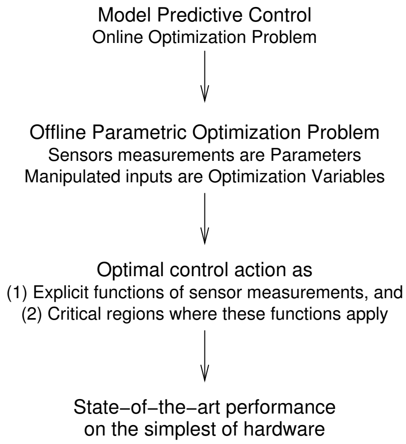


Fig. 6. Achieving state-of-the-art control performance on simple hardware

$$\hat{\phi} = \min_{x(t),v(t)} \frac{1}{2} x(t_f)^T P_1 x(t_f) + \frac{1}{2} \int_{t_o}^{t_f} [x(t)^T Q_1 x(t) + v(t)^T R_1 v(t)] dt$$

s.t.

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 v(t) \\ 0 &\geq \psi^g(x_f) = D_1 \cdot x(t_f) + b_2 \\ 0 &\geq g(x,v) = C_1 \cdot x(t) + C_2 \cdot v(t) + b_1 \\ x(t_o) &= x_o, \quad t_o \leq t \leq t_f \end{aligned} \tag{12}$$

where $x \in X \subseteq \Re^n$ are the states and $v \in U \subseteq \Re^{n_v}$ are the control manipulating inputs. Consider: $g : \Re^n \times \Re^m \times \Re^{n_v} \mapsto \Re^q$ and $\psi^g : \Re^n \times \Re^m \times \Re^{n_v} \mapsto \Re^{Q_g}$. The pair (A_1, A_2) is assumed to be stabilizable and the pair (A_1, B_1) detectable. Matrices $Q, P \succeq 0$, $R \succ 0$ constitute the quadratic performance index.

By treating the initial state conditions as parameters, problem (12) is recast as a multiparametric dynamic optimization problem (mp-DO). The solution this mp-DO problem is given by a set of expressions for the *optimal* value of the performance index and the *optimal* profiles of the control inputs and the states as a function of the initial conditions. This parametric controller has the following benefits comparing to the parametric controller for discrete-time dynamic systems: (A) The constraints are satisfied over the complete time horizon rather than at discrete time points. (B) The complexity of the control law derivation is contingent solely upon (i) the number of constraints, (ii) the system dynamics and (iii) the number of control variables.

3.3 2-state SISO illustrative example

Consider the open-loop unstable SISO plant:

$$\Sigma(s) = \frac{0.003396(s + 0.8575)}{(s-1)(s-0.6313)} \tag{13}$$

subject to the path constraint:

$$\frac{1.5s + 1}{s^2 - 1.63135s + 0.6313} \leq 2.4 \tag{14}$$

A receding horizon optimal control problem is formulated as in (12) where the terminal cost P_1 is evaluated from the solution of the Riccati equation. The solution of the continuous time formulation presented in section 3.2 results in the state space partition shown in Figure 7 where the mathematical functions of the region boundaries and the comparison with the partition of the discrete-time parametric controller (parco) is also displayed.

The control policy in critical region $CR01$ is given by the function:

$$\begin{aligned} \hat{v}(t) = &= -[1.24\, e^{-10.76\, t + 10.76\, t_o} (1.0912\, x_{o1} + 0.93\, x_{o2}) \\ &-0.21\, e^{-0.8570\, t + 0.857\, t_o} (0.1329\, x_{o1} + 1.4309\, x_{o2})]; \end{aligned}$$

The execution of the derived control law is also shown in Figure 7, where the system is steered to the origin starting from a perturbed point of $x = [7 \quad -13]^T$. It is clear from Figure 7 that the discrete-time parco described in section 3.1 results in a larger number of critical polyhedral regions comparing to the continuous-time parco described in section 3.2 that generates only three regions with non-linear boundaries.

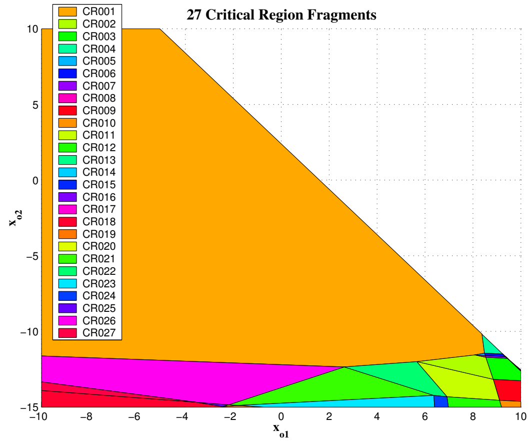
3.4 Hybrid Systems

The mathematical representation of a process system that operates in a transient mode and is subject to constraints and logical rules is considered as follows (Sakizlis *et al.*, 2002*a*):

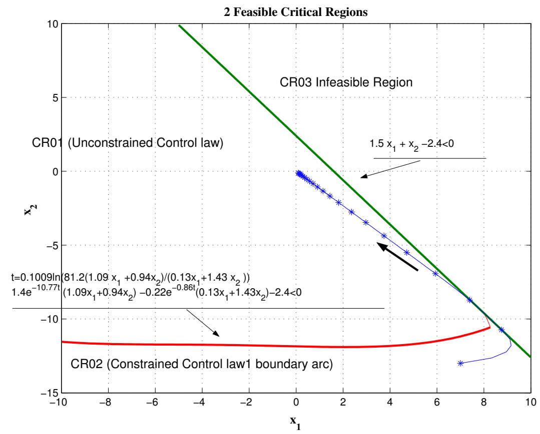
$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 v(t) + A_3 \delta(t) \\ y(t) &= B_1 x(t) + B_2 v(t) + B_3 \delta(t) \\ C_o y(t) + C_1 x(t) &+ C_2 v(t) + C_3 \delta(t) \leq 0 \end{aligned} \tag{15}$$

where $x \in \Re^n, y \in \Re^m$ are the states and the measurements of the dynamic system; $v \in \Re^q$ are the control manipulated inputs; $\delta \in \Delta \equiv \{0,1\}^b$ are binary variables that represent discrete decisions and logical conditions about the system operation.

The hybrid control problem is then formulated as follows:



(a) Critical Regions for discrete-time parco



(b) Critical Regions for continuous-time parco

Fig. 7. Comparison of state space partition of the discrete-time and continuous time parametric controller for the 2-state SISO example

$$\hat{\phi} = \min_{x,v,\delta} \frac{1}{2} x(t_f)^T P_1 x(t_f) + ||P_2 x(t_f)|| + ||S\delta(t)||$$
$$+ \int_{t_o}^{t_f} [\frac{1}{2} x(t)^T Q_1 x(t) + ||Q_2 x(t)||$$
$$+ \frac{1}{2} v(t)^T R_1 v(t) + ||R_2 v(t)||] dt$$
$$s.t. \quad \dot{x}(t) = A_1 x(t) + A_2 v(t) + A_3 \delta(t)$$
$$y(t) = B_1 x(t) + B_2 v(t) + B_3 \delta(t) \qquad (16)$$
$$\text{I. C.: } x(t_o) = x_o(t^*)$$
$$C_o y(t) + C_1 x(t) + C_2 v(t) + C_3 \delta(t) \leq 0$$
$$t_o \leq t \leq t_f$$

where terms $\int (1/2) \cdot x(t)^T Q_1 x(t) dt$, $(1/2) \cdot x(t_f)^T P_1 x(t_f)$ and $\int (1/2) v(t)^T R_1 v(t) dt$ correspond to the 2− norm of the state and output deviations respectively; terms $P_2 x(t_f), \int Q_2 x(t), \int R_2 v(t) dt$ represent either infinity $||Qx(t)||_\infty, ||Rv(t)||_\infty$ or 1− $||Qx(t)||_1, ||Rv(t)||_1$ norms.

After carrying out the appropriate substitutions and manipulations the optimization problem (16) is transformed to an equivalent finite dimensional problem of the following form:

$$\hat{\phi} = \min \frac{1}{2} \{L_1 + L_2 u + L_3 x_o + L_4 \zeta + u^T L_5 u$$
$$+ x_o^T L_6 u + x_o^T L_7 x_o + x_o^T L_8 \zeta + u^T L_9 \zeta + \zeta^T L_{10} \zeta\}$$
$$G_1 u + G_2 \zeta \leq G_3 + G_4 x_o \qquad (17)$$
$$u \in \Re^{q \cdot N_q \cdot M} \quad x_o \in X_o \subset \Re^n$$

where $\delta_l = \zeta_{li} \forall t \in [t_{i-1}, t_i]$, $i = 0, 1, ...N_t$, $l = 1, ...b, \zeta \in \{0,1\}^{b \cdot N_t}$. After removing nonlinearities from (17) by using appropriate algbraic manipulations, the problem becomes the following multi-parametric Mixed-Integer Quadratic Program (mp-MIQP):

$$\hat{\phi} = \min \frac{1}{2} \{\bar{L}_1 + \bar{L}_2 z + \bar{L}_3 x_o + \bar{L}_4 \zeta + z^T \bar{L}_5 z +$$
$$+ x_o^T \bar{L}_6 x_o\}$$
$$\bar{G}_1 z + \bar{G}_2 \zeta \leq \bar{G}_3 + \bar{G}_4 x_o \qquad (18)$$
$$z \in \Re^{q \cdot N_q \cdot M} \quad \zeta \in \{0,1\}^{b \cdot N_t} \quad x_o \in X_o \subset \Re^n$$

where $z = u + L_5^{-1} \cdot L_6^T \cdot x_o$. The procedure for solving (18) parametrically is based upon decomposing the mp-MIQP into a multi-parametric quadratic program (mp-QP) where the integers are fixed and an upper parametric bound is obtained, and a mixed-integer nonlinear program which treats the parameters x_o as free optimization variables and provides a new integer realization.

The final parametric solution provides the relation between the control and the current state of the system that ensures the optimum system regulation. This control law is proved to be (Theorem 1) piecewise affine with respect to the states and has the form:

$$u_c(x_o) = a_c \cdot x_o + b_c; \quad \zeta_c(x_o) = d_c$$
$$CR_c^1 \cdot x_o(t^*) + CR_c^2 \leq 0 \qquad (19)$$
$$\text{for } \; c = 1, ...N_c$$

where N_c is the number of regions in the state space. Matrices a_c, CR_c^1 and vectors b_c, CR_c^2, d_c are determined from the solution of the parametric programming problem and the index c designates that each region admits a different control law and different binary realizations.

4. CONCLUDING REMARKS

An overview of the parametric programming approach for the solution of on-line optimization problems has been presented. The on-line optimization problems for discrete and continuous time formulations and for hybrid systems were presented. Optimal value of the control variables is computed off-line as a function of the state variables or measurements. This is given by a set of control profiles on the space of state variables and the corresponding regions on the space of the state variables. On-line optimization is then carried out by taking measurements from the plant, identifying the region that is valid on the space of state variables and then calculating the control actions by simple function evaluations of the control profiles. The computational hardware requirements for such an implementation are very simple.

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