NONLINEAR OPTIMAL CONTROL USING DYNAMIC PROGRAMMING IN CELL SPACE — APPLICATION TO NONLINEAR CSTR

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Abstract: Optimal control problems for nonlinear systems are generally difficult to study. Closed-form solutions are often limited to linear systems and quadratic performance indices. This paper proposes a numerical approach to solve nonlinear optimal control problems using Bellman's principle of optimality in discrete time and space. The approach is based on Simple Cell Mapping (SCM), a numerical procedure to approximately describe nonlinear state-space dynamics. The paper discusses the construction of a general control database and performance index database along with a backward search algorithm to formulate optimal control policy. The cell space dynamic programming methodology is investigated on a nonlinear CSTR model. Numerical studies dealing with the influence of space and time discretization on computational feasibility are presented.

Keywords: Optimal control, nonlinear systems, cell mapping, dynamic programming.

1. INTRODUCTION

Optimal control of nonlinear dynamic systems has been an active research area for several decades. In general, the optimal control problem poses formidable difficulties, which are only compounded by nonlinearities and constraints placed on state and control variables. Since many chemical engineering systems fall under this category, they present challenging problems. For example, the Continuously Stirred Tank reactor (CSTR) is known to exhibit highly nonlinear behavior. Since process operations are under increasing demand to adapt and perform over a wide range of operating conditions, the limitations of traditional linear control theory are becoming more pronounced. As a result, research in nonlinear control theory has become one of the most important and relevant areas at present.

In theory, optimal control problems may be studied by analytical techniques such as calculus of variations, Pontryagin maximum principle, Bellman's principle of optimality and the Hamilton-Jacobi-Bellman equation (Lee, 1964; Lapidus and Luus, 1967). However, in the presence of strong nonlinearities and constraints, very few closed-form solutions are found. Alternatively, numerical techniques tend to be both cumbersome and computationally prohibitive (Kushner and Dupuis, 1992; Dupuis and James, 1998).

In this paper, we investigate the application of dynamic programming in discrete time and space to solve an optimal control problem for a CSTR. This numerical procedure is based on Bellman's principle of optimality in discrete domain. Discretization of the independent variable is commonly used for numerical procedures using nonlinear programming (Biegler, 1984; Logsdon and Biegler, 1989; Tieu *et al.*, 1995; Luus, 1994). This paper uses an approach based on the discretization of dependent variables also.

The foundation for the approach is provided by cell-to-cell mapping methods introduced by Hsu (1980). These methods have been primarily used for perform-

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ing global analysis of nonlinear systems. Several extensions to optimal control problems have been proposed since (Hsu, 1985; Crespo and Sun, 2002). These methods have yet to be investigated for process systems related to chemical engineering.

The idea of discretization of state and input variables is not new. Dynamic programming over discrete grids has been suggested by Bellman and Kalaba (1965) and Lapidus and Luus (1967). However, Bellman cited the "menace of expanding grid" and the "curse of dimensionality" as the causes of practical limitations, which continued to plague most numerical techniques. At present, the cell-mapping based methods offer a systematic and efficient methodology to explore nonlinear dynamics. Although the "menace" and the "curse" are still alive, they are rendered less potent due to the rapidly advancing computing power as well as cheap storage and fast data retrieval. At least for low dimensional systems, the cell mapping methodology offers a computationally feasible alternative.

The remainder of the paper is organized as follows. The concept of Simple Cell Mapping (SCP) is explained followed by a discussion of dynamic programming in cell space. The nonlinear CSTR is employed for a simulation example.

2. OPTIMAL CONTROL

A typical optimal control problem of dynamic systems is defined as follows: for a fixed time horizon $t \in [0, t_f]$, find the control signal policy $u(t)$ such that a performance index J, is maximized (or minimized),

$$J(x(0), t_f) = \phi(x(t_f)) + \int_0^{t_f} \psi(x(t), u(t))\, dt \qquad (1)$$

where ϕ is the arrival performance index and ψ is the incremental performance index. The optimization is typically subject to system equation in the form of a nonlinear ODE, initial condition and constraints as shown below,

$$\frac{dx}{dt} = f(x, u) \qquad (2)$$

$$x(0) = x_0 \qquad (3)$$

$$h(x, u) = 0 \qquad (4)$$

$$g(x, u) \geq 0 \qquad (5)$$

where x is the state vector and u is the control signal vector, both of which may be subjected to additional upper and lower bounds.

3. CELL-TO-CELL MAPPING

Consider the nonlinear dynamic system in Eq. (2). If the control signal is assumed constant, for a trajectory emanating from the initial condition $x(t)$, the arrival point after a time Δt is uniquely determined by

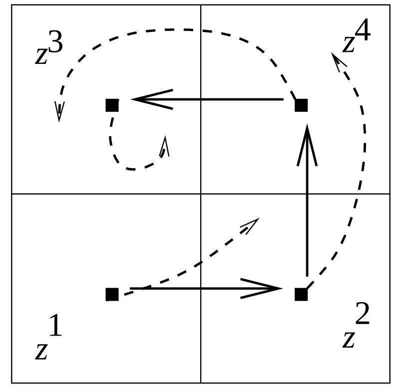

Fig. 1. Simple Cell Mapping (SCM)

$$x(t + \Delta t) = x(t) + \int_t^{t+\Delta t} f(x, u)\, d\tau \qquad (6)$$

In principle, the system equation may be integrated to define a point mapping in the state space for investigating the system dynamics parameterized by the mapping time step Δt. The discretization of time or the independent variable is commonly used for numerical procedures and also to reconcile with discretely observed systems. Eq. (6) may be written as a discrete system,

$$x_{k+1} = F(x_k, u_k, k) \qquad (7)$$

where $k = t/\Delta t$.

Consider a finite region $R \subset \mathbb{R}^n$, where the system dynamics, subject to constraints and bounds, are likely to be observed. Let R be partitioned into a collection of finite number of connected sets called cells, $\{z^i,\ i = 1, 2, \ldots, N\}$. State space outside the region of interest is a single infinite sized cell called the sink cell z_0. The continuous state space $\mathbb{R}^n$ is approximated by the discrete cell space $Z = \{z^i\}_{i=0}^N$, and $Z \to \mathbb{R}^n$ as $N \to \infty$. Bounds automatically define R and constraint surfaces further limit the likely regions of R. The inclusion of constraints and bounds on state variables actually helps in defining a smaller cell space than otherwise is necessary.

Cell-to-cell mapping considers the system evolution in discrete time as a mapping among the cells. State transitions from point to point described by (7) possess analogous cell transitions in cell space. Transitions from cells $\{z^i,\ i = 1, 2, \ldots, N\}$ into the sink cell z_0, are considered terminal. Simple Cell Mapping (SCM) uses a single point (usually the geometric center) of a cell to represent the entire volume of the cell (Hsu, 1980). The departure from a given cell z^j is evaluated by Eq. (7) to locate the arrival point in cell z^i, known as the image cell. The cell mapping function is defined as

$$z^i = C(z^j, \Delta t) \qquad (8)$$

which is Eq. (7) invoked at cell center.

Figure 1 illustrates the concept of SCM. The state space trajectories from four integrations from four cell centers are shown in dotted curves. The resulting cell trajectory is $z^1 \rightarrow z^2$, $z^2 \rightarrow z^4$, $z^4 \rightarrow z^3$ and $z^3 \rightarrow z^3$. In SCM terminology, a cell is classified as a *periodic* cell if it maps into itself in p mapping steps or $p \times \Delta t$ time period. For example, cell z^3 is a periodic cell with $p = 1$ containing a steady-state. Periodicity is used to identify steady-states and limit cycles in state space. The sink cell is a periodic cell with period equal to one.

Each group of periodic cells (called an attractor) has a set of *transient* cells, forming the domain of attraction, which map into the attractor within r mapping steps. For example cells z^1, z^2 and z^4 form the domain of attraction for cell z^3.

Consider the control vector u as a discrete representation $U = \{u^i\}_{i=1}^M$. Thus, the allowable control moves are limited to the set U and are executed as step functions at time intervals $k\Delta t$. A similar representation is used in Model Predictive Control (MPC).

For a constant u^i, the state space, partitioned into N cells, is perturbed for one mapping step by evaluating N integrations from N cell centers. The location of the end points of a trajectory assigns an image cell to each cell. The general control database Ω, consists of SCM implementations for all allowable controls in U, i.e., $N \times M$ mappings.

4. OPTIMAL CONTROL TABLE

The discrete version of the performance index is represented as follows,

$$J(x_0, H) = \phi(x_H) + \sum_{k=1}^{H} \psi(x_k, u_{k-1}) \tag{9}$$

where $H = t_f / \Delta t$. The terminal performance index ϕ is evaluated at the center of the cell containing $x(t_f)$ and the incremental performance index ψ is similarly evaluated at the centers of the respective cells and the corresponding control vectors.

The incremental performance index database Λ, contains the values of the incremental performance index for all possible cells and the allowable controls, which is the general control database Ω. The optimal control policy is achieved by maximizing the sum of the terminal performance index and the accumulative performance index, which is simply the sum of the incremental performance index at discrete times.

According to the principle of optimality (Bellman and Kalaba, 1965): "an optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The principle of optimality suggests that the search for minimizing or maximizing the performance index must proceed backwards in time from H to 1 (Lapidus and Luus, 1967). For instance, the optimal value of the performance index carried through H steps is

$$J^*(x_0, H) = \max \left\{ \phi(x_H) + \sum_{k=1}^{H} \psi(x_k, u_{k-1}) \right\} \tag{10}$$

which may also be written as

$$J^*(x_0, H) = \max \{\psi(x_1, u_0)\} + \max \left\{ \phi(x_H) + \sum_{k=2}^{H} \psi(x_k, u_{k-1}) \right\} \tag{11}$$

Thus, we reveal the fundamental recurrent nature of dynamic programming,

$$J^*(x_0, H) = \max \{\psi(x_1, u_0) + J^*(x_1, H-1)\} \tag{12}$$

Going backwards in time, if $H-1$ optimal controls have been determined, the next optimal control may be determined by considering the new step plus the already computed policy for the previous $H-1$ steps.

The backward search algorithm in cell space over the general control database Ω and the incremental performance index database Λ can be summarized now. Let S^* denote the cells representing the optimal target value of the state vector and V^* denote the corresponding set of optimal control vectors. At the last stage H, the set S^* contains the cell representing the value x_H.

Starting at step H, initialize the cumulative performance index $J^* = \phi(x_H)$ and an intermediate target set $S = S^*$,

(1) Locate the cells in the general control database Ω, which possess the elements of S as image cells.
(2) Look up the value of the incremental performance index from the performance index database Λ, for all the maps found.
(3) Compute the accumulative performance index for each map and locate the maximum to update J^*.
(4) Expand the set S^* and V^* with the respective cells corresponding to J^*.
(5) Reinitialize S with the newest entry of S^*.

The backward search yields the optimal control table for all cells in the cell space. For a particular case of x_0, the initial cell z_0 is identified and the corresponding J^* is identified and thus the optimal control u_0 and optimal cell z_1 are located from the table. Continuing forward, we eventually recover the optimal control policy V^*.

5. SIMULATION EXAMPLE

We consider the optimal control of a nonlinear CSTR. The dimensionless model is the following set of ODEs (Lapidus and Luus, 1967),

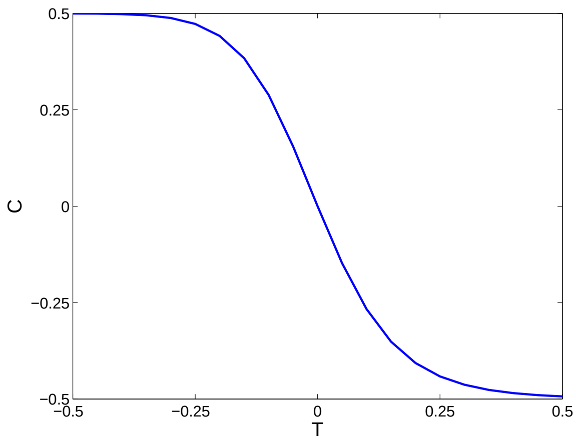

Fig. 2. Locus of steady-state points.

$$\frac{dT}{dt} = -(T+0.25)+(C+0.5)e^{\left(\frac{25T}{T+2}\right)} \\ -u(T+0.25) \quad (13)$$

$$\frac{dC}{dt} = 0.5 - C - (C+0.5)e^{\left(\frac{25T}{T+2}\right)} \quad (14)$$

where u is the control signal with a maximum of 2 (fully cooled) and a minimum of 0 (adiabatic), and T and C are deviations from steady-state dimensionless temperature and concentration respectively. The model does not exhibit steady-state multiplicity, Figure 2 shows the locus of steady-state points.

Under proportional control, the control signal is given by $u = 1+K_cT$, where K_c is the proportional gain. It is desired to find an optimal control policy so that when the reactor is started at $T = 0.05$ and $C = 0$, the origin $T = 0$ and $C = 0$ is reached in a fixed time horizon $t_f = 2$ at a minimal cost defined as follows,

$$J = \int_0^{t_f} T^2 + C^2 + u^2 \, dt \quad (15)$$

The state space $T \in [-0.1, 0.1]$ and $C \in [-0.1, 0.1]$ is discretized into 51 intervals on each dimension for a cell space Z, comprising 2,602 cells including the sink cell. A mapping time step of $\Delta t = 0.1$ is employed. The control signal is discretized into intervals of 0.5 over $u \in [0, 2]$ to define the allowable control set U comprising five elements.

The general control database Ω is computed using SCM. The mapping is performed by the fourth order Runge-Kutta integration scheme. The computation time required to perform 10,305 mappings is 2.5 secs on a Pentium 600 MHz machine using compiled code. The incremental performance index database is also computed in the same step. The storage requirement for the databases is approximately 256 KB. The backward search algorithm consumes 0.15 sec of CPU time. The computational demand and memory requirements are deemed fairly minimal for practical applications.

Figure 3 shows the results of the optimal control policy. The first two panels show the profiles of temperature and concentration respectively, while the third panel displays the control signal. The optimal trajectories of T and C in response to the control signal are shown in squares, which represent the optimal cell trajectories. Results of proportional control with $K_c = 17$ are also superimposed in solid curves. The optimal policy reaches the set point in nine steps.

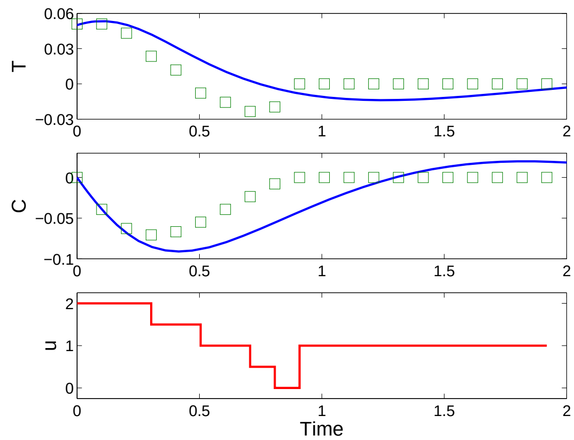

Fig. 3. Optimal control of CSTR along with proportional control.

6. CONCLUSIONS

A numerical procedure to solve optimal control problems for nonlinear systems using Bellman's principle of optimality in discrete time and space is presented. The point evolution of the system in state space is approximated as coarse grained dynamics of cell mappings in discretized cell space. The Simple Cell Mapping method is used to compile a general control database for all allowable controls in a discrete set. The approach bears close resemblance to state and control variable discretization schemes proposed in mid 1960s for dynamic programming. In contrast, the cell mapping approach avoids the difficulties of interpolations necessary therein. The accuracy of the solution in cell space is directly linked to the cell size and mapping time step size. Fine discretization is generally desired to minimize the approximation errors resulting from the center point method. However, it comes with added expense in computation and storage. Long mapping steps are detrimental since they may cause overshoots in the controlled response, which is a direct consequence of discrete control action. Application to a nonlinear CSTR shows promising results in terms of computational cost and controller performance.

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