Robust Predictive Control for NARX Models from Input-Output Data

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Abstract: This paper considers the problem of designing predictive control laws for nonlinear auto-regressive exogenous (NARX) models based on measured input-output data without explicitly identifying the model parameters. We explore the case when outputs are corrupted by additive measurement noise. An upper bound on the optimal value function for the robust case is derived, and the mismatch between the predicted and actual output is also theoretically studied. The recursive feasibility and practical stability of the robust data-driven predictive control (DDPC) scheme are guaranteed. The simulation results finally quantify the effectiveness of the proposed method with the experimental data gathered from a powder compaction process performed on a rotary tablet press.

Keywords: data-driven predictive control (DDPC), Willems' fundamental lemma, NARX systems, robust control, process control, powder compaction process.

1. INTRODUCTION

The nonlinear auto-regressive exogenous (NARX) representation is widely used to model and simulate different applications (Billings, 2013). Besides, due to the advancement of sensor technology and computational tools, data-driven schemes in systems and control have recently gained considerable attention, see, for example, Markovsky and Rapisarda (2008); De Persis and Tesi (2020); van Waarde et al. (2022); Coulson et al. (2019); Mishra et al. (2022); Berberich et al. (2021); Baros et al. (2022); Pan et al. (2022); Hiremath et al. (2022); Markovsky and Dörfler (2021); Yin et al. (2021); Mishra et al. (2023). In this paper, we design a data-driven predictive controller (DDPC) for NARX models in the presence of additive measurement noise. To tackle this problem, we invoke an extension (see, (Mishra et al., 2021, Theorem 1)) of a result from the behavioral system theory known as the fundamental lemma, which was developed by Willems and co-workers (Willems et al., 2005, Theorem 1). This result provides sufficient conditions when the space of all input-output trajectories of a deterministic linear time-invariant (LTI) system coincides with the column space of the Hankel matrix built from a persistently exciting (PE) measured input-output trajectory. With this general overview, we survey some of the previous research works to explain the motivations behind our proposed method and to present its main contributions with respect to the existing results.

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Recent attention has been directed towards the idea of designing control laws for unknown systems based on input-output data. The data-driven output matching problem for LTI systems has been solved in Markovsky and Rapisarda (2008). Data-driven stabilizing controllers for LTI systems have been synthesized in De Persis and Tesi (2020); van Waarde et al. (2022). Data-driven observers for LTI systems have been designed in Mishra et al. (2022); Turan and Ferrari-Trecate (2022). Data-driven stabilizer controllers are also derived in recent years for different classes of nonlinear systems (Bisoffi et al., 2020; Luppi et al., 2022; Guo et al., 2022; Strässer et al., 2021; De Persis et al., 2023). The fundamental lemma, however, was used in Coulson et al. (2019) to design predictive controllers. More precisely, an algorithm known as DeePC (data-enabled predictive control) was developed in Coulson et al. (2019) to solve a predictive control problem based on measured data. The aforementioned work received significant attention from researchers in different fields and has been extended in several directions. For example, theoretical guarantees related to recursive feasibility, closed-loop stability, and robustness for LTI systems have been studied in Berberich et al. (2021). Another version of DeePC has been proposed, called ODeePC (Online DeePC), where online measurements are adapted to capture the real-time behavior of time-varying systems in Baros et al. (2022). It has also been extended to stochastic systems by developing appropriate fundamental lemma-like results (Pan et al., 2022; Hiremath et al., 2022). Note that the fundamental lemma (Willems et al., 2005, Theorem 1) provides only sufficient conditions to parametrize the set of all input-output trajectories based on a single persistently exciting trajectory. However, recently, necessary and sufficient conditions for the fundamental lemma have been given in (Mishra and Markovsky, 2021, Theorem 4). For an overview, we refer to a recent survey Markovsky and Dörfler (2021).

Since the fundamental lemma is concerned with deterministic LTI systems, DeePC was basically developed for the same class of systems. However, the dynamics of almost all real systems are intrinsically nonlinear including considerable uncertain terms caused by different internal or environmental forces and factors Krstic et al. (1995). Therefore, it is a relevant subject to extend DDPC methods beyond the deterministic LTI case. An extension of Willems' fundamental lemma to a class of discrete-time feedback linearizable nonlinear systems is studied in Alsalti et al. (2023), thus providing a data-based representation of their input-output trajectories. The problem of simulation and predictive control based on noise-free data for polynomial NARX models are investigated in Mishra et al. (2021) and in Azarbahram et al. (2024), respectively. It has been shown empirically, without guarantees, that DeePC algorithm in Coulson et al. (2019) also works for simple nonlinear systems with measurement noise by using some regularizers and slack variables. However, a comprehensive algorithm with theoretical guarantees that directly targets the DDPC of NARX models, among many other classes of nonlinear dynamics/models, is yet to be addressed.

This motivates us to represent our contributions as follows. We formulate for the first time a DDPC problem for a class of NARX models. This is a topic of interest since these models can cover wide-ranging real-world scenarios and applications that include nonlinearity in input-output representation. We introduce the robust DDPC method to handle the corrupted output measurements. We first show that the optimal value function in this case is upper-bounded. In order to study the recursive feasibility and practical stability of the robust scenario, the mismatch between the predicted and actual output is also shown to be upper-bounded. The practical stability of the origin is finally studied.

The remaining article is organized as follows. Section II presents the formulation of the problem, including notation and preliminaries. The main results are then discussed in Section III. The simulation results are illustrated in Section IV. Finally, Section V concludes the paper.

2. PRELIMINARIES AND PROBLEM STATEMENT

The notation used throughout the article is reported in the footnote. [2] Consider the following NARX model

$$y(k) = \sum_{i_1=1}^{\rho_1}\sum_{\iota_1=1}^{l_1} \alpha_{i_1\iota_1} y^{i_1}_{(k-\iota_1)} + \sum_{i_2=1}^{\rho_2}\sum_{\iota_2=1}^{l_2} \beta_{i_2\iota_2} u^{i_2}_{(k-\iota_2)}, \quad (1)$$

where $\alpha_{i_1\iota_1}$ and $\beta_{i_2\iota_2}$ are unknown real scalars for all possible i_1, i_2, ι_1 and ι_2. Furthermore, $y(k) \in \mathbb{R}$ is the measured output and $u(k) \in \mathbb{R}$ is the control input. The positive predefined scalars ρ_1, ρ_2, l_1 and l_2 stand for the powers and delays of the measured output and control input, respectively. Besides, $y^{i_1}(k-\iota_1)$ and $u^{i_2}(k-\iota_2)$ also stand, respectively, for the i_1-th and i_2-th powers of delayed measured output (w.r.t., ι_1) and delayed control input (w.r.t., ι_2). Let $\Lambda_{i_1} = [\alpha_{i_1 1}, \ldots, \alpha_{i_1 l_1}]^{\mathrm{T}}$, and $B_{i_2} = [\beta_{i_2 1}, \ldots, \beta_{i_2 l_2}]^{\mathrm{T}}$. Then, we define the following vectors

$$\begin{aligned} \xi^{i_1}(k) &= [y^{i_1}(k-1), y^{i_1}(k-2), \ldots, y^{i_1}(k-l_1)]^{\mathrm{T}}, \\ \eta^{i_2}(k) &= [u^{i_2}(k-1), u^{i_2}(k-2), \ldots, u^{i_2}(k-l_2)]^{\mathrm{T}}. \end{aligned} \quad (2)$$

Given (2), the NARX model (1) is rewritten as

$$y(k) = \sum_{i_1=1}^{\rho_1} \Lambda_{i_1}^{\mathrm{T}} \xi^{i_1}(k) + \sum_{i_2=1}^{\rho_2} B_{i_2}^{\mathrm{T}} \eta^{i_2}(k). \quad (3)$$

Our goal is to design a predictive scheme for the NARX model (3) subject to user-specified input and output constraints. This can be expressed in the context of a finite horizon optimal control problem (OCP) as

$$J_L^*(\eta(k), \xi(k)) = \min_{\bar{u}(k), \bar{y}(k)} \sum_{\mu=0}^{L-1} \gamma(\bar{u}_\mu(k), \bar{y}_\mu(k)), \quad (4a)$$

s.t.

$$\bar{y}_\mu(k) = \sum_{i_1=1}^{\rho_1} \Lambda_{i_1}^{\mathrm{T}} \bar{\xi}^{i_1}_\mu(k) + \sum_{i_2=1}^{\rho_2} B_{i_2}^{\mathrm{T}} \bar{\eta}^{i_2}_\mu(k), \quad (4b)$$

$$\bar{\xi}^{i_1}_0(k) = \xi^{i_1}(k), \bar{\eta}^{i_2}_0(k) = \eta^{i_2}(k), \quad (4c)$$

$$\bar{y}_\mu(k) \in \mathcal{Y}, \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (4d)$$

$$\bar{u}_\mu(k) \in \mathcal{U}, \forall \mu \in \mathbb{N}_{[0,L-1]}. \quad (4e)$$

In this setting, $\bar{u}^*(k) = (\bar{u}^*_0(k), \ldots, \bar{u}^*_{L-1}(k))$ is the optimal control in case of an admissible feasible solution to the OCP. It is worth pointing out that an l-step predictive control scheme is developed in this paper, i.e., from $\bar{u}^*(k)$, $\{\bar{u}^*_0(k), \ldots, \bar{u}^*_{l-1}(k)\}$ and it's corresponding powers for $\bar{\eta}^{i_2 *}(k), i_2 = 2, \ldots, \rho_2$ is applied to the actual system (3). Accordingly, the horizon is then shifted l samples before the next iteration while making l measurements after applying the optimal control sequence $\{\bar{u}^*_0(k), \ldots, \bar{u}^*_{l-1}(k)\}$. The stage cost $\gamma(\bar{u}(k), \bar{y}(k))$ which penalizes the distance of predicted inputs and outputs w.r.t the desired equilibrium (u_s, y_s) is defined as

$$\gamma(\bar{u}_\mu(k), \bar{y}_\mu(k)) = |\bar{y}_\mu(k) - y_s|^2 + |\bar{u}_\mu(k) - u_s|^2.$$

Although the model predictive control problem (4a)-(4e) is now completely defined, the constraint in (4b) makes this problem impossible to solve since we do not have the parameters of the model for the evolution of system in open loop optimization at each iteration. Therefore, our

[2] To denote the sets of reals, non-negative reals, positive reals, non-negative integers, and positive integers, we use $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_+$, $\mathbb{N}$, and $\mathbb{N}_+$, respectively. For $I \subseteq \mathbb{R}_{0+}$, let $\mathbb{N}_I = \mathbb{N} \cap I$. The set of real $p \times q$ matrices is denoted by $\mathbb{R}^{p\times q}$. For any matrix A, A^{T} denotes its transpose. Additionally, $A^\dagger$ stands for the pseudo-inverse of matrix A. Furthermore, $\mathbf{0}_q$ and $\mathbf{1}_q$ are respectively vectors of all zero/one entries of dimension q. The i-th power of any vector $\varkappa \in \mathbb{R}^q$, denoted as $\varkappa^i$, is defined as the i-th power of each of its components. For matrices $A_1 \in \mathbb{R}^{p_1\times q}, A_2 \in \mathbb{R}^{p_2 \times q}, \ldots, A_r \in \mathbb{R}^{p_r\times q}$, we define $\mathcal{C}ol(A_1, A_2, \ldots, A_r) = [A_1^{\mathrm{T}}, A_2^{\mathrm{T}}, \ldots, A_r^{\mathrm{T}}]^{\mathrm{T}}$. A typical *offline* time series $\{\varkappa_d(k)\}_{k=0}^{N-1} \in (\mathbb{R}^q)^N$ is called the *historical data* where the subscript d stands for data. Furthermore, for any trajectory $\varkappa : \mathbb{N}_{[a,b]} \to \mathbb{R}^q$ we may use $\varkappa_{[a,b]}$ to denote $\mathcal{C}ol(\varkappa(a), \varkappa(a+1), \ldots, \varkappa(b))$ where $a, b \in \mathbb{N}$. Finally, we recall the persistency of excitation.

Persistency of excitation (Willems et al., 2005): A q-variate time series $\varkappa := (\varkappa(0), \varkappa(2), \cdots, \varkappa(N-1))$ is called PE of order $L \in \mathbb{N}$ if the Hankel matrix $\mathcal{H}_L(\varkappa)$ with L-block rows is full row rank, i.e., $\mathrm{rank}(\mathcal{H}_L(\varkappa)) = qL$.

main goal in this paper is to re-define this model-based OCP in the form of DDPC which only uses input-output data in open-loop prediction stage.

3. MAIN RESULTS

The key idea to continue with is to replace the constraint in (4b) by a non-parametric representation of model that is derived purely from data Mishra et al. (2021). Let $\boldsymbol{\xi}_d = \mathcal{C}\mathrm{ol}(\xi_d, \ldots, \xi_d^\rho)$, and $\boldsymbol{\eta}_d = \mathcal{C}\mathrm{ol}(\eta_d, \ldots, \eta_d^\rho)$ be defined over the typical offline time series $\{\xi_d^{i_1}(k), \eta_d^{i_2}(k)\}_{k=0}^{N-1} \in (\mathbb{R}^{l_1+l_2})^N$ for all $i_1 = 1, \ldots, \rho_1$ and $i_2 = 1, \ldots, \rho_2$ based on the historical data. By considering $L_1 = L + 1$, we also define $\mathcal{H}_{L_1}(\boldsymbol{\xi}_d) = \mathcal{C}\mathrm{ol}(\mathcal{H}_{L_1}(\xi_d), \ldots, \mathcal{H}_{L_1}(\xi_d^{\rho_1}))$, and $\mathcal{H}_{L_1}(\boldsymbol{\eta}_d) = \mathcal{C}\mathrm{ol}(\mathcal{H}_{L_1}(\eta_d), \ldots, \mathcal{H}_{L_1}(\eta_d^{\rho_2}))$. The data-driven non-parametric representation of NARX models of the form (3) is given in what follows.

Lemma 1: (Mishra et al., 2021, Theorem 1) Suppose that the *offline* time series $\{\boldsymbol{\eta}_d, \boldsymbol{\xi}_d\}$ is PE of order L_1. Let $\boldsymbol{\xi}_{[-1,L-1]}(k) = \mathcal{C}\mathrm{ol}(\xi_{[-1,L-1]}(k), \ldots, \xi_{[-1,L-1]}^{\rho_1}(k))$, and $\boldsymbol{\eta}_{[-1,L-1]}(k) = \mathcal{C}\mathrm{ol}(\eta_{[-1,L-1]}(k), \ldots, \eta_{[-1,L-1]}^{\rho_2}(k))$. Then, $y_{[0,L]}(k)$ is a trajectory of (3) w.r.t., $\boldsymbol{\xi}_{[-1,L-1]}(k)$ and $\boldsymbol{\eta}_{[-1,L-1]}(k)$, if and only if $\exists\, \boldsymbol{g} \in \mathbb{R}^{N-L_1+1}$ such that

$$\begin{bmatrix} \boldsymbol{\eta}_{[-1,L-1]}(k) \\ \boldsymbol{\xi}_{[-1,L-1]}(k) \\ y_{[0,L]}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{L_1}(\boldsymbol{\eta}_d) \\ \mathcal{H}_{L_1}(\boldsymbol{\xi}_d) \\ \mathcal{H}_{L_1}(y_d) \end{bmatrix} \boldsymbol{g}(k) \equiv \boldsymbol{\mathcal{H}}_{\xi\eta}\, \boldsymbol{g}(k). \quad (5)$$

For ease of notation, in the rest of this paper we consider the NARX model (3) with $\rho_1 = \rho_2 = \rho$ and $l_1 = l_2 = l$. This is quite a reasonable assumption since it is possible to add arbitrary extra powers and delays to input-output pairs with zero coefficients in the NARX model (1) to provide the same order for the purpose of theoretical analysis.

Now we are in the position to re-define the nominal model-based OCP (4a)-(4e) in the form of a DDPC problem which only uses data in open-loop prediction stage as follows

$$J_L^*(\eta(k), \xi(k)) = \min_{\bar{u}(k), \bar{y}(k), \boldsymbol{g}(k)} \sum_{\mu=0}^{L-1} \gamma(\bar{u}_\mu(k), \bar{y}_\mu(k)), \quad (6a)$$

s.t.

$$\bar{y}_\mu(k) \in \mathcal{Y}, \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (6b)$$

$$\bar{u}_\mu(k) \in \mathcal{U}, \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (6c)$$

$$\begin{bmatrix} \bar{\boldsymbol{\eta}}_{[-1,L-1]}(k) \\ \bar{\boldsymbol{\xi}}_{[-1,L-1]}(k) \\ \bar{y}_{[0,L]}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{L_1}(\boldsymbol{\eta}_d) \\ \mathcal{H}_{L_1}(\boldsymbol{\xi}_d) \\ \mathcal{H}_{L_1}(y_d) \end{bmatrix} \boldsymbol{g}(k), \quad (6d)$$

$$\bar{\xi}_{[-1]}^i(k) = \xi^i(k), \quad \bar{\eta}_{[-1]}^i(k) = \eta^i(k), \quad (6e)$$

$$\bar{\xi}_{[L-1]}^i(k) = y_s^i \mathbf{1}_l, \quad \bar{\eta}_{[L-1]}^i(k) = u_s^i \mathbf{1}_l, \quad (6f)$$

$$\forall i = 1, \ldots, \rho.$$

We realize that the constraint (4b) is now replaced with (6d). This provides a non-parametric representation of model purely from data that gives any trajectory of model (3) starting from $y(0)$ according to Lemma 1. The initial conditions for each iteration is set according to constraint (6e). The terminal equality constraint (6f) implies that the output and input of the system has reached to the desired equilibrium (y_s, u_s) at the last open-loop optimization stage.

In real-time applications, outputs of the system are subject to measurement noise and thus are inaccurate. Then, the data-dependent Hankel matrix does not span the system's trajectory space, and equation (5) does not precisely represent the model anymore. Therefore, the optimization problem (6a)-(6f) needs to be upgraded to account for inaccurate predictions. Otherwise, it may be infeasible or lead to system's instability. We consider model (1) with additive output noise for offline collected data as well as online measurements, i.e., $\tilde{y}(k) = y(k) + \epsilon(k)$ and $\tilde{y}_d(k) = y_d(k) + \epsilon_d(k)$. The noises at sample time k are denoted by $\epsilon(k)$ and $\epsilon_d(k)$ for online measurements and offline data, respectively. In order to build vectors $\tilde{\xi}^i(k)$ and $\tilde{\xi}_d^i(k)$, we assume that $\tilde{y}^i(k) \approx y^i(k) + \epsilon^i(k)$ and $\tilde{y}_d^i(k) \approx y_d^i(k) + \epsilon_d^i(k)$ for all $i = 2, \ldots, \rho$. Note that, the assumption $\tilde{y}^i(k) \approx y^i(k) + \epsilon^i(k)$ and $\tilde{y}_d^i(k) \approx y_d^i(k) + \epsilon_d^i(k)$ for all $i = 2, \ldots, \rho$ is a bit conservative because we are ignoring here the cross-terms. However, the key rationale behind this approximation is that we have the perturbed data in the noisy scenario. Notably, this approximation does not undermine the validity of the recursive feasibility and stability results, as the system behavior remains consistent under the bounded noise assumption. Thus, this assumption made here is merely to simplify the exposition.

Next, we define $\tilde{\xi}^i(k) = [\tilde{y}^i(k-1), \tilde{y}^i(k-2), \ldots, \tilde{y}^i(k-l)]^{\mathrm{T}}$. To handle the noisy measurements in the DDPC scheme, we need to re-design the equality constraint derived by the fundamental lemma in (6d). This must be a relaxed version of (6d) by adding some slack variables to the predicted outputs. The relaxation parameters are as a result compensated in the cost function. More precisely, the cost function in (6a) should also be modified. By leveraging the idea in Berberich et al. (2021), for a given initial input-output pair $(\eta(k), \tilde{\xi}(k))$ and a sequence of N-length offline data for which $\{\boldsymbol{\eta}_d, \tilde{\boldsymbol{\xi}}_d\}$ is PE of order L_1, we propose a robust modification for the DDPC problem in (6a)-(6f) as follows

$$J_L^*(\eta(k), \tilde{\xi}(k)) = \min_{\substack{\bar{u}(k), \bar{y}(k), \boldsymbol{g}(k) \\ \sigma^{[i]}(k), i=1,\ldots,\rho}} \sum_{\mu=0}^{L-1} \gamma(\bar{u}_\mu(k), \bar{y}_\mu(k)) + \sum_{i=1}^{\rho} \lambda_i \|\sigma^{[i]}(k)\|_2^2 + \phi\bar{\epsilon}\|\boldsymbol{g}(k)\|_2^2, \quad (7a)$$

s.t.

$$\bar{y}_\mu(k) \in \mathcal{Y}, \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (7b)$$

$$\bar{u}_\mu(k) \in \mathcal{U}, \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (7c)$$

$$\begin{bmatrix} \bar{\boldsymbol{\eta}}_{[-1,L-1]}(k) \\ [\bar{\xi}^1(k) + \sigma^{[1]}(k)]_{[-1,L-1]} \\ [\bar{\xi}^2(k) + \sigma^{[2]}(k)]_{[-1,L-1]} \\ \vdots \\ [\bar{\xi}^\rho(k) + \sigma^{[\rho]}(k)]_{[-1,L-1]} \\ [\bar{y}(k) + \sigma_{[1]}^{[1]}(k)]_{[0,L]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{L_1}(\boldsymbol{\eta}_d) \\ \mathcal{H}_{L_1}(\tilde{\xi}_d^1) \\ \mathcal{H}_{L_1}(\tilde{\xi}_d^2) \\ \vdots \\ \mathcal{H}_{L_1}(\tilde{\xi}_d^\rho) \\ \mathcal{H}_{L_1}(\tilde{y}_d) \end{bmatrix} \boldsymbol{g}(k), \quad (7d)$$

$$\bar{\xi}_{[-1]}^i(k) = \tilde{\xi}^i(k), \quad \bar{\eta}_{[-1]}^i(k) = \eta^i(k), \quad (7e)$$

$$\bar{\xi}_{[L-1]}^i(k) = y_s^i \mathbf{1}_l, \quad \bar{\eta}_{[L-1]}^i(k) = u_s^i \mathbf{1}_l, \quad (7f)$$

$$\|\sigma_\mu^{[i]}(k)\|_\infty \le \bar{\epsilon}(1 + \|\boldsymbol{g}(k)\|_1), \forall \mu \in \mathbb{N}_{[0,L-1]}, \quad (7g)$$

$$\forall\, i = 1, \ldots, \rho.$$

Let $\boldsymbol{\epsilon}^i(k) = \mathcal{C}\text{ol}(\epsilon^i(k-1), \ldots, \epsilon^i(k-l))$ for $i = 1, \ldots, \rho$. We point out here that no assumptions are made on the nature of noise however, it is required to be bounded, i.e., $|\epsilon^i(k)| \leq \bar{\epsilon}$ and $|\epsilon_d^i(k)| \leq \bar{\epsilon}$ with some positive constant $\bar{\epsilon}$. Notably, under the premise of small perturbations (less than 1), we can have a common noise bound for all powers. Accordingly, for all $i = 1, \ldots, \rho$, we have $\|\boldsymbol{\epsilon}^i(k)\|_\infty \leq \bar{\epsilon}$ and $\|\boldsymbol{\epsilon}_d^i(k)\|_\infty \leq \bar{\epsilon}$. We observe that the slack variables $\sigma^{[i]}(k) \in \mathbb{R}^l$ for $i = 1, \ldots, \rho$ are introduced in (7d) as a modified version of (6d). We have to note that the superscript notation $[i]$ for $\sigma^{[i]}(k)$ does not stand for the i-th power of $\sigma(k)$ and is in fact a vector including l elements that corresponds to different powers of $\bar{\xi}^i(k)$. These slack variables are bounded by new constraints compared to nominal DDPC scheme as we can see from (7g). We need this constraint later to establish an upper bound for the Lyapunov function and also in the recursive feasibility analysis of (7a). We also note that the cost function is modified by adding regularization terms for $\boldsymbol{g}(k)$ and $\sigma^{[i]}(k)$, respectively by weights $\bar{\epsilon}\phi$ and λ_i for $i = 1, \ldots, \rho$. It indicates that the regularization of $\boldsymbol{g}(k)$ depends on the noise level. Compared to the nominal DDPC scheme, $\bar{\xi}^i_{[-1]}(k)$ is now initialized after each iteration with $\tilde{\xi}^i(k)$ according to (7e) for the robust problem formulation. The final point to notice is the representation of Hankel matrices in (7d) which is defined over noisy offline data $\tilde{\boldsymbol{\xi}}_d$.

Remark 1: The optimization problem formulated in this work is nonconvex due to the nonlinear terms in the Fundamental Lemma developed in (6d) and (7d). Another source of nonconvexity is (7g). Nonconvex problems generally pose challenges in finding globally optimal solutions, as they may contain multiple local minima. A convexification of this problem is beyond the scope of this paper. In this work, we employed the "CasADi" framework to solve the nonconvex predictive control problem (Andersson et al., 2019). "CasADi" provides symbolic differentiation and optimization routines that enable the formulation and solution of nonconvex problems using interior-point or SQP (Sequential Quadratic Programming) methods. While these methods are not guaranteed to find the global optimum for nonconvex problems, they are well-suited for efficiently finding locally optimal solutions. To ensure the quality and feasibility of the obtained solutions in the simulation results, the problem was carefully initialized, and multiple runs with different initial conditions were performed to check for consistency in the results. The use of warm-starting techniques further helped to speed up convergence and improve solution quality. Additionally, the practical performance of the control algorithm was validated through extensive simulations, which demonstrated robustness and satisfaction of constraints across all tested scenarios.

We introduce the vectors $\boldsymbol{\xi}(k) = \mathcal{C}\text{ol}(\xi(k), \ldots, \xi^\rho(k))$, and $\boldsymbol{\eta}(k) = \mathcal{C}\text{ol}(\eta(k), \ldots, \eta^\rho(k))$. We then define two block column vectors over the past available input-output pairs at any sample time k according to the vectors defined in (2) as follows

$$\theta(k) = \begin{bmatrix} \boldsymbol{\eta}(k) \\ \boldsymbol{\xi}(k) \end{bmatrix} \in \mathbb{R}^{2\rho l}, \quad \vartheta(k) = \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} \in \mathbb{R}^{2l}. \tag{8}$$

The function $\mathcal{V}_L(\vartheta(k))$ is defined as the optimal value function of the form $\mathcal{V}_L(\vartheta(k)) = J_L^*(\eta(k), \xi(k)) = J_L^*(\vartheta(k))$. Throughout this paper, we consider $(u_s, y_s) = (0, 0)$ for the stability analysis. The optimal value function in (7a) is a function of past control inputs as well as noisy measured outputs denoted by $J_L^*(\eta(k), \tilde{\xi}(k))$ while the L-step cost function $J_L(\eta(k), \tilde{\xi}(k), \boldsymbol{g}(k), \sigma^{[i]}(k))$ is equal to $\sum_{\mu=0}^{L-1} \gamma(\bar{u}_\mu(k), \bar{y}_\mu(k)) + \sum_{i=1}^{\rho} \lambda_i \|\sigma^{[i]}(k)\|_2^2 + \phi\bar{\epsilon}\|\boldsymbol{g}(k)\|_2^2$. We see that the optimal value function depends on $\boldsymbol{g}^*(k)$. This results in the dependence of $J_L^*(\eta(k), \tilde{\xi}(k))$ on the past noisy measurements by (7e). Therefore, we have to find a new upper bound on the optimal value function. Similar to (8), we define a block column vector over the past available control input and noisy output pairs at any sample time k as follows

$$\tilde{\vartheta}(k) = \begin{bmatrix} \eta(k) \\ \tilde{\xi}(k) \end{bmatrix} = \begin{bmatrix} \eta(k) \\ \xi(k) + \boldsymbol{\epsilon}(k) \end{bmatrix} \in \mathbb{R}^{2l}. \tag{9}$$

The function $\mathcal{V}_L(\tilde{\vartheta})$ is defined as the optimal value function that is $J_L^*(\eta(k), \tilde{\xi}(k)) = J_L^*(\tilde{\vartheta}(k))$. The upper bound on the optimal value function $\mathcal{V}_L(\tilde{\vartheta}(k))$ is derived in what follows. We have to note that the lower bound of $\mathcal{V}_L(\tilde{\vartheta}(k))$ is trivial.

Lemma 2: Suppose that the sequence of data $\{\boldsymbol{\eta}_d, \tilde{\boldsymbol{\xi}}_d\}$ is PE of order L_1. Then, there exist positive constants c_1, c_2, and δ such that for any $\vartheta(k) \in \mathbb{B}_\delta$ with $\mathbb{B}_\delta = \{\vartheta(k) \mid \|\vartheta(k)\|_2 \leq \delta\}$, we have

$$\mathcal{V}_L(\tilde{\vartheta}(k)) \leq c_1 \|\vartheta(k)\|_2^2 c_2. \tag{10}$$

One of the main concerns in investigating the recursive feasibility and practical stability of our proposed robust DDPC scheme is the unavoidable mismatch between predicted output $\bar{y}^*(k)$ after applying the optimal control $\bar{u}^*(k)$ and the actual output $\hat{y}(k)$. In what follows, we study how this value is bounded in terms of the optimizer of (7a)-(7g), i.e., $\boldsymbol{g}^*(k)$, and $\sigma^{*[i]}(k)$ for all $i = 1, \ldots, \rho$. We first define the following

$$\breve{\xi}^i_{[-1,L-1]}(k) = \hat{\xi}^i_{[-1,L-1]}(k) - \mathcal{H}_{L_1}(\xi_d^i)\boldsymbol{g}^*(k). \tag{11}$$

We say that $\breve{\xi}^i_{[-1,L-1]}(k)$ is an output trajectory of the model with initial condition $\breve{\xi}^i_{[-1]}(k) = -\sigma^{*[i]}_{[-1]}(k) + \mathcal{H}_1(\boldsymbol{\epsilon}_d^i)\boldsymbol{g}^*(k) - \boldsymbol{\epsilon}^i(k)$. It is obvious that $\breve{y}^i_{[-1,L-1]}(k)$ is defined as the first element of each block row in $\breve{\xi}^i_{[-1,L-1]}(k)$, denoted by $\breve{\xi}^i_{1.[-1,L-1]}(k)$. Additionally, from (7d) we have

$$\begin{aligned} \bar{\xi}^{*i}_{[-1,L-1]}(k) = -\sigma^{*[i]}_{[-1,L-1]}(k) + \mathcal{H}_{L_1}(\xi_d^i)\boldsymbol{g}^*(k) \\ + \mathcal{H}_{L_1}(\boldsymbol{\epsilon}_d^i)\boldsymbol{g}^*(k). \end{aligned} \tag{12}$$

According to (7a)-(7g), we observe that $\mathcal{H}_{L_1}(\xi_d^i)\boldsymbol{g}^*(k)$ is a trajectory of model (3) with initial output condition $\tilde{\xi}^i_{[-1]}(k) + \sigma^{*[i]}_{[-1]}(k) - \mathcal{H}_1(\boldsymbol{\epsilon}_d^i)\boldsymbol{g}^*(k)$. We have to point out that $\tilde{\xi}^i_{1.[-1]}(k)$ is $\tilde{y}^i(k-1)$. Now, we can find an upper bound on the mismatch between the predicted output $\bar{y}^*(k)$, after applying the optimal control $\bar{u}^*(k)$, and the actual output $\hat{y}(k)$.

Lemma 3: The l_∞ norm of the difference between the i-th power of the predicted output $\bar{y}^{*i}_\mu(k), \forall \mu \in \mathbb{N}_{[0,L-1]}$ and the actual output μ sample time ahead, i.e., $\hat{y}^i(k+\mu)$ is upper bounded for all $i = 1, \ldots, \rho$ as follows

$$\begin{aligned} \|\hat{y}^i(k+\mu) - \bar{y}^{*i}_\mu(k)\|_\infty \leq |c_\mu| \big(\bar{\epsilon}(1 + \|\boldsymbol{g}^*(k)\|_1) \\ + |\sigma^{*[i]}_{1.[-1]}(k)|\big) + \bar{\epsilon}\|\boldsymbol{g}^*(k)\|_1 + \|\sigma^{*[i]}_\mu(k)\|_\infty. \end{aligned} \tag{13}$$

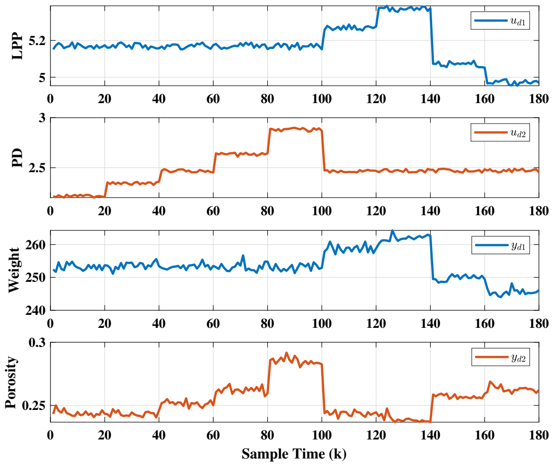

Fig. 1. Experimental data gathered from a powder compaction process performed on a rotary tablet press.

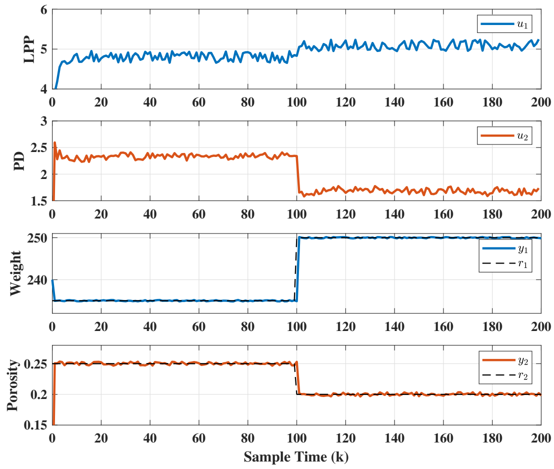

Fig. 2. The control inputs and tracking performance.

Theorem 1: Suppose that there exists $\boldsymbol{g}(k) \in \mathbb{R}^{N-L_1+1}$ such that (7d) holds and additionally assume that the OCP (7a)-(7g) is feasible at a given sample time k with $\mathcal{V}_L(\tilde{\vartheta}(k)) = J_L^*(\tilde{\vartheta}(k)) < \Omega$ for a positive constant Ω. Then, there exists $\bar{\epsilon}_0$ such that for all $\bar{\epsilon} < \bar{\epsilon}_0$, the robust DDPC formulation (7a)-(7g) is feasible at time $k + l$.

Lemma 4: The l_2 norm of the difference between the i-th power of the predicted output $\bar{y}_\mu^{*i}(k), \forall \mu \in \mathbb{N}_{[0,L-1]}$ and the actual output μ sample time ahead, i.e., $\hat{y}^i(k+\mu)$ is upper bounded for all $i = 1, \dots, \rho$ as follows

$$\begin{aligned}\|\hat{y}^i(k+\mu) - \bar{y}_\mu^{*i}(k)\|_2^2 \le{}& |c_\mu|^2 \left(4\|\sigma^{*[i]}_{1:[-1]}(k)\|_2^2\right.\\ &\left. + 8\bar{\epsilon}^2 + 8\bar{\epsilon}^2 \frac{c_\epsilon}{l}\|\boldsymbol{g}^*(k)\|_2^2\right) + 4\bar{\epsilon}^2 \frac{c_\epsilon}{l}\|\boldsymbol{g}^*(k)\|_2^2\\ &+ 4\|\sigma_\mu^{*[i]}(k)\|_2^2. \end{aligned} \tag{14}$$

Theorem 2: Suppose that there exists $\boldsymbol{g}(k) \in \mathbb{R}^{N-L_1+1}$ such that (7d) holds and additionally assume that the OCP (7a)-(7g) is feasible at a given initial sample time k with $\mathcal{V}_L(\tilde{\vartheta}(k)) = J_L^*(\tilde{\vartheta}(k)) < \Omega$ for any $\Omega > 0$. Suppose that in the cost function, the weights ϕ and λ_i for $i = 1, \dots, d$ are lower-upper bounded as $\phi_m \le \phi \le \phi_M$ and $\lambda_m \le \lambda_i \le \lambda_M$, for some existing positive constants $\phi_m, \phi_M, \lambda_m$ and λ_M. For some positive constant $\bar{h}_g$ we also have $h_g \bar{\epsilon} \le \bar{h}_g$. Then, the origin is practically stable for l-step implementation of robust DDPC in (7a)-(7g).

4. SIMULATION RESULTS

In this section, we show the effectiveness of our proposed robust DDPC method in terms of experimental data gathered from the powder compaction process. We define the predicted input-output pairs $\{\bar{u}_\mu(k), \bar{y}_\mu(k)\}$ (to build their power vectors correspondingly), $\sigma^{[i]}(k) \in \mathbb{R}^l$ for $i = 1, \dots, \rho$, and vector $\boldsymbol{g}(k)$ as the decision variables at each iteration to solve the optimization problems. The initial conditions for input-output pairs are also defined as the parameters of the problem to be updated at the beginning of each iteration.

Powder compaction performed on a rotary tablet press is a dry granulation method to transfer powder materials consisting of several components (drug, lubricant and other excipients) into compacts (tablets). This process is usually integrated in a multi-stage manufacturing line for the product design (Kleinebudde et al., 2017). Moreover, product quality indices refer to the dose, hardness, disintegration, and dissolution of the tablets. These attributes are maintained during processing by adaptation of the subsequent unit operations in the rotary tablet press. Feeding, blending and filling are executed continuously and determine the chemical composition. Compression and ejection, on the other hand, are performed semi-continuously and determine the mechanical properties of the compacts (e.g. hardness). The interconnection of the five process steps, i.e., feeding, blending, filling, compression and ejection leads to a complex control task. Here, the efficiency of our proposed robust DDPC method is evaluated based on a set of real data gathered from a powder compaction process performed on a rotary tablet press. The process is MIMO and the inputs are lower punch position (LPP), punch distance (PD), turret speed, impeller speed, and screw speed. The outputs are on the other hand weight, porosity, lubrication, and weight fraction. We collected $N = 180$ samples of input-output data shown in Fig. 1, that corresponds to a part of the whole process i.e., compression with two inputs (u_1: LPP in mm, u_2: PD in mm) and two outputs (y_1: weight in mg, y_2: porosity in $[0, 1]$). As we observe from Fig. 1, we mounted a small noise to maintain the PE condition for the gathered input data. We also notice fluctuations in the gathered output data which is caused by measurement noise. Accordingly, we chose an unknown two-input two-output ARX model with $l_1, l_2 = 3$ and the sequence of data is subsequently constructed to build the Hankel matrices and formulate the fundamental lemma in (7d). The constraints for the output values are $y_{1\,\min} = 0$ mg, $y_{1\,\max} = 300$ mg, $y_{2\,\min} = 0$, $y_{2\,\max} = 1$. The inputs are also constrained between 0 and 9.2 mm. For the online optimization problem, a pair of predefined time-varying reference trajectories is considered for the tracking performance (i.e., r_1 and r_2) and the prediction horizon is $L = 5$. The final results after applying the algorithm are shown in Fig. 2, where we can see the set points are appropriately adjusted to keep the tracking according to the reference trajectories to be tracked by the rotatory tablet press process.

5. CONCLUSION

This work is concerned with the problem of DDPC for NARX models based on measured input-output data corrupted by additive measurement noise. The theoretical guarantees are given for the upper bound on the optimal value function, the bounded mismatch between the predicted and actual output, and the recursive feasibility and practical stability of the proposed robust DDPC scheme. The effectiveness of the proposed method is shown in terms of simulations with the powder compaction process.

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