Distributionally Robust MPC for Nonlinear Systems

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Abstract: Classical stochastic model predictive control (SMPC) methods assume that the true probability distribution of uncertainties in controlled systems is provided in advance. However, in real-world systems, only partial distribution information can be acquired for SMPC. The discrepancy between the true distribution and the distribution assumed can result in sub-optimality or even infeasibility of the system. To address this, we present a novel distributionally robust data-driven MPC scheme to control stochastic nonlinear systems. We use distributionally robust constraints to bound the violation of the expected state-constraints under process disturbance. Sequential linearization is performed at each sampling time to guarantee that the system's states comply with constraints with respect to the worst-case distribution within the Wasserstein ball centered at the discrete empirical probability distribution. Under this distributionally robust MPC scheme, control laws can be efficiently derived by solving a conic program. The competence of this scheme for disturbed nonlinear systems is demonstrated through two case studies.

Keywords: Model predictive control,

1. INTRODUCTION

Model predictive control (MPC) has been successfully applied to real-world systems and is now widely adopted in the process industry (Dobos et al., 2009; Touretzky and Baldea, 2014; Qin and Badgwell, 2003). MPC outperforms other control approaches as it is able to predict the system's behavior while explicitly accounting for constraints within a prediction horizon (Mayne et al., 2000). MPC solves an open-loop control problem at each sampling time to determine a sequence of control actions, from which the first element of the sequence is implemented.

To control systems with uncertainties, there are two categories of approaches (Kouvaritakis and Cannon, 2016) proposed to guarantee closed-loop performance and constraint satisfaction: robust MPC (RMPC) (Houska and M.E, 2019) and stochastic MPC (SMPC) (Arcari et al., 2020; Hewing et al., 2020). RMPC determines the optimal control action with respect to the worst-case uncertainty within a bounded set (Mayne et al., 2005), whereas SMPC assumes or estimates the distribution of the uncertainty and selects the best control action for an expected objective function under probabilistic constraints. This can alliviate the conservativeness of RMPC by making use of this distributional information (Mesbah, 2016).

However, acquiring the true distribution of uncertainties for real-world applications is very challenging (Yang, 2021; Mark and Liu, 2020). Furthermore, the high computational time of SMPC (Mayne, 2015) and the discrepancy between the estimated distribution and true distribution (Heirung et al., 2018) limit the performance of SMPC for practical applications.

To address the aforementioned challenges, we propose an approach based on distributionally robust optimization (DRO) (Rahimian and Mehrotra, 2019) to solve stochastic optimal control problems in a computationally efficient manner while also dealing with distributional mismatch of the uncertainty. Based on the work by (Esfahani and Kuhn, 2018), we reformulate the optimization problem with an expected quadratic objective function and distributionally robust constraints into a convex conic program. The solution of this conic program determines optimal control laws with respect to the worst-case probabilistic distribution within a Wasserstein ball centered at the discrete empirical distribution (Zhong et al., 2021).

Several works have recently been conducted on distributionally robust control. In (Van Parys et al., 2016), stochastic optimal control problems with CVaR constraints with known first and second order moments are addressed by solving a semi-definite program (SDP). (Yang, 2021) synthesizes a distributionally robust policy with Wasserstein ambiguity sets for data-driven Markov decision processes. (Mark and Liu, 2020) extends tube-based SMPC to distributionally robust chance constraints. (Coppens and Patrinos, 2021) considers distributionally robust MPC problems with conic representable ambiguity sets.

⋆ Sponsor and financial support acknowledgment goes here. Paper titles should be written in uppercase and lowercase letters, not all uppercase.

However, all the above-mentioned relevant works are applied to linear systems. In this paper, we extend the work in (Zhong et al., 2021) to nonlinear systems. For the purpose of nonlinear system control, we adapt a more compact formulation for the nonlinear control problem, linearize the models of nonlinear systems around sampled states at each sampling time, and adaptively propagate the linearization errors within the prediction horizon.

We address a nonlinear distributionally robust optimal control problem with distributionally robust constraints by using the Wasserstein ball metric and reformulating the problem into a conic program for the linearized system. By solving this convex conic program, Purified-output-based (POB) affine control laws are determined for the nonlinear system such that distributionally robust constraints hold with finite sample guarantee. To the best of our knowledge, this is the first paper applying distributionally robust control to nonlinear systems.

1.1 Notation

Let $x_{[k:k+N]}$ denote the concatenated state vector $[x_k^\top, x_{k+1}^\top, \ldots, x_N^\top]^\top$ and $[x_{[k:k+N]}]_i$ be the i-th entry of this vector. We denote by $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ the sets of all positive semidefinite and positive definite symmetric matrices in $\mathbb{R}^{n_x \times n_x}$, respectively. The diagonal concatenation of two matrices A and B is denoted by $\mathrm{diag}(A, B)$. $A_{[i,j]}$ is the entry from the i-th row and j-th column in the matrix A, $A_{[i,j:k]}$ is the i-th row vector from the j-th to k-th columns in matrix A, and $A_{[i:,j]}$ is j-th column of the matrix A containing the i-th to the last row.

All random vectors are defined as measurable functions on an abstract probability space $(\Omega, X, \mathbb{P})$, where Ω is referred to as the sample space, X represents the σ-algebra of events, and $\mathbb{P}$ denotes the true but unknown probability measure. We denote by δ_ξ the Dirac distribution concentrating unit mass at ξ and by δ_k the state difference of the linearized and nonlinear system at sampling time k. The N-fold product of a distribution $\mathbb{P}$ on the uncertainty set Ξ is denoted by $\mathbb{P}^N$, which represents a distribution on the Cartesian product space Ξ^N. $\mathbb{M}(\Xi)$ is the space of all probability distributions supported on Ξ with finite expected norm. The training data set comprising N_s samples is denoted by $\widehat{\Xi}_{N_s} := \left\{\widehat{\xi}_i\right\}_{i\in\mathbb{Z}_{[1,N_s]}} \subseteq \Xi$.

2. PROBLEM FORMULATION

In this paper we consider the distributionally robust control problem for a nonlinear system of the form

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + Dw_k \\ y_k &= Cx_k + Ew_k, \end{aligned} \tag{1}$$

where we denote the state $x_k \in \mathbb{R}^{n_x}$, input $u_k \in \mathbb{R}^{n_u}$, output $y_k \in \mathbb{R}^{n_y}$ and noise $w_k \in \mathbb{R}^{n_w}$. The system dynamics f is assumed to be known exactly. Process noise and measurement noise are modelled via matrices D and E. For this disturbed nonlinear system, the design target of this paper is to find a control law that minimizes our objective function such that the probabilistic linear constraints involving both states and inputs

$$\sup_{\mathbb{Q}\in\mathbb{P}} \mathbb{E}^{\mathbb{Q}}[Fx_k + Gu_k] \leq \nVdash, \tag{2}$$

hold at each sampling time k, where $F \in \mathbb{R}^{n_c \times n_x}$ and $G \in \mathbb{R}^{n_c \times n_u}$ are matrices for mixed state and input constraints, and $\mathbb{P}$ is the ambiguity set containing all probability distributions consistent with the given partial distribution information.

The disturbed linearized system at sampling time k is defined as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + Dw_k + \delta_k \\ y_k &= Cx_k + Ew_k, \end{aligned} \tag{3}$$

where $A_k := \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_k \\ u=u_{k-1}}} \in \mathbb{R}^{n_x \times n_x}$ and $B_k := \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_k \\ u=u_{k-1}}} \in \mathbb{R}^{n_x \times n_u}$, and δ_k is the difference between the nonlinear system and the linearized system. The initial state of the system (3) at the sampling time k within the prediction horizon N is denoted as x_0.

The corresponding nominal linear system is defined as

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k + \delta_k \\ \hat{y}_k &= C\hat{x}_k, \end{aligned} \tag{4}$$

with the initial state $\hat{x}_k = x_r$ for the tracking point x_r.

Unlike classical state or output feedback systems, this work applies purified-output-based (POB) affine control laws (Ben-Tal et al., 2006) to nonlinear systems. These control laws map the accumulated discrepancies between the real output measurements and outputs of the nominal system to the system inputs.

Definition 1. (POB Affine Control). At sampling time t, given purified outputs from k to t, we define the *POB affine control laws* as

$$u_t = h_t + \sum_{\tau=k}^{t} H_{t,\tau} v_\tau \tag{5}$$

with $t \in \mathbb{Z}_{[k,k+N-1]}$ and purified outputs $v_t = y_t - \hat{y}_t = \sum_{\tau=k}^{t} CA^{t-\tau}D^\tau(x_0 - x_r) + Ew_t$.

POB control laws are described by the accumulated discrepancies between the disturbed system and the corresponding nominal system with the tracking point as the initial state (Skaf and Boyd, 2010). POB control laws as the feedback laws from disturbances to inputs, guarantee the set of admissible feedback parameters to be convex and closed (Goulart et al., 2006).

The goal of the controller design in this paper is therefore to find the proper values of h_t and $H_{t,\tau}$ with $t \in \mathbb{Z}_{[k,k+N-1]}$ such that the system state can be controlled to the reference point x_r, and the constraints (2) hold with high probability.

3. REFORMULATION OF PREDICTED LINEARIZED SYSTEMS

In this section we introduce the formulation of the predicted state under control laws (5) within the prediction horizon N. In an effort to simplify notation when constructing the optimization problem to determine the control law, we derive a compact form of the dynamical system. The predicted state for N-step prediction is described as

$$x_{[k:k+N]} = (B_x H_N(\tilde{D}_y + \tilde{E}_y) + \tilde{D}_x)\tilde{w}_{[k:k+N-1]}, \tag{6}$$

where the measurement of the initial state at k is described as x_0. The reformulation to get to the above expression is as follows.

To get the stacked state within the prediction horizon N in terms of disturbances, we firstly write the vector of purified outputs in terms of $x_0 - x_r$ for any purified output $v_t = CA^{N-1}(x_0 - x_r) + CA^{N-2}Dw_0 + \cdots + CDw_{N-2} + Ew_{N-1}$ starting from $t = k$ to $k + N - 1$:

$$\begin{aligned} v_{[k,k+N-1]} &= y_{[k,k+N-1]} - \hat{y}_{[k,k+N-1]} \\ &= (D_y + E_y)\, w_{[k,k+N-1]} + C_y(x_0 - x_r). \end{aligned} \tag{7}$$

Then, we denote the predicted system states as

$$x_{[k,k+N]} = A_x x_0 + B_x u_{[k,k+N-1]} + D_x w_{[k,k+N-1]} + A_{ext}\delta_k \tag{8}$$

and also reformulate the system inputs for $N-1$ steps from (5) and (7) into

$$u_{[k:k+N-1]} = H_N(\tilde{D}_y + \tilde{E}_y)\tilde{w}_{[k:k+N-1]} = \tilde{H}_N \tilde{w}_{[k:k+N-1]}, \tag{9}$$

where $\tilde{w}_{[k:k+N-1]} = \begin{bmatrix} 1 & w_{[k:k+N-1]}^\top \end{bmatrix}^\top$ is the extended disturbance vector, $\tilde{D}_y = \begin{bmatrix} 0 & 0 \\ A_y(x_0 - x_r) & D_y \end{bmatrix}$ and $\tilde{E}_y = \begin{bmatrix} 1 & 0 \\ 0 & E_y \end{bmatrix}$. H_N is the decision matrix which contains all the decision variables h_t and $H_{t,\tau}$ in (5). In the same way, the predicted state is formulated as

$$x_{[k:k+N]} = B_x u_{[k:k+N-1]} + \tilde{D}_x \tilde{w}_{[k:k+N-1]}, \tag{10}$$

where $\tilde{D}_x = [A_x x_0 + A_{ext}\delta \;\; D_x]$.

Lastly, by replacing the vector of inputs in (10) by (9), we derive the stacked system states in terms of disturbances in the form of (6).

For the stacked system states, the constraints within the prediction horizon N are described as

$$\sup_{\mathbb{Q}\in\mathbb{P}} \mathbb{E}^{\mathbb{Q}}[\tilde{F}x_{[k:k+N]} + \tilde{G}u_{[k:k+N-1]}] \leq \tilde{\not{F}}. \tag{11}$$

The matrices for the stacked system (10) and the expected linear constraints (11) can be found in the Appendix B.

4. DISTRIBUTIONALLY ROBUST MPC

4.1 Ambiguity Sets and the Wasserstein metric

Distributionally robust optimization is an optimization model which utilizes the partial information about the underlying probability distribution of the random variables in a stochastic model. To characterize the partial information about the true distribution $\mathbb{P}$, we define an ambiguity set (Wiesemann et al., 2014) which contains a set of probability measures on the measurable space (Ω, X). In this paper, this ambiguity set is modelled as a Wasserstein ball centered at the discrete empirical distribution. The Wasserstein ball is a discrepancy model wherein the distance on the probability distribution space is described by the Wasserstein metric. The Wasserstein metric defines the distance between all probability distributions $\mathbb{Q}$ supported on Ξ with finite p-moment $\int_\Xi \|\xi\|^p \mathbb{Q}(d\xi) < \infty$.

Definition 2. (Wasserstein Metric (Piccoli and Rossi, 2014)). The *Wasserstein metric* of order $p \geq 1$ is defined as $d_w : \mathbb{M}(\Xi) \times \mathbb{M}(\Xi) \to \mathbb{R}$ for all distribution $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbb{M}(\Xi)$ and arbitrary norm on $\mathbb{R}^{n_\xi}$:

$$d_{\mathrm{W}}(\mathbb{Q}_1, \mathbb{Q}_2) := \inf_{\Pi} \left\{ \left(\int_{\Xi^2} \|\xi_1 - \xi_2\|^p \, \Pi(\mathrm{d}\xi_1,\ \mathrm{d}\xi_2) \right)^{1/p} \right. \tag{12}$$

where Π is a joint distribution of ξ_1 and ξ_2 with marginals $\mathbb{Q}_1$ and $\mathbb{Q}_2$ respectively.

The ambiguity set $\mathbb{P}$ can then be formulated in terms of the Wasserstein metric (12) as

$$\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right) := \left\{ \mathbb{Q} \in \mathbb{M}(\Xi) : d_{\mathrm{W}}\left(\widehat{\mathbb{P}}_{N_s}, \mathbb{Q}\right) \leq \varepsilon \right\}, \tag{13}$$

where $\widehat{\mathbb{P}}_{N_s}$ is the discrete empirical probability distribution and ε is the radius of the Wasserstein ball. The empirical probability distribution $\widehat{\mathrm{P}}_{N_s} := \frac{1}{N_s}\sum_{i=1}^{N_s} \delta_{\widehat{\xi}_i}$ is a uniform distribution on the training data set $\widehat{\Xi}_{N_s} := \left\{\widehat{\xi}_i\right\}_{i\leq N_s} \subseteq \Xi$, and $\delta_{\widehat{\xi}_i}$ is the Dirac distribution which concentrates unit mass at the point $\widehat{\xi}_i \in \mathbb{R}^{n_\xi}$.

The radius ε tunes the size of the Wasserstein ball (13), which should be large enough to contain the true distribution but not unnecessarily large to prevent from including irrelevant distributions and making the problem over-conservative (Zhao and Guan, 2018). This variable is treated as a hyperparameter for the corresponding DRO. The solution of this DRO lies between the classical robust optimization on the support set and stochastic optimization, i.e. sample average approximation of the discrete empirical distribution.

4.2 Data-Based Distributionally Robust MPC

Given the system in (10), we construct a distributionally robust optimization to find the sequence of inputs in (9) such that system states can be controlled to the reference point x_r, and the distributionally robust constraints in (2) are satisfied for the ambiguity set $\mathbb{P} = \mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)$. The ambiguity set is centered at the discrete distribution of the sample of stacked disturbances for N steps, i.e. $\xi = w_{[k,k+N-1]}$. Since we assume the disturbances to be i.i.d., each sample of ξ_i takes $N-1$ values of disturbance w from the data set collected either online or offline. We therefore reformulate (2) into

$$\sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}^{\mathbb{Q}}[\ell(x_0, \xi, H_N)] \leq \tilde{\not{F}}, \tag{14}$$

where $\ell(x_0, \xi, H_N) = [(\tilde{F}B_x + \tilde{G})H_N(\tilde{D}_y + \tilde{E}_y) + \tilde{F}\tilde{C}_x]\tilde{\xi}$ with $\tilde{\xi} = \begin{bmatrix} 1 \\ \xi \end{bmatrix}$ is the function for predicted states within the prediction horizon N in terms of the initial state at sampling instant k, disturbances, and decision matrix.

Since the aim of this control problem is to find admissible control laws with regard to the distributionally robust constraints such that system states can be steered towards the reference state x_r while complying with the constraints (14), we characterize the objective function as the discounted sum of quadratic difference costs

$$J_N(x, H_N) := \sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}_{\mathbb{Q}} \left\{ \sum_{t=k}^{k+N-1} \beta^{t-k} \left[(x_t - x_r)^\top Q (x_t - x_r) + (u_t - u_r)^\top R (u_t - u_r) \right] + \beta^N (x_{k+N} - x_r)^\top Q_f (x_{k+N} - x_r) \right\}, \tag{15}$$

with β as the discount factor. It is further assumed that $Q, Q_f \in \mathbb{S}_+$and $R \in \mathbb{S}_{++}$ so that J_N is non-negative and also convex. The stochastic optimal control problem to determine affine control laws with N-step prediction horizon is formulated as

$$\begin{aligned} \inf_{H_N} \quad & J_N(x, H_N) \\ \text{s.t.} \quad & \sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}^{\mathbb{Q}}[\ell(x_0, \xi, H_N)] \leq \tilde{\not\kappa}. \end{aligned} \tag{16}$$

5. A TRACTABLE CONVEX CONE REFORMULATION

In this section, we introduce how to reformulate the stochastic optimal control problem in (16) with type-1 ($p = 1$) Wasserstein metric into a finite-dimensional convex conic program such that affine POB control laws can be efficiently determined.

Assumption 3. (i.i.d. Disturbance). We assume that in the discrete-time nonlinear system (1), the disturbance w_t is an i.i.d. random process with covariance matrix Σ_w and mean μ_w for all $t \in \mathbb{N}_0$.

The i.i.d. random process is a common assumption made in control literature, e.g. Arcari et al. (2020); Coppens and Patrinos (2021). It assumes a priori that only the first two moments of the random process are acquired as partial distributional information, which can either be estimated or prescribed a priori (Wan et al., 2014).

Assumption 4. (Moment Assumption). There exists a positive α such that $\int_\Xi \exp(\|\xi\|^\alpha)\, \mathbb{Q}(\mathrm{d}\xi) < \infty$ (Fournier and Guillin, 2015).

This assumption trivially holds for a bounded uncertainty set Ξ and finite measure $\mathbb{P}$.

Assumption 5. (Polytope Uncertainty Set). The space $\mathbb{M}(\Xi)$ of all probability distributions $\mathbb{Q}$ is supported on a polytope $\Xi := \{\xi \in \mathbb{R}^{n_\xi} : C_\xi \xi \leq d_\xi\}$ (Esfahani and Kuhn, 2018).

Theorem 6. (Tractable convex program). The optimal control problem (16) with discounted quadratic costs, distributionally robust constraints within a Wasserstein ball $\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)$ centered at the empirical distribution $\widehat{P}_{N_s}$ with N_s number of samples, and radius ε can be reformulated as the cone program (17) using the affine control laws (9) and under Assumptions 3, 4.

$$\begin{aligned} \inf_{\substack{\tilde{H}_N, \lambda_j, \\ s_{ij}, \gamma_{ij}}} \quad & \mathrm{Tr}\left\{ \left[(\tilde{D}_{xobj} + B_x \tilde{H}_N)^\top J_x (\tilde{D}_{xobj} + B_x \tilde{H}_N) + \tilde{H}_N^\top J_u \tilde{H}_N \right] \tilde{\Sigma}_w \right\} \\ & + \tilde{\mu}_w^\top \left[(\tilde{D}_{xobj} + B_x \tilde{H}_N)^\top J_x (\tilde{D}_{xobj} + B_x H_N) + \tilde{H}_N^\top J_u \tilde{H}_N \right] \tilde{\mu}_w \\ \text{s.t.} \quad & \lambda_j \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_{ij} \leq 1 \\ & b_j + \left\langle a_j, \widehat{\xi}_i \right\rangle + \left\langle \gamma_{ij}, d_\xi - C_\xi \widehat{\xi}_i \right\rangle \leq s_{ij} \\ & \left\| C_\xi^\top \gamma_{ij} - a_j \right\|_\infty \leq \lambda_j, \quad \gamma_{ij} \geq 0 \\ & \forall i \in \mathbb{Z}_{[1,N_s]}, \forall j \leq \mathbb{Z}_{[1, N_c \times N + N_{ct}]}, \end{aligned} \tag{17}$$

where $J_x := \mathrm{diag}\left(\mathrm{diag}\left(\beta^0, \ldots, \beta^{N-1}\right) \otimes Q, \beta^N Q_f\right)$, $J_u := \mathrm{diag}\left(\beta^0, \ldots, \beta^{N-1}\right) \otimes R$, $a_j = [(\tilde{F} B_x + \tilde{G}) H_N (\tilde{D}_y + \tilde{E}_y) + \tilde{F} \tilde{C}_x]_{[j,2:]}$ and $b_j = [(\tilde{F} B_x + \tilde{G}) H_N (\tilde{D}_y + \tilde{E}_y) + \tilde{F} \tilde{C}_x]_{[j,1]}$.The integers N_s, N_c and N_{ct} denote the number of samples and the the row size of the matrix F, and the the row size of the matrix F_T respectively. $\widehat{\xi}_i$ indicates a data point in the training data set, comprising the disturbance sequence consisted of N sampling times. $\tilde{D}_{xobj}$ is formulated as $\tilde{D}_{xobj} = [A_x x_0 + A_{ext}\delta - \tilde{x}_r \;\; D_x]$.

Proof. See Appendix A.

Remark 7. For any linear system satisfying Assumption 4 controlled by the control laws derived from (17) with ambiguity set $\mathbb{B}_{\varepsilon(N_s,\beta)}\left(\widehat{\mathbb{P}}_{N_s}\right)$, and training data set $\widehat{\Xi}_{N_s}$, the finite sample guarantee with confidence level $1 - \beta$

$$\mathbb{P}^{N_s}\left\{ \mathbb{E}^{\mathbb{P}}[\ell(x_0, \xi, H_N)] \leq \tilde{\not\kappa} \right\} \geq 1 - \beta \tag{17}$$

holds, where $\beta \in (0, 1)$. For more details, please refer to (Zhong et al., 2021)[Th. 2].

6. RESULTS

6.1 Configuration

In this work two case studies are implemented; the control of an inverted pendulum and the control of a semi-batch bioreactor with an irreversible exothermic reaction. Both case studies discretize continuous time models and have the prediction horizon $N = 5$. Each entry of w_k complies with the random process $a \sin(X)$, where $X \sim \mathbb{N}(0, 1)$, a equal to 2 for the inverted pendulum and 1 for the bioreactor. Therefore, we acquire $C_\xi = \mathrm{diag}(1, \ldots, -1, \ldots)$ and $d_\xi = [a, \ldots, a] \in \mathbb{R}^{2Nn_w}$. Control laws are updated at each sampling time with one sample for the inverted pendulum and two samples for the bioreactor.

6.2 Nonlinear inverted pendulum

In this case study, we illustrate how it is possible to mitigate the constraint violations with by increasing the number of samples or the Wasserstein ball radius. We consider the nonlinear inverted pendulum system represented in (Prasad et al., 2014).

For both simulations, the weighting matrices Q and Q_f are selected as $\mathrm{diag}(1500, 1, 1000, 1)$, and R is set as $\mathrm{diag}(1)$. Both the angular velocity of the pendulum and the measurement of the pendulum position are disturbed by the factor $1e{-}2$, i.e. $C = [\,1\;0\;0\;0\,]$, $D^\top = \begin{bmatrix} 0 & \text{1e-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $E = [\,0\;\text{1e-2}\,]$. The angular velocity is required to be upper bounded by $0.65\,1/\mathrm{s}$.

Case 1: Simulation with increased number of samples: In the first simulation, we show that the increased number of samples could reduce the rate of constraint violation. At each sampling time, we take N_s samples of the disturbance $\{\xi_i\}_{i=1}^{N_s}$, each of which consists of N samples w from the set of offline collected samples. N_s ranges from 1 to 8.

Case 2: Simulation with increased radius of the Wasserstein ball: As in the first simulation, we use samples

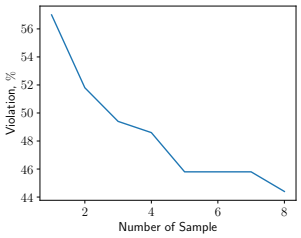

Fig. 1. Relation between sample number and constraint violations within first four seconds, averaged from 50 realizations.

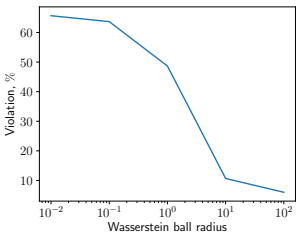

Fig. 2. Relation between Wasserstein ball radius and constraint violations within first two seconds, averaged from 50 realizations.

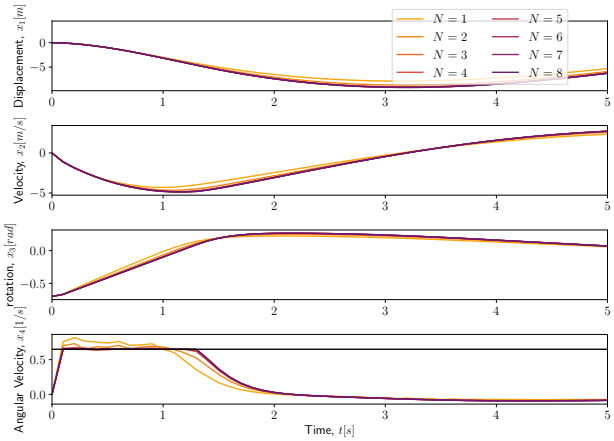

Fig. 3. Simulation results of (17) averaged from 50 realizations with sample number ranging from 1 to 8.

collected prior to the initialization to solve (17) at each sampling time. Since our interest in this section is to demonstrate the impact of the ball radius, simulations, each with 50 realizations, are carried out for various ball radii, ranging from 0.01 to 100 linearly in the log scale. The sampling number is fixed to 1 at each sampling time.

Results: For the case 1, as demonstrated in fig 3, the ball radius is fixed to 1. We simulate the state trajectory with sample numbers ranging from 1 to 8, each with 50 realizations. The control laws are determined at each sampling time with different samples collected prior to initialization to demonstrate the relation between the number of samples and constraint satisfaction. We can read from the figure that the averaged trajectory of 50 realizations with only 1 sample tends to significantly violate constraints from the starting point of each simulation. In contrast, as the number of collected samples increases, constraints are satisfied more robustly. Furthermore, results from fig 1 illustrate that the constraint violation is monotonically decreasing along the sample number. The rate of constraint violations for the first 0.7 seconds monotonically decreases from 58% to 44% as the number of samples applied at each sampling instant increases from 1 to 8.

For the case 2, as displayed in fig 4, the rotational velocity of the pendulum tends to violate the prescribed upper bound for the first 0.7 seconds when the radius of the Wasserstein ball has radius smaller than 1. The state trajectories violate constraints extensively since the center of the ball is roughly located only with one sample, and hence it is very likely that this ball does not contain the true distribution. As the ball radius increases, less violation is observed since the chance of containing the true distribution increases. When the radius increases to 10, constraints are satisfied for 90% of the simulations. We can also observe from figure 2 that the confidence of probabilistic constraints satisfaction monotonically increases as the Wasserstein ball expands when the number of sample is fixed.

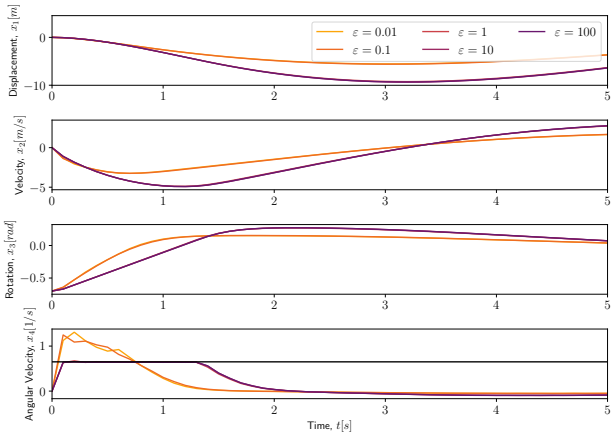

Fig. 4. Simulation results of (17) averaged from 50 realizations with Wasserstein ball radius ranging from 0.01 to 100 on the inverted pendulum.

6.3 Semi-batch bioreactor model

As the second case study, we consider semi-batch bioreactor in the paper (Bradford et al., 2020). The balance equations are described as follow:

$$\begin{aligned}
\frac{dx_1}{dt} &= u_m \cdot \frac{I}{I + k_s + \frac{I^2}{k_i}} \cdot x_1 \cdot \frac{x_2}{x_2 + K_N} - u_d \cdot x_1, \\
\frac{dx_2}{dt} &= -Y_{\frac{N}{X}} \cdot u_m \cdot \frac{I}{1 + k_s + \frac{I^2}{k_i}} \cdot x_1 \cdot \frac{x_2}{C_N + K_N} + u_1, \\
\frac{dx_3}{dt} &= k_m \cdot \frac{I}{I + k_{sq} + \frac{I^2}{k_{iq}}} \cdot x_1 - \frac{k_d x_3}{x_2 + K_{Np}},
\end{aligned} \tag{18}$$

where x_1, x_2, and x_3 are the concentrations of biomass, nitrate, and phycocyanin, respectively. The input of this system is the nitrate inflow rate u_1. The parameter values can be found in (Bradford et al., 2020)[Table 1] with the light intensity fixed to $150\,\mu\text{mol} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$.

The control problem is to have maximal yield of phycocyanin under uncertainties while complying with constraints. The prediction horizon is $5\,h$ with the sampling rate $1\,h$ up to $150\,h$. For both simulations, the weighting matrices Q and Q_f are selected as $\text{diag}(0,0,0)$ and $\text{diag}(0,0,-100)$ respectively, and R is set as $\text{diag}(0)$. The concentration of nitrate is disturbed by the factor 1, and the purified observation is noised by the factor 1e-2, i.e. $C = [1\ 0\ 0]$, $D^\top = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $E = [0\ \text{1e-2}]$. The nitrate concentration is required to be upper bounded by $800\,\text{mg/L}$ and the system input is constrained softly between 0 to 40.

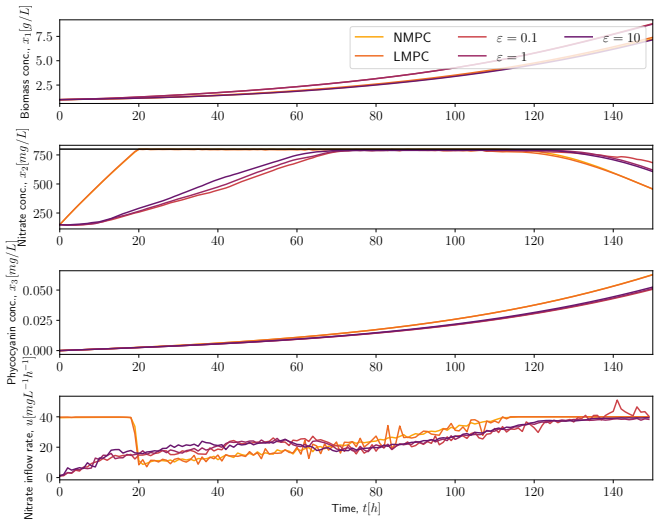

Fig. 5. Simulation of disturbed semi-batch bioreactor with nitrate concentration disturbance under difference MPC controllers.

To illustrate the effectiveness of our algorithm , we compare our result with nominal linearized and nonlinear MPCs. For our DRMPC controllers, we randomly select two samples from the data set collected offline and set the ball radius ranging from 0.1 to 10. With the same constraints , objective function and disturbances' generation, we run 30 Monte-Carlo simulations for each algorithm.

Algorithm	violate rate %
LMPC	16.8
NMPC	22.5
DRMPC $\epsilon = 0.1$	4.36
DRMPC $\epsilon = 1$	1.13
DRMPC $\epsilon = 10$	0.0

Table 1. Violation rate of different MPC algorithms

Results: In the figure 5 illustrates that DRMPC can control the bioproduct of the disturbed nonlinear system to above 5 mg/L. However the linearized and nonlinear MPC violate the constraint more than 10 times than DRMPC with radius $\epsilon = 1$. By tuning the Wasserstein ball radius to 10, constraint violation can be avoided entirely in this case study.

7. CONCLUSION

In this paper, we propose a novel data-driven distributionally robust MPC method for nonlinear systems using the Wasserstein ball as the ambiguity set. Our approach relies on building an ambiguity set defined by the Wasserstein metric which allows to characterize the uncertainty even when limited information on the probability distributions is available. In this approach we reformulate the distributionally robust optimal control problem into a tractable convex cone program with finite sample guarantee and apply POB control laws derived from linearized systems to the corresponding nonlinear systems. Monte-Carlo simulations of numerical case studies on two systems are conducted to illustrate the effectiveness of the algorithm introduced in the paper.

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Appendix A. PROOF OF THEOREM 6

Proof. We shall now prove the equivalence of the objective function and constraints in (16) and (17) respectively. We first show the objective function of both optimization problems are equivalent. For the convenience of notation, we reformulate the predicted states in the objective function (15) dependent on system inputs. (15) is then equivalent to

$$\inf_{H_N} \sup_{\mathbb{P}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}_{\mathbb{P}}(\tilde{w}_{[k:k+N-1]}^\top J_{total} \tilde{w}_{[k:k+N-1]}^\top), \qquad \text{(A.1)}$$

where $J_{total} = [(\tilde{D}_{xobj} + B_x\tilde{H}_N)^\top J_x(\tilde{D}_{xobj} + B_x\tilde{H}_N) + \tilde{H}_N^\top J_u \tilde{H}_N]$, $\tilde{D}_{xobj} = [A_x x_0 + A_{ext}\delta - \tilde{x}_r \; D_x]$ and $\tilde{H}_N = H_N(\tilde{D}_y + \tilde{E}_y)$. Then, under Assumption 3 and known or computed from the data already, the objective is equivalent to $\inf_{H_N}(\tilde{\mu}_w^\top J_{total}\tilde{\mu}_w + \operatorname{Tr} J_{total}\Sigma_w)$ as the the expectation of the quadratic cost (Seber and Lee, 2012, THEOREM 1.5). Here $\tilde{\mu}_w$ and Σ_w denote the expectation and covariance of $\tilde{w}_{[k:k+N-1]}$ respectively.

The next step is to show that constraints in (16) and (17) are equivalent. Given that constraints in (16) requiring that the expectation of the linear constraints of states and inputs to be bounded under the worst-case distribution, we reformulate the these stochastic linear constraints into several convex deterministic constraints. Since the satisfaction of the constraints in (14) is equivalent to each entry of the column vector ℓ smaller than one, we show that the inequality constraints (14) can be reformulated into a convex constraint.

First, each j-th entry in the expected linear constraints can be described as the j-th constraint of $\sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}^{\mathbb{Q}}[\tilde{F}x_{k:k+N} + \tilde{G}u_{[k:k+N-1]}] \leq \tilde{\not{F}}$. By reformulating the stacked state and input vector in terms of the initial state x_0, disturbances ξ, and the decision matrix H_N, $\ell(x_0, \xi, H_N)$, the left-hand side of the expected linear constraints $\tilde{F}x_{k:k+N} + \tilde{G}u_{[k:k+N-1]}$ is equivalent to $\ell(x_0, \xi, H_N) = [(\tilde{F}B_x + \tilde{G})H_N(\tilde{D}_y + \tilde{E}_y) + \tilde{F}\tilde{C}_x]\tilde{\xi}$ with $\tilde{\xi} = \begin{bmatrix}1\\ \xi\end{bmatrix}$.

Then, given that the stacked state is represented by the initial state x_0, disturbance sequence and inputs in (8), the j-th component of the vector ℓ can be separated into two terms, one of which contains disturbance sequence and the another does not. Hence $[\ell(x_0, \xi, H_N)]_j$ is further described as $b_j + a_j^\top \xi$, where $b_j = [(\tilde{F}B_x + \tilde{G})H_N(\tilde{D}_y + \tilde{E}_y) + \tilde{F}\tilde{C}_x]_{[j,1]}$ and $a_j = [(\tilde{F}B_x + \tilde{G})H_N(\tilde{D}_y + \tilde{E}_y) + \tilde{F}\tilde{C}_x]_{[j,2:]}$. Clearly, the constraints in (14) are equivalent to the distributionally robust constraints of piecewise maximum of affine functions $b_j + a_j^\top \xi, \forall j \leq \mathbb{Z}_{[1,N_c\times N+N_{ct}]}$.

Finally, by leveraging the result from (Esfahani and Kuhn, 2018, Corollary 5.1), the distributionally robust constraints in (16) are rewritten into "best-case" constraints

$$\inf \lambda_j \varepsilon + \frac{1}{N}\sum_{i=1}^{N} s_{ij} \leq 1 \qquad \text{(A.2)}$$

along with several additional inequalities. Hence, any feasible solutions of the tractable reformulation (17) guarantee constraints satisfaction of (2). We thus prove the equivalence of the distributionally robust optimization problem (16) and cone program in the form of (17).

Appendix B. NOTATIONS

$C_y = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ CD & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 \\ CA^{N-2}D & \cdots & CD & 0 \end{bmatrix}$, $E_y = \begin{bmatrix} E & & \\ & \ddots & \\ & & E \end{bmatrix}$, $D_y = \begin{bmatrix} DA^0 \\ \vdots \\ DA^{N-1} \end{bmatrix}$, $A_x = \begin{bmatrix} A^0 \\ \vdots \\ A^N \end{bmatrix}$, $B_x = \begin{bmatrix} 0 & \cdots & 0 \\ A^0B & \ddots & \vdots \\ \vdots & \ddots & 0 \\ A^{N-1}B & \cdots & A^0B \end{bmatrix}$, $D_x = \begin{bmatrix} 0 & \cdots & 0 \\ A^0D & \ddots & \vdots \\ \vdots & \ddots & 0 \\ A^{N-1}D & \cdots & A^0D \end{bmatrix}$, $A_{ext} = \begin{bmatrix} 0 & \cdots & 0 \\ A^0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ A^{N-1} & \cdots & A^0 \end{bmatrix}$. $\tilde{F} = \begin{bmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F_T \end{bmatrix}$, $\tilde{G} = \begin{bmatrix} G & & & \\ & G & & \\ & & \ddots & \\ & & & G \\ 0 & 0 & \dots & 0 \end{bmatrix}$, where F and G are matrices corresponding to the expected linear constraints within the prediction horizon N, and F_T is the matrix for the terminal state. $\tilde{\mathbb{F}} = \begin{bmatrix} \mathbb{F} \\ \vdots \\ \mathbb{F} \end{bmatrix}$ is the stacked column vector on the right hand side of the linear constraints (11).