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Design of a Control Lyapunov Function for Stabilizing Specified States ⋆

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Abstract: This paper focuses on the design of control Lyapunov function for control affine systems to guarantee the stability for the states of interest in a specified region. Without restrictive assumptions found in previous approaches, a min-max optimization problem is formulated to solve for a quadratic Lyapunov function. A derivative-free coordinate search method is employed to optimize a non-differentiable objective function. Approximation of the objective function as a piecewise linear function gradually reduces search space, leading to an effective Lyapunov function. A CSTR example with actuator saturation is illustrated to demonstrate the efficacy of the proposed approach.

Keywords: Lyapunov stability, Minimax techniques, Control affine system, Coordinate search

1. INTRODUCTION

Though nonlinear models are commonly used for describing chemical/biochemical process dynamics, designing a feedback control law that stabilizes such systems is not an easy task. Even for special cases, such as a control affine system, only a few approaches are available under restrictive assumptions. For instance, nonlinear model predictive control can only guarantee the stability when a local linear model is stable and the nonlinear programming has a feasible solution for an initial state (Chen and Allgöwer, 1998). Backstepping provides a stable and recursive control for the lower triangular system with known control law and Lyapunov function for its subsystems (Krstic et al., 1995). Feedback linearization techniques transform a nonlinear system to an equivalent linear system via a diffeomorphism transformation for the state, input, and output. Nevertheless, such transformation may hold only for a small region and the input constraints are not considered explicitly (Isidori, 1995).

Control Lyapunov function (CLF) based approaches have recently received increasing attentions due to their explicit consideration of the stability prior to the regulator design. Once this CLF is constructed, design of a feedback law (e.g., $u_t = Kx_t$) can be straightforward (Sontag, 1989). Though much of the previous work has focused on improving control algorithm or enhancing the initial region where closed loop stability can be achieved based on a known CLF (e.g., see Mhaskar et al. (2006); Mahmood and Mhaskar (2008); Zhong et al. (2008); Primbs and Giannelli (1998)), the main bottleneck to the success of these methods lies in the construction of Lyapunov function. One simple approach is to linearize a nonlinear system and solve the Riccati equation and obtain a quadratic CLF, but the inherent discrepancy between linear and nonlinear dynamics makes the analysis difficult.

Not only a systematic rule to find a CLF for nonlinear systems is currently unknown but the region of attraction (ROA) of the resulting CLF also needs further investigation. A sum of squares programming based approach is proposed to find a CLF for polynomial dynamics system without considering input constraints (Tan, 2006). Construction of a CLF for the system under Jurdjevic-Quinn conditions is also proposed (Mazenc and Malisoff, 2005). Zubov's partial differential equation is solved to design a CLF (Dubljevic and Kazantzis, 2002), which cannot specify a ROA. A genetic programming based approach is employed to design a CLF for one-input systems only (Tsuzuki et al., 2006). Some optimization based frameworks, similar to the proposed scheme in this work, are proposed to compute a CLF (Davision and Kurak, 1971; Johansen, 2000). However, these approaches do not consider state regions of interest for stabilization. In addition, these are computationally expensive.

This work presents a method that constructs a CLF, which stabilizes the states in a specified region without restrictive assumptions as in the previous work. Construction of a CLF is formulated as a constrained min-max problem and its solution method is also presented. The rest of the paper is organized as follows: some preliminaries on CLF are given in Section 2; Section 3 describes the formulation and parametrization of CLF and its optimization procedure. Simulation results are given in Section 4. Finally, concluding remarks are provided.

2. PRELIMINARY

2.1 Control Lyapunov function

We consider the following nonlinear control affine system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

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where $x \in \Re^n$ is the state vector, $f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T \in \Re^n$, $g(x) = [g_1(x), g_2(x), \ldots, g_m(x)] \in \Re^{n\times m}$, and input vector $u \in \Re^m$ satisfies $a_i \leq u_i \leq b_i$, $i = 1, 2, \ldots, m$.

A function $V : \Re^n \to \Re$ is a control Lyapunov function if (i) V is positive definite and (ii) the following inequality holds:

$$\inf_{u\in\Re^m} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u < 0 \quad \forall x \neq 0 \tag{2}$$

2.2 Region of attraction

Since the global asymptotical stability is very strict and cannot be achieved for most nonlinear systems with input constraints, the local asymptotical stability and its region of attraction (ROA) is analyzed. Usually, it is determined as a set related to a Lyapunov function. Denote the equilibrium point as x_0, if there exists $r > 0$ satisfying:

$$V(\tilde{x}) < r \Rightarrow \lim_{t\to\infty} ||x(t,\tilde{x}) - x(t,x_0)|| = 0 \tag{3}$$

where $x(t,\tilde{x})$ is the state at time t with an initial state $\tilde{x}$. Given the largest possible value for r, the estimation of ROA is defined as: $\Omega = \{x|V(x) < r_{max}\}$. Moreover, there is $\Omega \subset \Gamma$ such that

$$\Gamma = \{x| \inf_{u\in\Re^m} \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u < 0\} \tag{4}$$

Fig 1 shows the relationship between Ω and Γ.

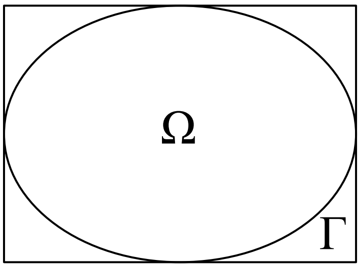

Fig. 1. The largest region of attraction.

2.3 Control law design

For the control affine system with bounded input, the following Lyapunov function-based feedback law is proposed in Lin and Sontag (1991); El-Farra and Christofides (2003):

Define the $a(x)$ and $b(x)$ as:

$$a(x) = \frac{\partial V}{\partial x} f(x) \tag{5}$$

$$b(x) = \frac{\partial V}{\partial x} g(x) \tag{6}$$

Then the control policy is prescribed as:

$$u(x) = \begin{cases} K(x)b^T(x) & \text{if } \|b(x)\| \neq 0 \\ 0 & \text{if } \|b(x)\| = 0 \end{cases} \tag{7}$$

where both $a(x)$ and $b(x)$ are vectors, u^{max} is the maximum magnitude of the Euclidean norm of the vector of manipulated inputs allowed by the constraints, and $K(x) = -\frac{a(x)+\sqrt{a(x)^2+(u^{max}\|b(x)\|)^4}}{\|b(x)\|^2[1+\sqrt{1+(u^{max}\|b(x)\|)^2}]}$.

3. PROPOSED APPROACH

Constructing the CLF with the largest possible ROA for all possible states ends up with a small ROA in general, thus limiting its application to practical problems. Oftentimes, it is only necessary to guarantee stability for certain operating regions. Motivated by this, this section proposes a min-max approach to obtaining a Lyapunov function given a subset of states, which we refer to "target region." Without loss of generality and for simplicity, the target region, Ψ, can be defined as a polytope $\Psi = \{x|\, Lx < c\}$.

The objective is to design a CLF and guarantee that its ROA includes a target region. For simplicity, we assume that there is no uncertainty in the system. In this case, once the target region is in the ROA, the Lyapunov theorem can guarantee the state always lies in the invariant set. Since the CLF at least should be locally positive definite, a simple choice is the symmetric matrix for P in

$$V(x) = x^T P x \tag{8}$$

where P is a positive definite matrix. Hence, the following optimization problem can be formulated to find V:

Problem 1

$$\begin{aligned} &\max_{x,\hat{x}} \hat{x}^T P \hat{x} - x^T P x \\ \text{subject to } &\inf_{u\in\Re^m} \frac{\partial V}{\partial t} f(x) + \frac{\partial V}{\partial t} g(x)u > 0 \\ &\hat{x} \in \Psi \end{aligned} \tag{9}$$

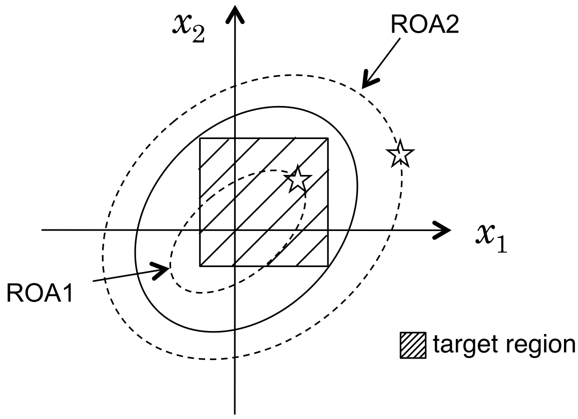

Fig. 2. Geometric illustration of Problem 1.

Solution to the *Problem 1* yields the maximum difference in Lyapunov function values of the states in the target regions and the unstable states. Fig. 2 is a geometric illustration of Problem 1. Given two matrices P_1 and P_2, the stars represent the state point in $\{x|V(x) \geq r_{\max}\}$ for each ROA, respectively. If the maximum difference of the CLF values is negative, this means the target region is included in ROA (ROA2 in Fig. 2) because the value of the CLF for any point outside the ROA is larger than that of the target region. Hence, P that minimizes *Problem 1* is found by formulating the following minimax optimization:

Problem 1'

$$\min_{P} \Phi(P) \tag{10}$$

$$\Phi(P) = \max_{x,\hat{x}} \hat{x}^T P \hat{x} - x^T P x \tag{11}$$

$$\text{subject to} \quad \inf_{u \in \Re^m} \frac{\partial V}{\partial t} f(x) + \frac{\partial V}{\partial t} g(x) u > 0 \tag{12}$$

$$\hat{x} \in \Psi \tag{13}$$

$$P > 0 \tag{14}$$

where minimization is used to find a negative value of Φ.

As will be shown later, the constraint of Eq. (12) is converted to an expression containing an absolute value function. This makes the minimax optimization non-differentiable, which is nontrivial to solve. Very few methods are suitable for this type of optimization problem. Convex approximation of the min-max objective function is proposed in Kiwiel (1987). This, however, cannot handle the constraint of variable P. Reference variable methods (Lu et al., 2008) and interior point algorithm (Rustem et al., 2008) are shown to solve nonlinear minimax problems with constraints, but they both require the objective function to be continuous and differentiable. Moreover, all of these methods cannot handle the constraints in P and x at the same time.

In summary, there are three issues that need to be addressed in solving *Problem 1'*:

- Eq. (12) imposes constraints on both P and x, which are difficult to handle with existing minimax optimization methods.
- The inequality constraint Eq. (14) is also difficult to handle. Usually, it should be transformed to n sub-inequality constraints by Sylvester's criterion.
- Because of the absolute operator in the constraint, the derivative cannot be obtained (See Remark 2).

In order to handle the non-differentiable constraints, a derivative-free optimization method based on a coordinate search algorithm is proposed. In determining P, one of the elements, P_{ij}, is chosen as a variable to be determined and its search space is computed (See Remark 3). After an approximate $\tilde{\Phi}$ is computed based on the old matrix, a new value for P_{ij} is selected and the max of the objective function is evaluated to obtain $x_0^{\max}$ and $\hat{x}_0^{\max}$. It is easy to iteratively improve $x_0^{\max}$ and $\hat{x}_0^{\max}$ by calculating the upper envelope of the old approximation and a new piecewise function $\tilde{\Phi}^k$ because the objective is piecewise linear in P_{ij}. If a new P_{ij} yields significant decrease in the objective value, the new P_{ij} is chosen and another parameter in P is searched for. The algorithm is terminated when there is no further decrease in the objective value. With the parameters denoted as P_l, $l = 1, 2, \ldots, \frac{n^2+n}{2}$, where n is the number of states, the details of this approach are described as follows:

Step 0: Set $l = 1$ and calculate the objective function Φ_{min} based on initial parameters.

Step 1: Set $k = 1$. Calculate the upper bound $\overline{P_l}$ and lower bound $\underline{P_l}$ of P_l as in Remark 3. Approximate the objective function in $[\underline{P_l}, \overline{P_l}]$ as piecewise linear function $\tilde{\Phi}^k$ **(See Remark 4)**. Let $\tilde{\Phi} = \tilde{\Phi}^k$ and switch to a new matrix P^k by changing P_l only.

Step 2: Evaluate the function value $\Phi(P^k)$ and obtain the $[x^k, \hat{x}^k]$:

$$[x^k, \hat{x}^k] = \arg\max_{x^k, \hat{x}^k} \Big[(\hat{x}^T P^k \hat{x} - x^T P^k x) + \lambda \inf_{u \in \Re^m} x^T P^k (f(x) + g(x)u) \Big] \tag{15}$$

where λ is the Lagrange multiplier.

If $\Phi(P^k) < 0$, stop the algorithm and set $P = P^k$.

If $\Phi_{min} - \Phi(P^k) > \delta$, where δ is a positive threshold, set $\Phi_{min} = \Phi(P^k)$, $l = l + 1$ and go to Step 1.

Step 3: Based on the current $[x, \hat{x}]$, compute the piecewise linear function $\tilde{\Phi}^k$ in the interval of $[\underline{P_l}, \overline{P_l}]$. Then, $\tilde{\Phi} = \max(\tilde{\Phi} \cup \tilde{\Phi}^k)$ and $q = \min_{P_l} \tilde{\Phi}$. If $\Phi_{min} - q < \epsilon$, where ϵ is a small positive value, let $l = l + 1$ and go to Step 1.

Step 4: Set $k = k + 1$ and

$$P^k = \arg\min_{P_l} \Big[\lambda \inf_{u \in \Re^m} (x^k)^T P (f(x^k) + g(x^k)u) + ((\hat{x}^k)^T P \hat{x}^k - (x^k)^T P x^k) \Big] \tag{16}$$

then switch to Step 2.

Remark 1: Since this algorithm cannot guarantee the global optimum, the initial guess plays an important role. In order to obtain a good guess, we suggest to use a linearized model and solve the Riccati equation. For the linearized model:

$$\dot{x} = Ax + Bu \tag{17}$$

If the (A, B) is stabilizable, the Riccati equation, Eq. (18), has a unique solution for a positive definite matrix Q.

$$A^T P + PA - PBB^T P + Q = 0 \tag{18}$$

Although the stable region of the nonlinear system is unknown, a local region near the equilibrium point is stabilized by this method.

Remark 2: In *Problem 1'*, minimizing the objective function with respect to the input u is also worthwhile to discuss.

With the following input saturation constraints:

$$a_t \leqslant u_t \leqslant b_t \quad t = 1, 2, \ldots, m, \tag{19}$$

Because the constraint in Eq. (12) is linear in u given x and multiplied by a negative coefficient, a bang-bang control is adopted to enlarge the ROA. That is, the value of input determined only by the sign of $Pxg(x)$. The input will take the minimum value and maximum value for the positive and negative values of $Pxg(x)$, respectively. Given the input, Eq. (12) becomes

$$\begin{aligned} S &= \inf_{u \in \Re^m} Px(f(x) + g(x)u) \\ &= x^T P f(x) - \frac{(b_t - a_t)}{2} \sum_{t=1}^{m} \left| \sum_{i=1}^{n} \sum_{j=1}^{n} x_i P_{ij} g_{jt}(x) \right| \\ &+ \frac{(b_t + a_t)}{2} \sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i P_{ij} g_{jt}(x) \end{aligned} \tag{20}$$

The minimization is replaced by the absolute magnitude, which is easily handled by any nonlinear programming solver.

Remark 3: In Step 1, the lower bound and upper bound for each single parameter need to be computed. For any off-diagonal element P_{ij} that needs to be determined, the following elementary operations can be used to transform it to:

$$\hat{P} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & P_{ij} \\ \cdots & \cdots & P_{ij} & \hat{P_{nn}} \end{pmatrix} \quad (21)$$

For the element P_{ii}, it can be transformed to

$$\hat{P} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & P_{ii} \end{pmatrix} \quad (22)$$

Since elementary operations can keep the positive definiteness of P during the procedure, Cholesky decomposition can be used as follows:

$$\hat{P} = LL' = \begin{pmatrix} L_{11} & 0 & \ldots & 0 \\ L_{21} & L_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & \ldots & l_{n1} \\ 0 & L_{22} & \ldots & L_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & L_{nn} \end{pmatrix} \quad (23)$$

where L is a triangular matrix. From Cholesky-Banachiewicz or Cholesky-Crout algorithm (see Golub and Loan (1996)) the following recursive formula for $L_{i,j}$ can be obtained.

$$L_{ij} = \frac{1}{L_{jj}}(\hat{P}_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk}) \quad i > j \quad (24)$$

$$L_{ii} = \sqrt{\hat{P}_{ii} - \sum_{k=1}^{i-1} L_{ik}^2} \quad (25)$$

Hence, for the element P_{ii}, from Eq. (25), we have

$$P_{ii} > \sum_{k=1}^{n-1} L_{nk}^2 \quad (26)$$

For the P_{ij}, at first, using Eq. (26)

$$P_{ii} - \sum_{k=1}^{n-2} L_{nk}^2 > L_{n,n-1}^2$$

$$\sqrt{\hat{P}_{nn} - \sum_{k=1}^{n-2} L_{nk}^2} > L_{n,n-1} > -\sqrt{\hat{P}_{nn} - \sum_{k=1}^{n-2} L_{nk}^2} \quad (27)$$

Notice that $P_{ij} = \hat{P}_{n,n-1}$, then plugging (27) into Eq. (24) and taking the place of i,j by n and $n-1$, respectively, which yields

$$\sum_{k=1}^{j-1} L_{ik}L_{jk} + L_{n-1,n-1}\sqrt{\hat{P}_{nn} - \sum_{k=1}^{n-2} L_{nk}^2} > P_{ij} \quad (28)$$

and

$$\sum_{k=1}^{j-1} L_{ik}L_{jk} - L_{n-1,n-1}\sqrt{\hat{P}_{nn} - \sum_{k=1}^{n-2} L_{nk}^2} < P_{ij} \quad (29)$$

Note that locating the elements to be optimized in the right-most column and second-last row and its transpose location using elementary operations, Cholesky-Banachiewicz and Cholesky-Crout algorithms can be effectively used to find the bounds on the optimization variables.

Remark 4: In general, the coordinate search has slower convergence rate than the gradient method. However, in the case of point-wise function $\Phi(P)$, no derivative is available and a large number of nonlinear constraints are difficult to handle with such kind of approaches. Moreover, note that fixing x and $\hat{x}$ yields piecewise linear function $\Phi(P)$ with a single variable P_{ij}. Thus, the objective function for a single variable can be estimated by the union of different x and $\hat{x}$ values as shown in Fig. 3. This

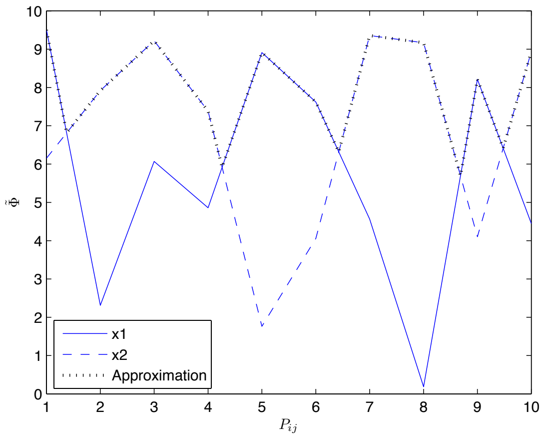

Fig. 3. The upper bound of two lines is used to approximate the objective function.

estimate function can be described only by a few points which is trivial to store and search, and it represents the lower bound of the true objective function. For the purpose of searching for a suitable P_{ij} that decreases the objective function, we only need to consider the points that lower the current objective value. A good candidate is the minimum point in this approximation which can be selected easily.

As shown in Fig. 4, the approximate function, $\tilde{\Phi}$ is not continuous due to the constraint in Eq. (12) that bounds P_{ij} for each $[x, \hat{x}]$. The discontinuity point at the end of the line $L1$ satisfying the following equation should not be selected:

$$\inf_{u \in \Re^m} Px(f(x) + g(x)u) = 0 \quad (30)$$

As shown in Fig. 4, the current minimum point may be located near the discontinuity point. In such a situation, a neighbouring point is randomly selected to check the objective function because the true value at the minimum location may be very large.

Remark 5: Compared to grid-based approaches where each state point on grids is used to check Eq. (2), the proposed approach includes Eq. (2) using the maximum operator in the objective function, thus reducing a large number of constraints, especially for the high dimensional case. However, for non-convex function with constraints, it is difficult to evaluate its global extreme value using existing nonlinear programming solvers. In such a case, many initial guesses of x need to be tried to approach the

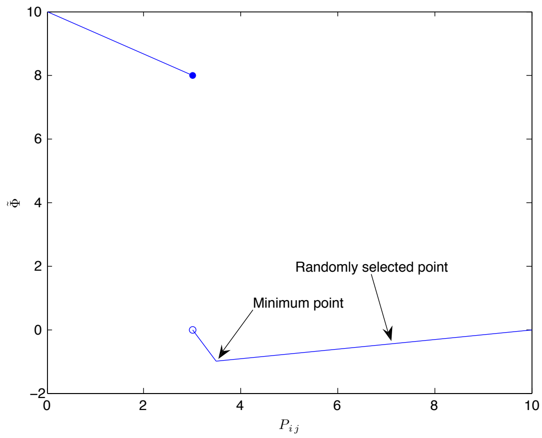

Fig. 4. The discontinuity in the estimated objective function.

optimal value as close as possible. Moreover, some points in the target area, should be selected to randomly check Eq. (2).

4. CASE STUDY

Consider a continuous stirred tank reactor example in Mhaskar et al. (2006), whose model takes the form of:

$$\dot{C}_A = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{-E/RT_R} C_A \quad (31)$$

$$\dot{T}_R = \frac{F}{V}(T_{A0} - T_R) + \frac{-\Delta H}{\rho C_p} k_0 e^{-E/RT_R} C_A + \frac{Q_\sigma}{\rho C_p V} \quad (32)$$

where $0 < C_{A0} < 2$ and $|Q_\sigma| < 0.0167$ are two inputs. The other parameters are shown in Table 1 where the subscript s indicates a steady-state value.

Table 1. Parameters of the process model

V	$0.1\ \mathrm{m^3}$	R	$8.314\ \mathrm{kJ/(kmol \cdot K)}$
C_{A0s}	$1.0\ \mathrm{kmol/m^3}$	T_{A0s}	310.0 K
ΔH	-4.78×10^4 kJ/kmol	k_0	$72 \times 10^9\ \mathrm{min^{-1}}$
E	8.314×10^4 kJ/kmol	C_p	$0.239\ \mathrm{kJ/(kg \cdot K)}$
ρ	$1000\ \mathrm{kg/m^3}$	F	$0.1\ \mathrm{m^3/min}$
T_{Rs}	395.33 K	C_{as}	$0.57\ \mathrm{kmol/m^3}$

The control objective is to stabilize all the states in the target region, $0.6 \leq C_A \leq 0.62$, $397.5 \leq T_R \leq 398$ and drive the system to the unstable equilibrium point: $T_{Rs} = 395.33$ K, $C_{as} = 0.57\ \mathrm{kmol/m^3}$. Linearizing the process model and choosing the Q as the identity matrix yields the solution of Ricatti equation as

$$P_1 = \begin{bmatrix} 14.6402 & 0.8897 \\ 0.8897 & 0.1041 \end{bmatrix}$$

Fig. 5 shows that the entire target region is not included its ROA. Given P_1 as an initial guess, the proposed method is applied to find the following new CLF:

$$P_2 = \begin{bmatrix} 41.3275 & 1.5066 \\ 1.5066 & 0.0928 \end{bmatrix}$$

The ROA associated with P_2 and the target region are shown in Fig. 6. Notice that the parameter λ, which effects the convergence of the final result, should be tuned very carefully.

The control policy suggested in Eq. (7) is applied to stabilize the point $(C_A, T_R) = (0.61, 397.9)$ with CLF P_2.

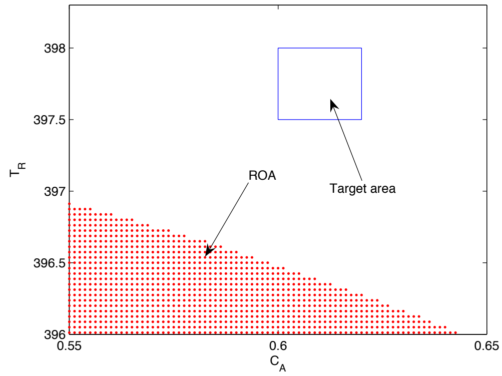

Fig. 5. The entire target region is not included in the ROA of Lyapunov function 1

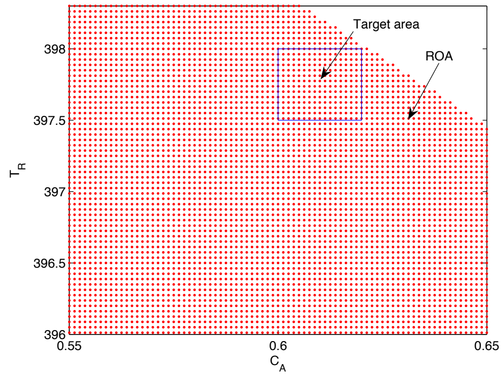

Fig. 6. The entire target region is included in the ROA of Lyapunov function 2

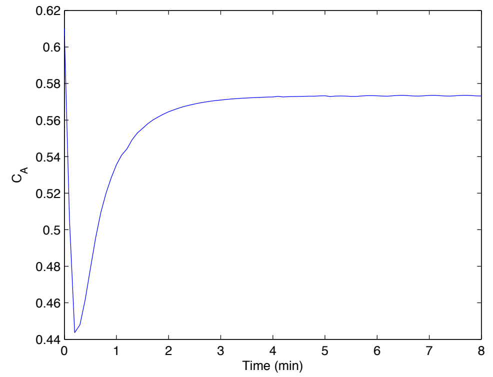

Fig. 7. Close-loop response of C_A

Figs. 7 and 8 show the state trajectories that converge to the specified steady state. Figs. 9 and 10 show both inputs are within the constraints.

5. CONCLUSION

In this work, a control Lyapunov function (CLF) design method, which can stabilize specified state region for control affine system with input constraints, is proposed. The problem for obtaining such a CLF is formulated as a minimax optimization problem and a derivative-free

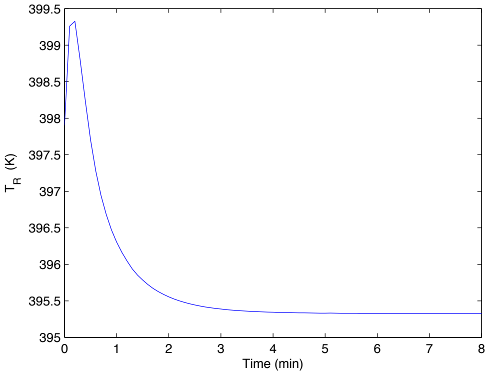

Fig. 8. Close-loop response of T_R

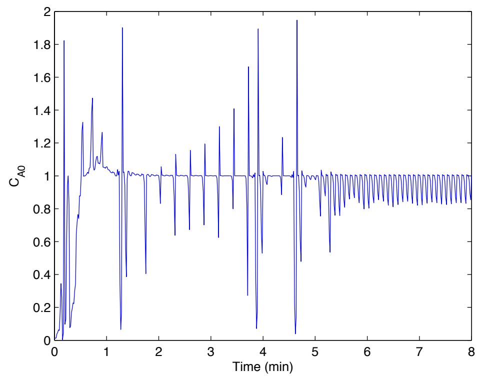

Fig. 9. Input trajectory: C_{A0}

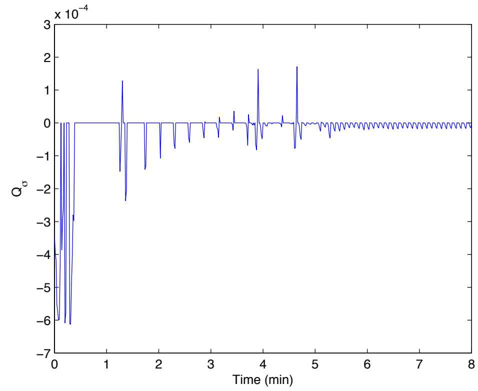

Fig. 10. Input trajectory: Q_σ

optimization scheme is proposed to obtain the optimal CLF starting from a simple initial guess.

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