PERFORMANCE ANALYSIS OF ROBUST AVERAGING LEVEL CONTROL

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Abstract

Frequent inlet flow changes, especially in the same direction, typically cause problems for averaging level controllers. To obtain optimal flow filtering while being robust towards future inlet flow upsets closed loop robust MPC has been used. It differs from other averaging controllers in that it does not return the tank level to a fixed set-point, but rather lets it depend on the current inlet flow. The performance and robustness of this robustly optimal behavior is analyzed and compared to that of the (non-robust) optimal averaging level controller. Both the analysis and the simulation results show that the robust controller obtains comparable flow filtering as the optimal controller even when inlet flow changes are sparse while handling frequent upsets considerably better.

Keywords

Averaging level control, non-linear control, surge tanks

Introduction

Buffer tanks are widely used within the process industry to prevent upstream flows from upsetting downstream processes. By smoothly controlling the outlet flow the capacity of the tank is used to surge inlet flow variations.

Early approaches to averaging level control are the PL-controller by (Luyben and Buckley, 1977), and the non-linear wide range controller proposed in (Shunta and Fehervari, 1976). In (Cheung and Luyben, 1979) a systematic way of tuning PI controllers to achieve good flow filtering is presented. Later approaches using PI controllers are (Shin et al., 2008) and (Kelly, 1998). A two-degree of freedom averaging level controller which permits designing the step and load disturbance response separately is suggested by (Wu et al., 2001).

In (McDonald et al., 1986) the continuous time optimal averaging level controller, minimizing the maximum rate of change of the outlet flow, is derived and analyzed. Its discrete time counterpart is presented in (Campo and Morari, 1989). Following an inlet flow step change the optimal control policy is to slowly ramp up the outlet flow so that it equals the inlet flow just as the tank reaches its upper or lower boundary.

With the tank level at, for example, its upper limit another positive step change to the inlet flow has to be directly transferred to the outlet, resulting in bad flow filtering. To prepare for the next flow upset the tank level is therefore returned to its set-point (usually 50%). This, however, gives a long settling time. If inlet flow changes are sparse with respect to time, so that the tank level has been returned to, or is at least close to, 50%, when the next flow upset occurs this strategy achieves very good flow filtering. If, however, two sequential upsets in the same direction occur within a short time interval, the flow filtering of the controller deteriorates. This is due to the fact that, when the second step occurs, the tank level is already close to its boundary leaving less tank capacity to surge the upset. One way to remedy this is of course by returning the level to 50% faster. That will however result in worse flow filtering of all inlet flow upsets.

Looking at inlet flow data for one of Perstorp AB's surge tanks, see Figure 1, it is apparent that frequent step changes do occur more or less all the time but especially for $t \in [30, 50]$ and $t \in [145, 170]$.

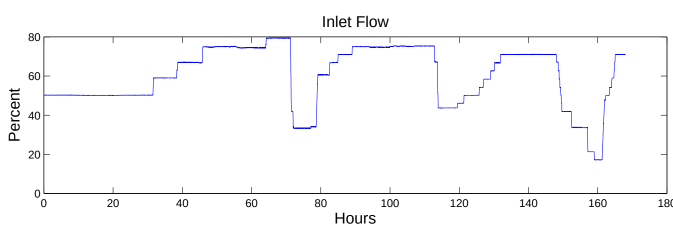

Figure 1. Inlet flow data from Perstorp AB last week of March 2011.

In (Rosander et al., 2011) another approach than returning the level to 50% was taken. Robust model predictive control (MPC) was used to obtain optimal flow filtering while being robust towards future changes of the inlet flow. The resulting controller does not return the tank level to 50% following an inlet flow change but rather adapts the tank level to the new level of the inlet flow. A similar behavior as is observed with proportional only control.

In this paper the differences in terms of robustness and filtering performance between the optimal and the robust controller are analyzed. The robust MPC controller is, however, difficult to analyze analytically and instead the insight gained from the robust MPC exercise is used to robustify the optimal averaging level controller. This controller obtains flow filtering, in principal, the same way as the robust MPC controller.

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System Description

We assume a cylindrical buffer tank with level $x(t)$ and inlet and outlet flows $q_{in}(t)$ and $q_{out}(t)$ respectively. Furthermore we assume that we can manipulate the outlet flow, $u(t) = q_{out}(t)$. Using mass balance we obtain the model

$$\dot{x}(t) = k_v\,(q_{in}(t) - u(t)) \tag{1a}$$

$$x_{min} \leq x(t) \leq x_{max} \tag{1b}$$

$$q_{min} \leq q_{in}(t),\, u(t) \leq q_{max} \tag{1c}$$

where it is assumed that all quantities are measured in percent, i.e., $q_{min} = x_{min} = 0\%$ and $q_{max} = x_{max} = 100\%$, if no extra safety limitations are put on the tank level. The parameter k_v describes how one percent of flow imbalance per time unit relates to change in tank level.

The assumption of equal range, (1c), is made to avoid risking emptying or overflowing the tank, due to lack of outlet flow capacity. With the outlet being the manipulated flow we could hence allow it to have greater range than the inlet flow, but for the scope of this paper that would only serve to make the notation overly complicated.

For the same reason we assume that the inlet flow is directly measurable. In any case the linear dynamics of the system allow for a straightforward estimation of the inlet flow using the Kalman filter as shown in (Khanbaghi et al., 2001).

Optimal Averaging Control

The performance of an averaging level controller is typically quantified by its capability to minimize

$$J = ||\dot{u}||_\infty = \max_t |\dot{u}(t)| \text{ or } J = ||\dot{u}||_2^2 = \int_0^\infty \dot{u}(t)^2\,dt$$

following an inlet flow step while guaranteeing the level constraints. We will in this paper focus on the former one ($\max|\dot{u}|$) but the latter will also be used to quantify the filtering performance when considered necessary.

For the sake of completeness the optimal controller derived in (McDonald et al., 1986) will be briefly recapitulated. It minimizes the max-criterion but permits $u(t)$ to not be differentiable in every point. This corresponds to using the criterion

$$J = \min_u \sup_{t,t'>0,\, t\neq t'} \left| \frac{u(t) - u(t')}{t - t'} \right| \tag{2}$$

Assuming that the system given by (1) is in steady state with the tank level x_0 and that the inlet flow q_{in} performs a step at time $t = 0$ from q_0 to q_1 the optimal control law is given by

$$u(t) = \begin{cases} q_0 + \dfrac{q_1 - q_0}{T} t, & t \in [0, T) \\ q_1, & t \in [T, \infty) \end{cases} \tag{3}$$

Note that the steady state assumption implies that $u(0) = q_{in}(0) = q_0$ and that after T time units the system will again be in steady state since $u(T) = q_{in}(T) = q_1$. With $u(t)$ given by (3) the performance criterion evaluates to

$$J = \frac{|q_1 - q_0|}{T} \tag{4}$$

Substituting (3) into (1a) and solving the differential equation for $t \in [0, T]$ gives

$$x(t) = x_0 + k_v \left((q_1 - q_0)t - \frac{q_1 - q_0}{2T} t^2 \right) \tag{5}$$

The time T can then be expressed in terms of the new steady state tank level, x_T, according to

$$\begin{aligned} x_T =& x_0 + k_v \left((q_1 - q_0)T - \frac{q_1 - q_0}{2T} T^2 \right) \Leftrightarrow \\ x_T - x_0 =& \frac{T k_v}{2}(q_1 - q_0) \Leftrightarrow T = \frac{2(x_T - x_0)}{k_v(q_1 - q_0)} \end{aligned} \tag{6}$$

By specifying the new steady state tank level, x_T, we thus implicitly specify the slope of the control law and consequently also the flow filtering performance. To achieve optimal flow filtering, T should be as large as possible which corresponds to picking x_T as

$$x_T = \begin{cases} x_{max}, & q_1 > q_0 \\ x_{min}, & q_1 < q_0 \end{cases} \tag{7}$$

The open loop solution (3) can also be cast as a combined feed-forward and feedback controller. We will present a different formulation than the one derived in (McDonald et al., 1986). By isolating t from (3) and inserting that expression into (5) we obtain

$$x(t) = x_0 + k_v \left((u(t) - q_0)T - \frac{T(u(t) - q_0)^2}{2(q_1 - q_0)} \right) \tag{8}$$

Isolating u and using (6) we obtain the closed loop formulation

$$u(x(t)) = q_1 \pm \sqrt{(q_1 - q_0)^2 - \frac{(q_1 - q_0)^2}{x_T - x_0}(x(t) - x_0)} \tag{9}$$

where $-\sqrt{\ldots}$ is used when $q_1 > q_0$ and $+\sqrt{\ldots}$ if $q_1 < q_0$. Note that, in addition to being a feedback solution, (9) does not depend on the parameter k_v which makes it less sensitive to modeling errors than the open loop solution (3). That the feedback controller indeed achieves $x(t) \to x_T$ is shown in Appendix A.

In common for both the open and closed loop formulation is, however, that following an inlet flow step the tank level will be on the boundary (due to (7)) and might thus struggle to surge the next flow upset. To counter this (McDonald et al., 1986) proposes augmenting (9) with a detuned PI controller which slowly brings back the level to its set-point x_s

$$\begin{aligned} u(x(t)) =& q_1 \pm \sqrt{(q_1 - q_0)^2 - \frac{(q_1 - q_0)^2}{x_T - x_0}(x(t) - x_0)} \\ &+ K_c \left(x_s - x(t) + \frac{1}{T_I} \int (x_s - x(\tau))\,d\tau \right) \end{aligned} \tag{10}$$

This will, however, slightly worsen the performance of the controller. How to select K_c and T_I to obtain a good compromise between the conflicting objectives of sufficiently fast set-point tracking and good flow filtering is not straightforward although McDonald et al. (1986) give some guidelines. A very neat solution to this is to formulate the filtering problem in an MPC framework as in (Campo and Morari, 1989). By using a terminal state constraint along with a sufficiently long prediction horizon it is in fact possible to bring back the tank level to its set-point without affecting the criterion. Both approaches do however give a rather long settling time.

In the sequel we will refer to both the open and closed loop formulation as the optimal controller.

Robust Averaging Level Control

This section briefly describes the robust closed loop MPC approach to flow filtering taken in (Rosander et al., 2011). To use the framework of robust MPC the model (1a) first needs to be discretized. Under the assumption that the inlet and outlet flows are constant during a sampling time period of length T_s the discrete system is

$$x(k+1) = x(k) + T_s k_v\,(q_{in}(k) - u(k)) \tag{11}$$

Optimal flow filtering while accounting for frequent inlet flow upsets is obtained by minimizing $||u(k)-u(k-1)||_\infty$ under the assumption that future inlet flows $q_{in}(k)$ are bounded but can change in every sample. To mitigate the feasibility problem associated with robust MPC the policy proposed by (Löfberg, 2003) is used

$$u(k) = \sum_{i\leq k} l_{k,i}q_{in}(i) + v(k) = L_k\mathbf{q_{in}} + v(k) \qquad (12)$$

where $L_k = (l_{k,0},\, l_{k,1},\ldots,l_{k,k},\, 0,\ldots,0)$ and $\mathbf{q_{in}} = (q_{in}(0),\, q_{in}(1),\ldots,q_{in}(N))^T$. As shown in (Goulart et al., 2006) the set of admissible states using (12) (with $i \leq k-1$) is equivalent to the one using $u(k) = l_k x(k)$. The reason for using the policy (12) instead of state feedback is that it gives a convex optimization problem

$$\begin{aligned}&\min_{L_{0:N},v(0:N)}\ \max_{q_{in}(1:N)} \|(L_k - L_{k-1})\mathbf{q_{in}} + v(k) - v(k-1)\|_\infty\\ &\text{subject to}\\ &x(k+1) = x(k) + T_s k_v\,(q_{in}(k) - u(k))\\ &x(0),\, q_{in}(0),\, L_{-1},\, v(-1) \text{ known}\\ &x(k)\in[x_{min},\, x_{max}]\ \forall q_{in}(k)\in[q_{min},\, q_{max}]\\ &q_{in}(k),\, u(k)\in[q_{min},\, q_{max}]\end{aligned} \qquad (13)$$

where the initial condition should be interpreted as $L_{-1} = 0$ and $v(-1)$ equal to $u(0)$ from the last sampling instance. That $q_{in}(k)$ is known when $u(k)$ is decided is motivated by the fact that the MPC controller runs at a lower sampling frequency than the inlet flow measurements.

Note that the affine mapping of past disturbances, (12), is not the actual control law but serves to obtain feasibility. The control used is still receding horizon MPC.

The behavior of the robust MPC controller, illustrated in Figure 2, is different from earlier approaches to averaging level control as it does not return the tank level to 50% following an inlet flow upset. Instead the new steady state tank level depends on the actual level of the inlet flow.

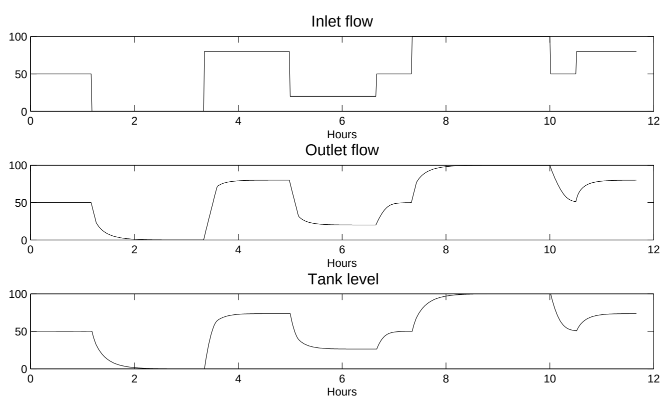

Figure 2. Simulation of closed loop robust MPC with a sampling period of one minute, a prediction horizon of 60 *minutes and* $k_v = \frac{1}{10}$.

Up to a prediction horizon of approximately 40 the optimization can be solved within a couple of seconds and could thus, in principal, be used for a real time implementation. The controller does, however, not always use a straight line (c.f. (3)) to even out flow imbalances, see for example the steps at $t = 3.33$ and at $t = 6.67$. If this behavior is indeed the optimal way of controlling while being robust towards frequent inlet flow changes or a side effect of the used policy (12) is not yet fully understood. Evident is however, that the MPC controller cannot be analytically analyzed. Instead we propose a robustification of the optimal controller, which yields a controller that in all essence behaves as the robust MPC while being both simpler to implement and allowing for an analytical analysis.

Robustified Optimal Controller

One way to robustify the optimal controller towards frequent inlet flow changes, as suggested by (McDonald et al., 1986), is to let x_T depend on the size of the flow imbalance but still bring back the tank level to $x_s = 50\%$. To the authors' understanding that must definitely add robustness to the controller. We know however from the exercise of applying robust closed loop MPC that to obtain robustly optimal filtering performance, also the steady state tank level should be adapted. A better strategy would thus be to change both x_T and x_s. A simple way of achieving this is to use an affine map according to

$$x_T = x_s = K_{SP}q_{in}(t) + b_{SP} \qquad (14)$$

The parameters K_{SP} and b_{SP} can be calculated using the fact that minimum and maximum inlet flow should correspond to minimum and maximum tank level, see Figure 2,

$$\begin{aligned}K_{SP} &= \frac{x_{max} - x_{min}}{q_{max} - q_{min}}\\ b_{SP} &= \frac{q_{max}x_{min} - q_{min}x_{max}}{q_{max} - q_{min}}\end{aligned} \qquad (15)$$

We will present the consequences using (14) have assuming that the system is in steady state when the inlet flow step occurs. This way the differences compared to the optimal controller become more evident. The steady state assumption yields that

$$\begin{aligned}x_0 &= K_{SP}q_0 + b_{SP}\\ x_T = x_s &= K_{SP}q_1 + b_{SP}\end{aligned} \qquad (16)$$

It then follows that T is given by

$$\begin{aligned}T =& \frac{2(x_T - x_0)}{k_v(q_1 - q_0)} =\\ =& \frac{2\left(K_{SP}q_1 + b_{SP} - (K_{SP}q_0 + b_{SP})\right)}{k_v(q_1 - q_0)} = \frac{2K_{SP}}{k_v}\end{aligned} \qquad (17)$$

which is constant regardless of the size of the flow imbalance (c.f. (6) which depends on the size of the flow imbalance). One effect of the robustification is thus that all upsets are filtered equally fast. Using (17) we obtain that for a flow imbalance, while in steady state, the open loop policy is given by

$$u(t) = \begin{cases} q_0 + \frac{k_v(q_1-q_0)}{2K_{SP}}t, & t\in\left[0,\, \frac{2K_{SP}}{k_v}\right)\\ q_1, & t\in\left[\frac{2K_{SP}}{k_v},\, \infty\right)\end{cases} \qquad (18)$$

Another effect of (14) is that the PI controller, at least in theory, can be removed since the non-linear feedforward/feedback controller achieves $x(t) \to x_T$. The behavior of (9) in the presence of noisy level and inlet flow measurements has however not been fully investigated and for a real life implementation it could thus be wise to keep the PI controller. In this paper we will only use the nonlinear controller which for a step in steady state is given by

$$u(x(t)) = q_1 \pm \sqrt{(q_1 - q_0)^2 - \frac{(q_1 - q_0)}{K_{SP}}(x(t) - x_0)} \qquad (19)$$

Either one of (19) and (18) will be referred to as the robust controller in the sequel.

To illustrate the different behavior obtained by using (10) and (19) the system, using $k_v = 1\,h^{-1}$, is simulated for an inlet flow step change from 50% to 60%, see Figure 3. The PI controller in (10) is tuned to have a double pole in $-\frac{1}{\lambda}$ with $\lambda = 10$ hours. The robust controller quickly filters the upset and is ready to surge the next one. The optimal controller on the other hand uses more tank capacity and consequently also filters the upset better.

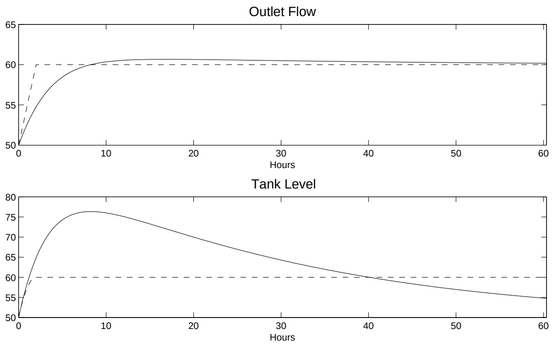

Figure 3. Step responses to an inlet flow change from 50% *to* 60% *for the optimal controller,* (10), *(solid) and the robust controller,* (19), *(dashed).*

Inlet Flow Change when not in Steady State

Both the optimal and robust controller can easily handle inlet flow steps also when the system is not in steady state ($\dot{x}(t) \neq 0$). What really matters is the flow imbalance $q_{in}(t) - u(t)$ which if the system is in steady state is $q_1 - q_0$. For an upset at $t = t'$ when not in steady state two changes to the previously presented derivation are needed

$$\begin{aligned} q_0 &\triangleq u(t') \\ x_0 &\triangleq x(t') \end{aligned} \tag{20}$$

Illustration of Robustness

To illustrate how the two controllers handle frequent inlet flow changes we simulate the system for an inlet flow sequence starting at 50% and making new steps of size 10% every five hours. A similar sequence actually occurs for the feed in Figure 1 at $t \in [30, 45]$. The PI in (10) was tuned with $\lambda = 10$ hours. Figure 4 shows the resulting tank level, the outlet flow and the derivative of the outlet flow. Initially the optimal controller performs better but its performance deteriorates and the step from 70% to 80% has to be very rapidly transferred to the outlet flow to avoid level violation. [2] The robust controller on the other hand filters all the upsets equally well. The combination of the used tuning and inlet flow sequence exploits the weakness of using a fixed set-point and a somewhat tighter tuning would perform better. It is however the case that for every PI tuning there exists an inlet flow sequence that will yield a similar result as the one shown in Figure 4.

Evaluation of Flow Filtering Performance

The benefit of not returning to a fixed set-point when filtering frequent flow steps in the same direction was elaborated in the previous section. In this section we will analyze the filtering performance of the controllers when flow upsets are sparse.

For this comparison we will use the open loop descriptions of the optimal and robust controller. Furthermore we make some simplifying assumptions:

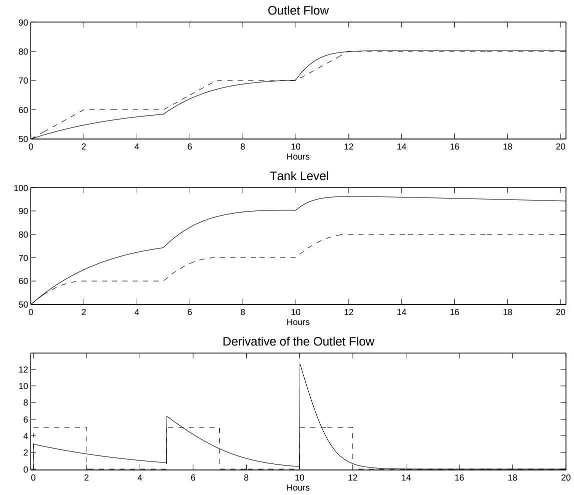

Figure 4. The optimal controller (solid) and the robust controller (dashed) for inlet flow steps of size 10% *every five hours and* $k_v = 1$.

(i) The effect the PI controller has on the criterion is neglected but it is still assumed that the tank level is brought back to the set-point. [3]
(ii) We only consider one single inlet flow step change and not, for example, a time-series description. The system is assumed to be in steady state when the step occurs.
(iii) In the derivations x_{min}, x_{max}, etc will be used but when numerical values are presented 0% and 100% are used if not stated otherwise.

Note that assumption (ii) is more restrictive for the optimal controller than the robust one since it has considerably longer settling time.

The assumption of steady state yields that the optimal controller initially has the tank level at x_s. In the analysis we will assume that the set-point is right in between the tank level boundaries

$$x_s = \frac{x_{max} + x_{min}}{2} \tag{21}$$

It then follows that T for the optimal controller is given by

$$T = \frac{x_{max} - x_{min}}{k_v |q_1 - q_0|} \tag{22}$$

Furthermore the construction of K_{SP} and b_{SP} implies that

$$\begin{aligned} x_{max} &= K_{SP} q_{max} + b_{SP} \\ x_{min} &= K_{SP} q_{min} + b_{SP} \end{aligned} \tag{23}$$

which together with (17) or (22) gives the criteria

$$\begin{aligned} J_{ROB} &= \frac{k_v}{2K_{SP}} |q_1 - q_0| \\ J_{OPT} &= \frac{k_v}{K_{SP}(q_{max} - q_{min})} (q_1 - q_0)^2 \end{aligned} \tag{24}$$

The subscripts ROB and OPT are used to refer to the robust and the optimal controller respectively.

Known q_0 *and* q_1

Let us start the comparison by considering some fixed values of q_0 and q_1. In Table 1 the performance for three combinations are shown. The control scheme which uses the larger part of the tank to surge the upset also obtains the best flow filtering. In the last case both controllers use

[2] Other averaging controllers, such as a detuned PI controller typically display better robustness towards frequent upsets, but at the cost of worse flow filtering of infrequent upsets.

[3] For the MPC formulation of (Campo and Morari, 1989) this can be fulfilled.

Table 1. Performance of the optimal and robust controller for some fixed inlet flow steps using $k_v = 1$.

q_0	q_1	J_{ROB}	J_{OPT}
20%	40%	10	4
20%	90%	35	49
30%	80%	25	25

the same tank volume (50%) and thus also obtain equal performance.

Expected Performance for Random q_0 and q_1

We will in this section analyze the statistically expected performance of the two controllers. To follow standard mathematical notation we will let Q_0 and Q_1 be random variables and q_0 and q_1 be actual realizations of these random variables. The average performance of the controllers is then given by

$$\mathrm{E}\left[J(Q_0, Q_1)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(q_0, q_1)\varphi(q_0, q_1)dq_0dq_1 \qquad (25)$$

where $J(q_0, q_1)$ is either J_{ROB} or J_{OPT} and $\varphi(q_0, q_1)$ is the joint probability density function of Q_0 and Q_1. To clarify: We do not assume a stochastic process describing $q_{in}(t)$ but will still consider step changes to the inlet flow but with the addition that when the step occurs it is distributed according to the distribution $\varphi(q_0, q_1)$.

Let us investigate the case where Q_0 and Q_1 are independently distributed

$$\varphi(q_0, q_1) = \varphi_{Q_0}(q_0)\varphi_{Q_1}(q_1) \qquad (26)$$

and all values between q_{min} and q_{max} are equally probable. This corresponds to that Q_0 and Q_1 are independently uniformly distributed

$$\varphi_{Q_0}(q_0) = \varphi_{Q_1}(q_1) = \frac{1}{q_{max} - q_{min}} \qquad (27)$$

Starting with the optimal controller we obtain

$$\begin{aligned}\mathrm{E}\left[J_{OPT}\right] &= \frac{k_v}{K_{SP}(q_{max} - q_{min})}\times \\ &\int_{q_{min}}^{q_{max}}\int_{q_{min}}^{q_{max}} (q_1 - q_0)^2 \frac{1}{q_{max} - q_{min}}\frac{1}{q_{max} - q_{min}} dq_0dq_1 = \\ &= \frac{k_v(q_{max} - q_{min})}{6K_{SP}}\end{aligned} \qquad (28)$$

and analogously for the robust controller

$$\begin{aligned}\mathrm{E}\left[J_{ROB}\right] &= \frac{k_v}{2K_{SP}}\times \\ &\int_{q_{min}}^{q_{max}}\int_{q_{min}}^{q_{max}} |q_1 - q_0| \frac{1}{q_{max} - q_{min}}\frac{1}{q_{max} - q_{min}} dq_0dq_1 = \\ &\Big/ \int_{q_{min}}^{q_{max}} |q_1 - q_0|dq_0 = \int_{q_{min}}^{q_1} (q_1 - q_0)dq_0 + \int_{q_1}^{q_{max}} (q_0 - q_1)dq_0 \Big/ \\ &= \frac{k_v(q_{max} - q_{min})}{6K_{SP}}\end{aligned} \qquad (29)$$

Surprisingly the two controllers achieve the same performance. We know from Table 1 that the robustified controller performs better for larger upsets while the optimal one is better for smaller ones. Obviously if we average over all possible values the controllers' performances become equal.

The assumption of independently uniformly distributed Q_0 and Q_1 does admittedly not describe real data particularly well. Typically there exists some conditioning, e.g., (26) does not hold, and values $\gtrsim 60\%$ are often more likely since factories typically are run close to their maximum capacity. It is, however, difficult to provide a general description of such distribution but for a specific problem φ could of course be estimated from historical data and (25) used to obtain an indication of which controller that is preferable.

Optimal Model Predictive Control

The discrete time formulation of the optimal level control derived in (Campo and Morari, 1989) will be used in the results section and is therefore briefly presented with the notation used in this paper. Following a step from q_0 to q_1 the optimal control law is

$$\begin{cases} u(k+1) &= u(k) + u^\star,\ |u^0| \le |u^\star| \\ u(k+1) &= u(k) + u^0,\ |u^0| > |u^\star| \end{cases} \qquad (30)$$

where

$$u^\star = \frac{2(q_1 - q_0)}{k^\star + 1} - \frac{2(x_T - x_0)}{k_v T_s k^\star(k^\star + 1)} \qquad (31)$$

$$k^\star = \left\lceil \frac{2(x_T - x_0)}{k_v T_s (q_1 - q_0)} \right\rceil \qquad (32)$$

and

$$u^0 = \frac{2(q_1 - q_0)}{N + 1} + \frac{2(x(k) - x_s)}{k_v T_s N(N + 1)} \qquad (33)$$

where N is the prediction horizon used in the MPC formulation (c.f. (13)) and $\lceil\,\rceil$ means rounding to the nearest larger integer. By selecting N such that

$$N \ge N_{crit} = \frac{1}{2|u^\star|}\left(2|q_1 - q_0| + \sqrt{\alpha}\right) \qquad (34)$$

where

$$\alpha = \left(|u^\star| - 2|q_1 - q_0|\right)^2 + \frac{8}{k_v T_s}|(x_0 - x_s)u^\star| \qquad (35)$$

it is possible to bring back the tank level to 50% without affecting the criterion. To handle a step when not in steady state the changes presented in (20) apply. Note that the above formulation does not explicitly consider limitations on $u(k)$. If this is incorporated there does not exist an explicit solution to the optimization problem.

Simulation Results

Using $k_v = 1$ the system was simulated for the feed shown in Figure 1. To limit the computational burden (especially for the robust MPC controller) the data was however down-sampled to one hour intervals. The robust controller was compared to the continuous and the discrete time optimal controller as well as the robust MPC controller.

An extensive search was made to find the λ for the PI in (10) which minimized the criterion for the considered inlet flow sequence. Similarly the optimal prediction horizon, N, was found for the optimal predictive controller. These values were $\lambda^\star = 4.5$ and $N^\star = 3.8$ hours. The optimal MPC controller was also simulated using an adaptive prediction horizon. For every flow imbalance (34) was evaluated and $N = N_{crit}$ was used. That way the controller brings back the tank level as fast as possible without affecting the criterion. The robust MPC used a prediction horizon of 20 hours and was implemented using YALMIP, (Löfberg, 2004).

The quantitative performance of the controllers are summarized in Table 2 where we see that almost all controllers perform equally well. The use of $N = N_{crit}$ gives however too long a settling time and has to transfer the steps at $t \approx 46$ and $t \approx 160$ very abruptly to avoid violating the level constraints. Note that the tuning using $\lambda^\star$ and $N^\star$ is optimal for the considered inlet flow sequence and for another feed realization the control performance will typically deteriorate. This deterioration can be quite significant, for example, using $\lambda \geq 5.5$ for the feed in Figure 1 even leads to violation of the tank level constraints. The qualitative behavior of the controllers is

Table 2. Tuning and performance criteria for the different controllers.

Controller	Tuning	$\max\lvert\dot{u}\rvert$	$\int \dot{u}^2 dt$
Optimal	$\lambda = \lambda^\star = 4.5$ hours	21.1	2333
Robust	$K_{SP} = 1$, $b_{SP} = 0$	21.5	3018
Optimal MPC[1]	$N = N_{crit}$	2014	49061
Optimal MPC[2]	$N = N^\star = 3.8$ hours	21.0	3007
Robust MPC	$N = 20$ hours	22.2	2600

illustrated in Figure 5 where the outlet and resulting tank level for the optimal, robust and optimal MPC[2] controller for $t \in [110, 135]$ is shown. The inlet flow for the present time is shown (dotted) together with the outlet flows. At $t \approx 117$ we see one effect of the tuning $\lambda^\star$: When surging the upset at $t = 114$ the controller all but empties the tank. If the upset had been even the slightest larger the level constraint had been violated.

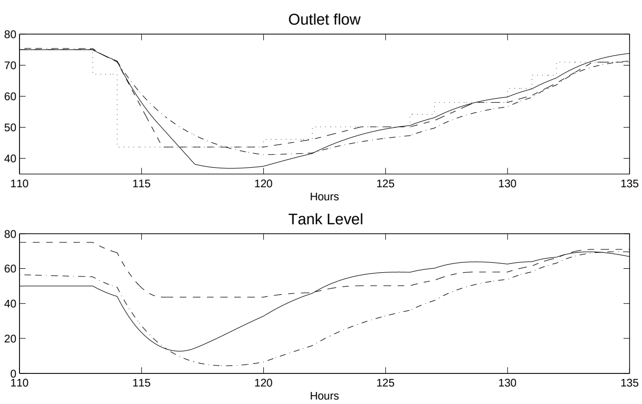

Figure 5. Outlet and tank level for the optimal MPC[2] (solid), robust (dashed) and optimal (dash-dotted) controllers.

Conclusions

Using insight gained from robust MPC, changes to the optimal level controller to improve the filtering of frequent inlet flow upsets, were proposed. The performance of the robustified controller was shown to be comparable to that of the optimal one even when inlet flow changes are sparse. The robust controller does, however, not suffer from the deterioration of performance associated with the optimal controller for frequent inlet flow upsets.

Admittedly, the derived non-linear robust controller may still be perceived as unnecessarily complex for a real industrial implementation. Better alternatives, which we are currently investigating, are probably an affine feedback law relating level and valve opening or possibly a standard PI but whose set-point is allowed to vary according to (14). Another extension is to use knowledge of the distribution of the inlet flow to develop application specific alternatives to the affine mapping (15).

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Appendix A. Convergence of (9)

We will prove that $x(t) \to x_T$ for the flow imbalance $q_1 - q_0$, $q_1 > q_0$ which occurs at $t = t'$. The case $q_0 < q_1$ follows analogously. By the construction (20) we have that

$$\begin{aligned} x(t') &= x_0 \\ u(t') &= q_0 < q_1 \end{aligned} \tag{A.1}$$

Furthermore we assume that $x(t') = x_0 < x_T$ (which we will show is fulfilled). When the step has occurred (1a) gives that

$$\dot{x}(t) \propto q_1 - \left(q_1 - \sqrt{(q_1 - q_0)^2 - \frac{(q_1-q_0)^2}{x_T - x_0}\left(x(t) - x_0\right)} \right) = \sqrt{(q_1 - q_0)^2 \underbrace{-\frac{(q_1-q_0)^2}{x_T - x_0}}_{<0}\left(x(t) - x_0\right)} > \Big/ \text{if } x(t) < x_T \Big/ > \sqrt{(q_1 - q_0)^2 - \frac{(q_1-q_0)^2}{x_T - x_0}\left(x_T - x_0\right)} = 0 \tag{A.2}$$

We thus have that $x(t') < x_T$, $\dot{x}(t) > 0$ if $x(t) < x_T$ and that $\dot{x}(t) = 0$ if $x(t) = x_T$. We thus have that $x(t) \to x_T$ monotonically. Provided that the controller is introduced when the system is in steady state it follows from the monotonicity that the assumption $x_0 < x_T$ holds.