Input to State Stability and Related Notions

Eduardo D. Sontag*
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903

Abstract

The analysis of input/output stability is one of the fundamental issues in control theory. External inputs might represent disturbances, estimation errors, or tracking signals, and outputs may correspond to the entire state, or to a more general quantity such as a tracking or regulation error, or the distance to a target set of states such as a desired periodic orbit.

For linear systems, one characterizes i/o stability through finiteness of gains (operator norms). A nonlinear generalization is provided by *input to state stability (ISS)*. This paper summarizes some of the main theoretical results concerning ISS and related notions such as integral ISS (energy bounds), output to input stability (a notion of "minimum-phase" system), and input/output to state stability (a notion of detectability). It also describes, as an illustrative application, an observer design methodology for certain kinetic networks which is based on ISS ideas.

Keywords

Input-to-state stability, ISS, Detectability, Observers, Minimum-phase, Lyapunov functions, Dissipation inequalities, H_∞ control

Introduction

Analyzing how external signals influence system behavior is one of the fundamental issues in control theory. In particular, a central concern is that of input/output stability, that is, stability from inputs u to outputs y in a system.

$$u(\cdot) \;\rightarrow \boxed{\;x(\cdot)\;} \rightarrow\; y(\cdot)$$

Inputs u might represent disturbances, estimation errors, or tracking signals, while outputs y may correspond to the entire state, or a more general quantity such as a tracking or regulation error, or the distance to a target set of states such as a desired periodic orbit.

The classical approach to i/o stability questions, for linear systems, relies upon transfer functions, which are closely related to more "modern" formulations in terms of operator norms (H_∞ control and the like). However, these approaches have a limited utility when used in a nonlinear context. A new paradigm which emerged within the last 10 or so years, for understanding input/output stability for general nonlinear systems, is that of *input to state stability (ISS)*.

This paper summarizes some of the main theoretical results concerning ISS and related notions such as input to output stability (IOS), integral ISS (iISS, which deals with "energy," as opposed to uniform bounds), mixed notions of integral and uniform stability, output to input stability (which is a notion of "minimum-phase" system), and input/output to state stability (a notion of detectability).

Also described are some illustrative applications, including an observer design methodology for kinetic networks based on ISS ideas.

The paper is written in a very informal style, and readers should consult the references for precise statements and proofs of results.

Input to State Stability

There are two desirable, and complementary, features of stability from inputs u to outputs y, one asymptotic and the other one on transients:

- *asymptotic stability*, which can be summarized by the implication "u small $\Rightarrow$ y small," and
- *small overshoot*, which imposes a boundedness constraint on the behavior of internal states x.

Of course, "small" and "boundedness" must be precisely defined, and to that goal we turn next. In order to explain the main ideas as simply as possibly, we begin with the case when the output y is the full state x (which will lead us to "input to state stability"); later we explain the general case ("input to output stability"). In addition, and also in the interest of exposition, we deal with notions relative to equilibria, instead of stability with respect to more arbitrary attractors.

We start by recalling the basic concept of internal stability for linear systems

$$\dot{x} = Ax + Bu\,, \quad y = Cx\,. \tag{1}$$

Internal stability means that A is a Hurwitz matrix, i.e., $x(t) \to 0$ as $t \to +\infty$ for all solutions of $\dot{x} = Ax$, or equivalently, that $x(t) \to 0$ whenever $u(t) \to 0$. For internally stable systems, one has the explicit estimate

$$|x(t)| \;\le\; \beta(t)|x^0| + \gamma\,\|u\|_\infty$$

where

$$\beta(t) = \left\|e^{tA}\right\| \to 0 \;\text{ and }\; \gamma = \|B\| \int_0^\infty \left\|e^{sA}\right\| ds$$

*Research supported in part by US Air Force Grant F49620-98-1-0242. URL: `http://www.math.rutgers.edu/\protect\nobreakspace{}sontag`, email: `sontag@control.rutgers.edu`

and $\|u\|_\infty$ = (essential) sup norm of u restricted to $[0,t]$. For t large, $x(t)$ is bounded by $\gamma\,\|u\|_\infty$, independently of initial conditions; for small t, the effect of initial states may dominate. Note the superposition of transient and asymptotic effects. Internal stability will now be generalized to "ISS," with the linear functions of $|x^0|$ and $\|u\|_\infty$ replaced by nonlinear ones.

We consider systems of the form

$$\dot{x} \;=\; f(x,u)\,,\;\; y \;=\; h(x)$$

evolving in finite-dimensional spaces $\mathbb{R}^n$, and we suppose that inputs u take values in $\mathbb{R}^m$ and outputs y are $\mathbb{R}^p$-valued. An *input* is a Lebesgue-measurable locally essentially bounded $u(\cdot):[0,\infty)\to\mathbb{R}^m$. We employ the notation $|x|$ for Euclidean norms, and use $\|u\|$, or $\|u\|_\infty$ for emphasis, to indicate the essential supremum of a function $u(\cdot)$, usually (depending on context) restricted to an interval of the form $[0,t]$. The map $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0)=0$. The map $h:\mathbb{R}^n\to\mathbb{R}^p$ is locally Lipschitz and satisfies $h(0)=0$.

The internal stability property for linear systems amounts to the "$L^\infty\to L^\infty$ finite-gain condition" that

$$|x(t)| \;\le\; c|x^0|e^{-\lambda t} \,+\, c\sup_{s\in[0,t]}|u(s)| \qquad (2)$$

holds for all solutions (assumed defined for all $t>0$), where c and $\lambda>0$ and appropriate constants. What is a reasonable nonlinear version of this?

Two central characteristic of the ISS philosophy are (a) the use of nonlinear gains rather than linear estimates, and (b) the fact that one does not ask what are the exact values of gains, but instead asks *qualitative* questions of *existence*. This represents a "topological" vs. a "metric" point of view (the linear analogy would be to ask only "is the gain $<\infty$?" or "is an operator bounded?").

Our general guiding principle may be formulated thus:

notions of stability should be invariant under (nonlinear) changes of variables.

By a *change of variables* in $\mathbb{R}^\ell$, let us mean here any transformation $z=T(x)$ with $T(0)=0$, where $T:\mathbb{R}^\ell\to\mathbb{R}^\ell$ is a homeomorphism whose restriction $T|_{\mathbb{R}^\ell\setminus\{0\}}$ is a diffeomorphism. (We allow less differentiability at the origin in order to state elegantly a certain converse result later.)

Let us see where this principle leads us, starting from the $L^\infty\to L^\infty$ finite-gain condition (2) and taking both state and input coordinate changes $x=T(z)$, $u=S(v)$. For any input u and initial state x^0, and corresponding trajectory $x(t)=x(t,x^0,u)$, we let $x(t)=T(z(t))$, $u(t)=S(v(t))$, $z^0=z(0)=T^{-1}(x^0)$.

For suitable functions $\underline{\alpha},\overline{\alpha},\overline{\gamma}\in\mathcal{K}_\infty$, we have:

$$\underline{\alpha}(|z|)\;\le\;|T(z)|\;\le\;\overline{\alpha}(|z|)\;\;\forall\,z\in\mathbb{R}^n$$

$$|S(v)|\;\le\;\overline{\gamma}(|v|)\;\;\forall\,v\in\mathbb{R}^m\,.$$

The condition $|x(t)|\le c|x^0|e^{-\lambda t}+c\sup_{s\in[0,t]}|u(s)|$ becomes, in terms of z,v:

$$\underline{\alpha}(|z(t)|)\;\le\;c\,e^{-\lambda t}\overline{\alpha}(|z^0|)\,+\,c\sup_{s\in[0,t]}\overline{\gamma}(|v(s)|)\quad\forall\,t\ge0\,.$$

Using again "x" and "u" and letting $\beta(s,t):=ce^{-\lambda t}\overline{\alpha}(s)$ and $\gamma(s):=c\overline{\gamma}(s)$, we arrive to this estimate, with $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}_\infty$:

$$\underline{\alpha}\left(|x(t)|\right)\;\le\;\beta(|x^0|,t)\,+\,\gamma\left(\|u\|_\infty\right)\,.$$

(It is shown in (Sontag, 1998a) that, for any $\mathcal{KL}$ function β, there exist $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ with

$$\beta(r,t)\;\le\;\alpha_2\left(\alpha_1(r)e^{-t}\right)\;\;\forall\,s,t\,,$$

so the special form of β adds no extra information.) Equivalently, one may write (for different β,γ)

$$|x(t)|\;\le\;\beta(|x^0|,t)\,+\,\gamma\left(\|u\|_\infty\right)$$

or one may use "max" instead of "+" in the bound.

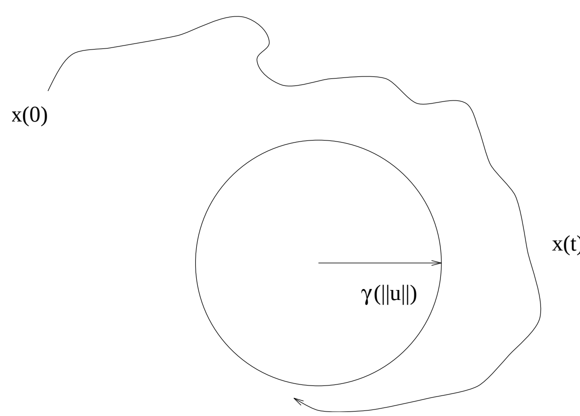
A system is *input to state stable (ISS)* if such an estimate holds, for some $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}_\infty$. More precisely, for each x^0, u, the solution $x(t)=x(t,x^0,u)$ is defined for all $t\ge0$, and the estimate holds.

Asymptotic Gain Characterization

For $u\equiv0$, the estimate reduces to $|x(t)|\le\beta(|x^0|,t)$, so ISS implies that the unforced system $\dot{x}=f(x,0)$ is globally asymptotically stable (with respect to $x=0$), or as one usually says, "GAS", and in particular stable.

In addition, an ISS system has a well-defined *asymptotic gain*: there is some $\gamma\in\mathcal{K}_\infty$ so that, for all x^0 and u:

$$\limsup_{t\to+\infty}|x(t,x^0,u)|\;\le\;\gamma\left(\|u\|_\infty\right)\,.$$



A far less obvious converse holds:

Theorem. *("Superposition principle for ISS") A system is ISS if and only if it admits an asymptotic gain and the unforced system is stable.*

This result is nontrivial, and constitutes the main contribution of the paper (Sontag and Wang, 1996), which establishes as well many other fundamental characterizations of the ISS property. The proof hinges upon a relaxation theorem for differential inclusions, shown in that paper, which relates global asymptotic stability of an inclusion $\dot{x} \in F(x)$ to global asymptotic stability of its convexification.

Dissipation Characterization of ISS

A smooth, proper, and positive definite $V : \mathbb{R}^n \to \mathbb{R}$ is an *ISS-Lyapunov function* for $\dot{x} = f(x,u)$ if, for some $\gamma, \alpha \in \mathcal{K}_\infty$,

$$\dot{V}(x,u) = \nabla V(x)\, f(x,u) \;\leq\; -\alpha(|x|) + \gamma(|u|) \quad \forall\, x, u$$

i.e., one has the dissipation inequality

$$V(x(t_2)) - V(x(t_1)) \;\leq\; \int_{t_1}^{t_2} w(u(s), x(s))\, ds$$

along all trajectories of the system, with "supply" function $w(u,x) = \gamma(|u|) - \alpha(|x|)$.

The following is a fundamental result in ISS theory:

Theorem. (Sontag and Wang, 1995a) *A system is ISS if and only if it admits an ISS-Lyapunov function.*

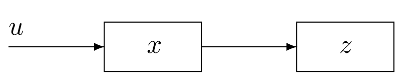
Sufficiency is easy: a differential inequality for V provides an estimate on $V(x(t))$, and hence on $|x(t)|$. Necessity follows by applying the converse Lyapunov theorem from (Lin et al., 1996) for GAS uniform over all $\|d\|_\infty \leq 1$, to a system of the form $\dot{x} = g(x,d) = f(x, d\rho(|x|))$, for an appropriate "robustness margin" $\rho \in \mathcal{K}_\infty$. This is in effect a smooth converse Lyapunov theorem for locally Lipschitz differential inclusions.

ISS is Natural for Series Connections

As a further motivation for the concept of ISS, and as an illustration of the characterizations given, we remark that any cascade (series connection) of ISS systems is again ISS. Consider a cascade connection of ISS systems

$$\begin{aligned} \dot{z} &= f(z,x) \\ \dot{x} &= g(x,u) \end{aligned}$$

where the z-subsystem is ISS with x as input and the x-subsystem is ISS.



The fact that cascades of ISS systems are ISS is one of the reasons that the concept is so useful in recursive design. (In the particular case in which the system $\dot{x} = g(x)$ has no inputs, we conclude that cascading an ISS with a GAS system we obtain a system which is GAS with respect to the state $(x,z) = (0,0)$.)

This fact can be established in several manners, but a particularly illuminating approach is as follows. We start by picking *matching* (cf. (Teel and Sontag, 1995)) ISS-Lyapunov functions for each subsystem:

$$\begin{aligned} \dot{V}_1(z,x) &\leq \theta(|x|) - \alpha(|z|) \\ \dot{V}_2(x,u) &\leq \tilde{\theta}(|u|) - 2\theta(|x|)\,. \end{aligned}$$

Then, $W(x,z) := V_1(z) + V_2(x)$ is an ISS-Lyapunov function:

$$\dot{W}(x,z) \;\leq\; \tilde{\theta}(|u|) - \theta(|x|) - \alpha(|z|)$$

and so a cascade of ISS systems is indeed ISS.

Generalization to Small Gains

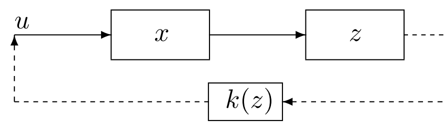
In particular, when $u = 0$, one obtains that a cascade of a GAS and an ISS system is again GAS. More generally, one may allow inputs u fed-back with "small gain": if $u = k(z)$ is so that $|k(z)| \leq \tilde{\theta}^{-1}((1-\varepsilon)\alpha(|z|))$, i.e.

$$\tilde{\theta}(|u|) \leq (1-\varepsilon)\alpha(|z|)$$

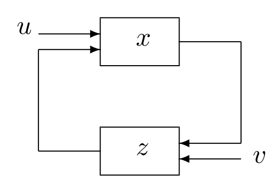
then

$$\dot{W}(x,z) \leq -\theta(|x|) - \varepsilon\alpha(|z|)$$

and the closed-loop system is still GAS.



Even more generally, under suitable conditions on gains (Small-Gain Theorem (Jiang et al., 1994) of Jiang, Praly, and Teel) the closed loop system obtained from an interconnection of two ISS systems $\dot{x} = f(x,z,u)$ and $\dot{z} = g(z,x,v)$, is itself ISS with respect to (u,v).



Series Connections: An Example

As a simple illustration of the cascade technique, consider the angular momentum stabilization of a rigid body controlled by two torques acting along principal axes (for instance, a satellite controlled by two opposing jet pairs). If $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of a body-attached frame with respect to inertial coordinates, and $I = \mathrm{diag}(I_1, I_2, I_3)$ are the principal moments of inertia, we obtain the equations:

$$I\dot{\omega} \;=\; \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} I\omega + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} v\,.$$

We assume $I_2 \neq I_3$; then, introducing new state and input coordinates via $(I_2 - I_3)x_1 = I_1\omega_1$, $x_2 = \omega_2$, $x_3 =$

ω_3, $I_2 u_1 = (I_3 - I_1)\omega_1\omega_3 + v_1$, and $I_3 u_2 = (I_1 - I_2)\omega_1\omega_2 + v_2$, we obtain a system on $\mathbb{R}^3$, with controls in $\mathbb{R}^2$:

$$\begin{aligned}\dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2\,.\end{aligned}$$

Then the following feedback law globally stabilizes the system:

$$\begin{aligned}u_1 &= -x_1 - x_2 - x_2 x_3 + v_1 \\ u_2 &= -x_3 + x_1^2 + 2x_1 x_2 x_3 + v_2\end{aligned}$$

when $v_1 = v_2 \equiv 0$. The feedback was obtained arguing in this way: with $z_2 := x_1 + x_2$, $z_3 := x_3 - x_1^2$, the system becomes:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + \alpha(x_1, z_2, z_3) \\ \dot{z}_2 &= -z_2 + v_1 \\ \dot{z}_3 &= -z_3 + v_2\,.\end{aligned}$$

The x_1-subsystem is easily seen to be ISS, because $\deg_{x_1} \alpha \leq 2$ and hence the cubic term dominates, for large x_1. Thus the cascade is also ISS; in particular, it is GAS if $v_1 = v_2 \equiv 0$. (We also proved a stronger result: ISS implies a global robustness result with respect to actuator noise.)

Generalizations of Other Gains

ISS generalizes finite $L^\infty \to L^\infty$ gains ("L^1 stability") but other classical norms often considered are induced $L^2 \to L^2$ ("H_∞") or $L^2 \to L^\infty$ ("H_2").

Nonlinear transformations starting from "H_∞" estimates:

$$\int_0^t |x(s)|^2\, ds \;\leq\; c|x^{\mathrm{o}}|^2 + c\int_0^t |u(s)|^2\, ds \quad \forall\, t \geq 0$$

lead to (for appropriate comparison functions):

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \;\leq\; \kappa(|x^{\mathrm{o}}|) + \int_0^t \gamma(|u(s)|)\, ds \quad \forall\, t \geq 0\,.$$

Theorem. *There is such an "integral to integral" estimate if and only if the system is ISS.*

The proof of this unexpected result is based upon the nontrivial characterizations of the ISS property obtained in (Sontag and Wang, 1996); see (Sontag, 1998a).

On the other hand, "$L^2 \to L^\infty$" stability:

$$|x(t)| \;\leq\; c|x^{\mathrm{o}}|e^{-\lambda t} + c\int_0^t |u(s)|^2\, ds \quad \text{for all } t \geq 0$$

leads to (for appropriate comparison functions):

$$\underline{\alpha}(|x(t)|) \;\leq\; \beta(|x^{\mathrm{o}}|, t) + \int_0^t \gamma(|u(s)|)\, ds \quad \text{for all } t \geq 0\,.$$

This is the *iISS* (*integral* ISS) property to which we'll return later.

An Application: Observers for Kinetic Networks

We now describe some recent work which we recently carried out with our graduate student Madalena Chaves, see (Chaves and Sontag, Chaves and Sontag), dealing with the design of observers (deterministic Kalman filters) for chemical reaction networks of the "Feinberg-Horn-Jackson" type (cf. (Feinberg, 1987, 1995) as well as an exposition in (Sontag, Sontag)), when seen as systems $\dot{x} = f(x)$ with outputs $y = h(x)$.

For such a system, the dynamics $\dot{x} = f(x)$ are n-dimensional (n is the number of species), and assumed to be given by ideal mass action kinetics, the reaction graph is weakly reversible, and the "deficiency" is zero: $m - \ell - d = 0$, where m is the number of complexes in the network, ℓ is the number linkage classes (connected components in the reaction graph), and d is the dimension of the stoichiometric subspace. We assume also that there are no boundary equilibria on positive classes. It is possible to write such systems as follows (we assume for simplicity here that $\ell = 1$, i.e. the graph is connected):

$$\dot{x} \;=\; f(x) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\, x_1^{b_{1j}} x_2^{b_{2j}} \ldots x_n^{b_{nj}}\,(b_i - b_j),$$

where the constants a_{ij} are all nonnegative, and the matrix $A = (a_{ij})$ irreducible, each b_j is a column vector in $\mathbb{R}^n$ with entries $b_{1j}, b_{2j}, \ldots, b_{nj}$, which are nonnegative integers, and the matrix $B := [b_1, b_2, \ldots, b_m]$ has rank $m \leq n$, and no row of B vanishes. We are interested in trajectories which evolve in the positive orthant.

We consider output functions $h : \mathbb{R}^n \to \mathbb{R}^p$ given by vectors of monomials; this includes situations in which concentrations (x_1, x_2, etc.) or reaction rates (proportional to $x_1 x_2^3$, etc) are measured. That is, $h : \mathbb{R}^n \to \mathbb{R}^p$ (typically, $p < n$) is of this form:

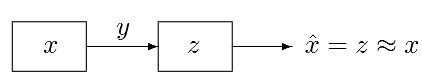
$$h(x) = \begin{pmatrix} x_1^{c_{11}} x_2^{c_{12}} \cdots x_n^{c_{1n}} \\ \vdots \\ x_1^{c_{p1}} x_2^{c_{p2}} \ldots x_n^{c_{pn}} \end{pmatrix},$$

where

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ & & \ddots & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

has nonnegative integer entries.

A (full-state) *observer* for $\dot{x} = f(x)$, $y = h(x)$ is a system $\dot{z} = g(z, y)$, with state-space $\mathbb{R}^n_+$, such that, for all $x(0)$ and $z(0)$ in $\mathbb{R}^n_+$, the composite system obtained by feeding $y = h(x)$ has solutions defined for $t > 0$, and $|z(t) - x(t)| \to 0$ as $t \to +\infty$.



Generally speaking, the problem of constructing nonlinear observers is extremely difficult. An obvious necessary condition for the existence of observers is detectability: A system $\dot{x} = f(x)$, $y = h(x)$ is *detectable* (on $\mathbb{R}^n_+$) if for all pairs of solutions $x_1(\cdot)$ and $x_2(\cdot)$ in $\mathbb{R}^n_+$:

$$h(x_1(t)) \equiv h(x_2(t)) \Rightarrow |x_1(t) - x_2(t)| \to 0 \text{ as } t \to \infty .$$

Let us introduce the stoichiometric subspace, i.e. the linear span of the "reaction vectors":

$$\mathcal{D} := \operatorname{span}\{b_i - b_j \mid i,j = 1,\ldots,m\} .$$

Theorem. *The system $\dot{x} = f(x)$, $y = h(x)$ is detectable if and only if*

$$\mathcal{D}^\perp \bigcap \ker C = \{0\} .$$

This condition is simple to check, involving only linear algebra computations. Our main result in (Chaves and Sontag, Chaves and Sontag) shows that this is in fact sufficient.

Theorem. *There exixts an observer if and only if the system is detectable.*

Moreover, in that case, we showed that the following system is an observer:

$$\dot{z} = f(z) + C'(y - h(z))$$

Note the formal analogy to Luenberger (deterministic Kalman filters) observers for linear systems (in which case the C' matrix is a replaced by a gain L which stabilizes $A + LC$).

In the context of the present paper, the most interesting feature of this observer's construction is the proof that it indeed provides unbiased estimates. The proof is based, roughly, upon the following idea. We first consider the system with inputs

$$\dot{z} = f(z) + C'(u - h(z))$$

and prove that this system is ISS, relative not to "$z = 0$ and $u = 0$" but each steady-state $z = x_0$ of $\dot{x} = f(x)$ and the associated input $u = h(x_0)$. This is established using an ISS-Lyapunov function (based on relative entropy). Next, we invoke the fact, discussed above, that a cascade of ISS systems is again ISS, plus the fact that $x(t) \to x_0$ for some equilibrium, to conclude the observer property. As a "bonus" from the construction, one gets an automatic property of robustness to small observation noise.

The paper (Chaves and Sontag, Chaves and Sontag) illustrated the observer with the example (studied in (Sontag, Sontag)) of the class of systems corresponding to the kinetic proofreading model for T-cell receptor signal transduction due to McKeithan in (McKeithan, 1995). Let us show here the simplest case of that class of models, with $n = 3$. The kinetics correspond to

$$X_1 + X_2 \underset{k_-}{\overset{k_+}{\rightleftarrows}} X_3 .$$

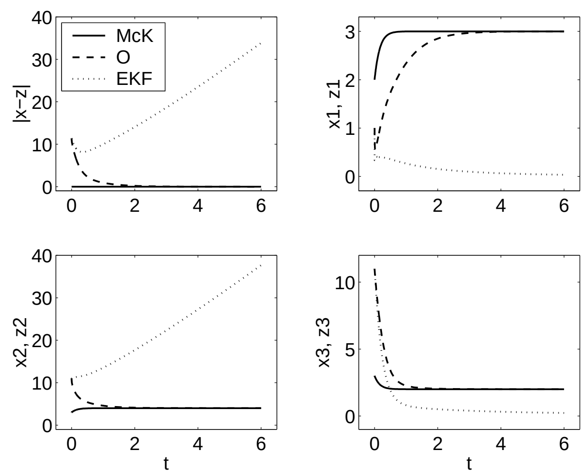
The equations for the system are as follows:

$$\begin{aligned}
\dot{x}_1 &= f_1(x) = -k_+ x_1 x_2 + k_- x_3 \\
\dot{x}_2 &= f_2(x) = -k_+ x_1 x_2 + k_- x_3 \\
\dot{x}_3 &= f_3(x) = k_+ x_1 x_2 - k_- x_3
\end{aligned}$$

and we pick for example the following measurement function: $y = h(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 x_3 \end{pmatrix}$. Then, our theory results in the following observer:

$$\begin{aligned}
\dot{z}_1 &= f_1(z) + (x_1 x_2^2 - z_1 z_2^2) + (x_1 x_3 - z_1 z_3) \\
\dot{z}_2 &= f_2(z) + 2(x_1 x_2^2 - z_1 z_2^2) \\
\dot{z}_3 &= f_3(z) + (x_1 x_3 - z_1 z_3)
\end{aligned}$$

This observer turns out to work surprisingly well (at least in sumulations), even when measurements are very noisy or when there are unobserved step or periodic disturbances on the states of the system. As an example, we show here a simulation which displays the convergence of the observer to the true solution, while an Extended Kalman Filter diverges for the same example. (Here, k_+=0.5, k_-=3, and the initial conditions are $x(0)$=$(2, 3, 3)'$ and $z(0)$=$(1, 11, 11)'$.)



Remark: Reversing Coordinate Changes

The "integral to integral" version of ISS arose, in the above discussion, from coordinate changes when starting from L^2-induced operator norms. Interestingly, this result from (Grune et al., 1999) shows that the reasoning can be reversed:

Theorem. *Assume $n \neq 4, 5$. If $\dot{x} = f(x,u)$ is ISS, then, under a coordinate change, for all solutions one has:*

$$\int_0^t |x(s)|^2 \, ds \;\leq\; |x^0|^2 + \int_0^t |u(s)|^2 \, ds .$$

The cases $n = 4, 5$ are still open. The proof is based on tools from "h-cobordism theory," developed by Smale and Milnor in the 1960s in order to prove the validity of the generalized Poincaré conjecture.

Integral-Input to State Stability

Recall that the "$L^2 \to L^\infty$" operator gain property led us, under coordinate changes, to the iISS property expressed by the estimate:

$$\underline{\alpha}\,(|x(t)|) \;\le\; \beta(|x^0|,t) \;+\; \int_0^t \gamma(|u(s)|)\,ds\,.$$

There is a dissipation characterization here as well.

A smooth, proper, and positive definite $V:\mathbb{R}^n \to \mathbb{R}$ is an *iISS-Lyapunov function* for $\dot{x}=f(x,u)$ if for some *positive definite* continuous α and $\gamma \in \mathcal{K}_\infty$

$$\nabla V(x)\,f(x,u) \;\le\; -\alpha(|x|)+\gamma(|u|) \quad \forall\, x\in\mathbb{R}^n,\, u\in\mathbb{R}^m$$

—observe that we are not requiring now $\alpha \in \mathcal{K}_\infty$. (Intuitively: even for constant u one may have $\dot{V}>0$, but $\gamma(|u|)\in\mathcal{L}^1$ means that $\dot{V}$ is "often" negative.)

A recent result from (Angeli et al., 2000a) is this:

Theorem. *A system is iISS if and only if it admits an iISS-Lyapunov function.*

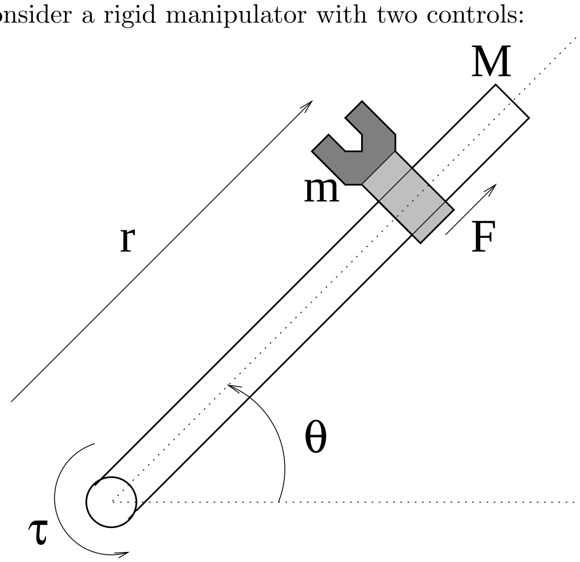
Since any $\mathcal{K}_\infty$ function is positive definite, every ISS system is also iISS, but the converse is false. For example, a bilinear system

$$\dot{x}=(A+\sum_{i=1}^m u_iA_i)x+Bu$$

is iISS if and only if A is a Hurwitz matrix, but in general it is not ISS—e.g., if $B=0$ and $A+\sum_{i=1}^m u_i^0A_i$ is not Hurwitz for some u^0. As another example, take $\dot{x}=-\tan^{-1}x+u$. This is not ISS, since bounded inputs may produce unbounded trajectories; but it is iISS, since $V(x)=x\,\tan^{-1}x$ is an iISS-Lyapunov function.

An Application of iISS Theory

Let us illustrate the iISS results through an application which, as a matter of fact, was the one that originally motivated much of the work in (Angeli et al., 2000a). Consider a rigid manipulator with two controls:



The arm is modeled as a segment with mass M and length L, and the hand as a point with mass m. Denoting by r the position and by θ the angle of the arm, the resulting equations are:

$$(mr^2+ML^2/3)\;\ddot{\theta}+2mr\dot{r}\dot{\theta}=\tau\,, \quad m\ddot{r}-mr\dot{\theta}^2=F$$

where F and τ are the external force and torque. In a typical passivity-based tracking design, one takes

$$\begin{aligned}\tau &:= -k_{d_1}\dot{\theta}-k_{p_1}(\theta-\theta_d)\\ F &:= -k_{d_2}\dot{r}-k_{p_2}(r-r_d)\end{aligned}$$

where r_d and θ_d are the desired signals and the gains $(k_{d_1},\ldots)$ are >0. For constant reference θ_d, r_d, there is tracking: $\theta\to\theta_d$, $\dot{\theta}\to 0$, and analogously for r.

But, what about time-varying θ_d, r_d? Can these destabilize the system? The answer is yes: there are bounded inputs which produce "nonlinear resonance," so the system cannot be ISS (not even bounded-input bounded-state). Such examples are presented in (Angeli et al., 2000a). On the other hand, one reason that standard tracking design is useful is that many inputs are *not* destabilizing, and one would like to find a way to formulate qualitatively that aspect. One way is by showing that the system is iISS.

The closed-loop system is 4-dimensional, with states $(q,\dot{q})$, $q=(\theta,r)$ and $u=(k_{p_1}\theta_d, k_{p_2}r_d)$:

$$\begin{aligned}(mr^2+ML^2/3)\ddot{\theta}+2mr\dot{r}\dot{\theta} &= u_1-k_{d_1}\dot{\theta}-k_{p_1}\theta\\ m\ddot{r}-mr\dot{\theta}^2 &= u_{-}k_{d_2}\dot{r}-k_{p_2}r\,.\end{aligned}$$

To prove that it is iISS, we consider the mechanical energy V, and note the following passivity-type estimate:

$$\frac{d}{dt}V(q(t),\dot{q}(t)) \;\le\; -c_1|\dot{q}(t)|^2+c_2|u(t)|^2$$

for sufficiently small $c_1>0$ and large $c_2>0$.

In general, we say that a system is *h-dissipative* with respect to an output function $y=h(x)$ (continuous and with $h(0)=0$) if, for some $\mathcal{C}^\infty$ positive definite, proper $V:\mathbb{R}^n\to\mathbb{R}$, and for some γ,α as above,

$$\nabla V(x)\,f(x,u) \;\le\; -\alpha(h(x))+\gamma(|u|) \quad \forall\, x\in\mathbb{R}^n,\, u\in\mathbb{R}^m$$

and that it is *weakly h-detectable* if, for all trajectories, $y(t)=h(x(t))\equiv 0$ implies that $x(t)\to 0$ as $t\to\infty$.

This is proved in (Angeli et al., 2000a):

Theorem. *A system is iISS if and only if it is weakly h-detectable and h-dissipative for some output h.*

With output $\dot{q}$, our example is weakly zero-detectable and dissipative, since $u\equiv 0$ and $\dot{q}\equiv 0$ imply $q\equiv 0$. Thus the system is indeed iISS, as claimed.

Mixed Notions

Changes of variables transformed "finite L^2 gain" to an "integral to integral" property, which turns out to be equivalent to ISS. Finite gain as operators between L^p and L^q spaces, with $p \neq q$ both finite, lead instead to this type of "weak integral to integral" estimate:

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \;\leq\; \kappa(|x^0|) \,+\, \alpha\left(\int_0^t \gamma(|u(s)|)\, ds\right)$$

for appropriate $\mathcal{K}_\infty$ functions (note the additional "α"). See (Angeli et al., 2000b) for more discussion on how this estimate is reached, as well as this result:

Theorem. *A system satisfies a weak integral to integral estimate if and only if it is iISS.*

Another interesting variant results by studying *mixed* integral/supremum estimates:

$$\underline{\alpha}(|x(t)| \;\leq\; \beta(|x^0|, t) \,+\, \int_0^t \gamma_1(|u(s)|)\, ds \,+\, \gamma_2(\|u\|_\infty)$$

for suitable $\beta \in \mathcal{KL}$ and $\underline{\alpha}, \gamma_i \in \mathcal{K}_\infty$. This result is also from (Angeli et al., 2000b):

Theorem. *The system $\dot{x} = f(x,u)$ satisfies a mixed estimate if and only if it is iISS.*

We also remark a "separation principle" recently obtained for iISS. In (Angeli, Ingalls, Sontag, and Wang, Angeli et al.), a system is said to be *bounded energy converging state* (BECS) if it is forward complete and, for all trajectories,

$$\int_0^\infty \sigma(|u(s)|)\, ds < \infty \;\Rightarrow\; \liminf_{t\to\infty} |x(t)| = 0$$

(for suitable $\sigma \in \mathcal{KL}$). The authors then prove that a system is iISS if and only if it is BECS and the 0-system $\dot{x} = f(x,0)$ has the origin as an asymptotically stable equilibrium.

Input/Output Stability

The discussion so far (except for the application to chemical observers) has been exclusively for notions involving stability from inputs to internal states. We now turn to external stability.

For linear systems (1), *external stability* means that the transfer function is stable, or, in terms of a state-space realization, that an estimate as follows holds:

$$|y(t)| \;\leq\; \beta(t)|x^0| + \gamma\, \|u\|_\infty\ ,$$

where γ is a constant and β converges to zero (β may be obtained from the restriction of A to a minimal subsystem). Observe that, even though we only require that y, not x, be "small" (relative to $\|u\|_\infty$), the initial internal states still affect the estimate in a "fading memory" manner, via the β term. (For example, in PID control, when considering the combination of plant, exosystem and controller, the overshoot of the regulated variable will be determined by the magnitude of the constant disturbance, and the initial state of the integrator.)

Under coordinate changes, and arguing just as earlier, external stability leads us to *input to output* stability (IOS) for systems with outputs $\dot{x} = f(x,u)$, $y = h(x)$. This is the property that, for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, the following estimate must hold along all solutions:

$$|y(t)| \;\leq\; \beta(|x^0|, t) \,+\, \sup_{s\in[0,t]} \gamma(|u(s)|)\,.$$

A dissipation (Lyapunov-) type characterization of this property is as follows. An *IOS-Lyapunov function* is a smooth $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that, for some $\alpha_i \in \mathcal{K}_\infty$, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$:

$$\alpha_1(|h(x)|) \;\leq\; V(x) \;\leq\; \alpha_2(|x|)$$

and

$$V(x) > \alpha_3(|u|) \;\Rightarrow\; \nabla V(x)\, f(x,u) < 0\,.$$

For systems that are bounded-input bounded-state stable, we have (see (Sontag and Wang, Sontag and Wang)):

Theorem. *A system $\dot{x} = f(x,u)$, $y = h(x)$ is IOS if and only if it admits an IOS-Lyapunov function.*

One may re-interpret this result as the existence of a new output map $\widetilde{y} = \alpha_1^{-1}(V(x))$ which dominates the original output ($y \leq \widetilde{y}$) and which is monotonically decreasing (no overshoot) as long as inputs are small. This is, in fact, one generalization of a central argument used in regulator theory (Francis equations).

A "separation theorem" providing an asymptotic gain characterization of IOS, similar to that given earlier for ISS, can be found in (Ingalls et al., 2001). A version of the above theorem for systems with are not BIBS is given in (Ingalls and Wang, 2001).

Zero-Detectability: IOSS

Detectability is yet another property which is central to systems analysis. For linear systems (1), (zero-) detectability means that the unobservable part of the system is stable, i.e.,

$$y(t) = Cx(t) \equiv 0 \;\&\; u(t) \equiv 0 \;\Rightarrow\; x(t) \to 0 \;\text{ as } t \to \infty$$

or equivalently:

$$u(t) \to 0 \;\&\; y(t) \to 0 \;\Rightarrow\; x(t) \to 0$$

$$\xrightarrow{u \to 0} \boxed{\Rightarrow \quad x \to 0} \xrightarrow{y \to 0}$$

(see, for instance, the textbook (Sontag, 1998b)) and can be also expressed by means of an estimate of the following form:

$$|x(t)| \;\leq\; \beta(t)|x^0| + \gamma_1\, \|u\|_\infty + \gamma_2\, \|y\|_\infty$$

where the γ_i's are constants and β converges to zero (now β is obtained from a suitable matrix $A - LC$, where L is an observer gain) and the sup norms are interpreted as appying to restrictions to $[0,t]$.

Under cordinate changes, one is led to *input/output to state* stability (IOSS). This is the property defined by the requirement that an estimate of the following type hold along all trajectories:

$$|x(t)| \;\leq\; \beta(|x^0|,t) \;+\; \sup_{s\in[0,t]} \gamma(|u(s)|) \;+\; \sup_{s\in[0,t]} \gamma(|y(s)|)$$

(for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$). The terminology IOSS is self-explanatory: formally, there is "stability from the i/o data to the state".

Dissipation Characterization of IOSS

A smooth, proper, and positive definite $V : \mathbb{R}^n \to \mathbb{R}$ is an *IOSS-Lyapunov function* if, for some $\alpha_i \in \mathcal{K}_\infty$,

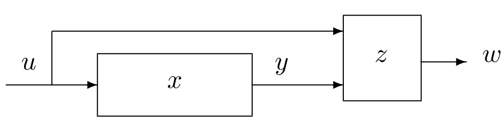
$$\nabla V(x)\, f(x,u) \;\leq\; -\alpha_1(|x|) + \alpha_2(|u|) + \alpha_3(|y|)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

This is from (Krichman, 2000) and (Krichman, Sontag, and Wang, Krichman et al.):

Theorem. *A system $\dot{x} = f(x,u)$, $y = h(x)$ is IOSS if and only if it admits an IOSS-Lyapunov function.*

As a corollary, IOSS is equivalent to the existence of a *norm-estimator*: driven by the i/o data generated by the original system, it estimates an upper bound on the internal state.



This is defined as a system $\dot{z} = g(z,u,y)$, $w = \ell(z)$, whose inputs are the i/o pairs of the original system, which is ISS with respect to u, y as inputs (so that there is robustness to signal errors), and, for some $\rho \in \mathcal{K}$ and $\beta \in \mathcal{KL}$,

$$|x(t)| \;\leq\; \beta(|x^0|+|z^0|,t) \;+\; \rho(|w(t)|) \quad \forall\, t \geq 0$$

for all initial states x^0 and z^0. (See the paper (Krichman, Sontag, and Wang, Krichman et al.) for the precise definition.)

An asymptotic gain characterization of IOSS also exists, see (Angeli, Ingalls, Sontag, and Wang, Angeli et al.).

Output to Input Stability and Minimum-Phase Systems

Recall that a linear system (1), let us say for simplicity single-input and single-output, is said to be *minimum-phase* if the inverse of its transfer function is stable, i.e. if all the zeroes of its transfer function have negative real part. The minimum-phase property is ubiquitous in control design, for instance because it allows to solve control problems by simple inversion; it is also needed for convergence of several adaptive control algorithms, and it allows stabilization by output feedback.

In the late 1980s, a notion of minimum-phase (and associated "zero dynamics") was introduced by Byrnes and Isidori (see (Byrnes and Isidori, 1988) as well as (Isidori, 1995)). This concept has proved very succesful in allowing the extension of many linear systems results to nonlinear systems. Basically, a minimum-phase system is one for which the zero-dynamics subsystem (which is obtained by clamping the output at zero) is GAS.

Often, however, an enhancement of this GAS property is needed, in effect impossing on the zero-dynamics an ISS property with respect to the output and its derivatives, see for instance (Praly and Jiang, 1993). The paper (Liberzon et al., 2000) showed that it is possible to define this enhancement directly, and with no recourse to "normal forms" or even zero-dynamics, by requiring an "output to input stability" property. A system is said to be *output-input stable* (OIS), or more precisely "derivatives of output to state and input stable" if, for some positive integer k, an estimate as follows:

$$|u(t)| + |x(t)| \leq \beta(|x^0|\,,t) + \gamma(\|y\|_\infty + \ldots \|y^{(k)}\|_\infty)$$

holds along all trajectories corresponding to smooth controls, for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ (the norms of y and its derivatives are understood as those of restrictions to $[0,t]$).

We refer the reader to (Liberzon et al., 2000) for details, as well as an application in adaptive control and the proof that the OIS property is equivalent (for scalar input and output real-analytic systems) to the existence of a relative degree plus an OSS property with respect to output derivatives.

Other Cascading Results

We remarked several times on the fact that cascades of ISS systems are ISS, and the role that such a property plays. It is easy to provide examples of the fact that the cascade of an iISS system $\dot{x} = f(x,z)$ with an ISS system $\dot{z} = f(z,u)$ need not be iISS, and this motivated work reported in (Arcak, Angeli, and Sontag, Arcak et al.) dealing with "matching" conditions which insure such cascade well-behavior, as well as the work reported in (Angeli et al., 2001) which deals with a notion of ISS with respect to input *derivatives* (there is a certain formal duality to the output to input stability property).

Closing Comments

The developments in ISS theory during the last decade have allowed a complete characterization of most of the important properties identified so far (ISS itself, IOS, IOSS, iISS, etc). Nonetheless, the area remains very open, as major concepts still need clarification. Among the main questions are the need to further study and characterize *incremental* IOSS (not merely distinguishing from $x = 0$, but distinguishing every pair of states, as needed for observers), and the need to deal with feedback designs which provide an ISS property with respect to observation errors.

This brief survey has focused on basic theoretical constructs, instead of on applications. Let us turn now to some further references related to ISS-related theory as well as pointers to applications.

Textbooks and research monographs which make use of ISS and related concepts include (Freeman and Kokotovi'c, 1996; Isidori, 1999; Krstić and Deng, 1998; Krstić et al., 1995; Khalil, 1996; Sepulchre et al., 1997).

After the definition in (Sontag, 1989a) and the basic characterizations in (Sontag and Wang, 1995a), the main results on ISS are given in (Sontag and Wang, 1996). See also (Coron et al., 1995; Sussmann et al., 1994) for early uses of asymptotic gain notions. "Practical" ISS is equivalent to ISS with respect to compact attractors, see (Sontag and Wang, 1995b).

Several authors have pointed out that *time-varying* system versions of ISS are central to the analysis of asymptotic tracking problems, see e.g. (Tsinias and Karafyllis, 1999). In (Edwards et al., 2000), one can find further results on Lyapunov characterizations of the ISS property for time-varying (and in particular periodic) systems, as well as a small-gain theorem based on these ideas.

Perhaps the most interesting set of open problems concerns the construction of feedback laws that provide ISS stability with respect to observation errors. Actuator errors are far better understood (cf. (Sontag, 1989a)), but save for the case of special structures studied in (Freeman and Kokotovi'c, 1996), the one-dimensional case (see e.g. (Fah, 1999)) and the counterexample (Freeman, 1996), little is known of this fundamental question. Recent work analyzing the effect of small observation errors (see (Sontag, 1999)) might provide good pointers to useful directions of research (indeed, see (Liberzon, 2000) for some preliminary remarks in that direction). For special classes of systems, even output feedback ISS with respect to observation errors is possible, cf. (Nešić and Sontag, 1998).

Both ISS and iISS properties have been featured in the analysis of the performance of switching controllers, cf. (Hespanha and Morse, 1999a) and (Hespanha and Morse, 1999b).

Coprime factorizations are the basis of the parameterization of controllers in the Youla approach. As a matter of fact, as the paper's title indicates, their study was the original motivation for the introduction of the notion of ISS in (Sontag, 1989a). Some further work can be found in (Sontag, 1989b), see also (Fujimoto and Sugie, 1998), but much remains to be done.

There are now results on averaging for ISS systems, see (Nešić and Teel, Nešić and Teel), as well as on singular perturbations, see (Christofides and Teel, 1996).

Discrete-time ISS systems are studied in (Kazakos and Tsinias, 1994) and in (Jiang et al., 1999); the latter paper provides Lyapunov-like sufficient conditions and an ISS small-gain theorem, and more complete characterizations and extensions of many standard ISS results for continuous time systems are given in (Jiang and Wang, Jiang and Wang).

Discrete-time iISS systems are the subject of (Angeli, 1999b), which proves the very surprising result that, in the discrete-time case, iISS is actually no different than global asymptotic stability of the unforced system (this is very far from true in the continuous-time case, of course). In this context, of interest are also the relationships between the ISS property for a continuous-time system and its sampled versions. The result in (Teel et al., 1998) shows that ISS is recovered under sufficiently fast sampling; see also the technical estimates in (Nešić et al., 1999).

Stochastic ISS properties are treated in (Tsinias, 1998).

A very interesting area regards the combination of clf and ISS like-ideas, namely providing necessary and sufficient conditions, in terms of appropriate clf-like properties, for the existence of feedback laws (or more generally, dynamic feedback) such that the system $\dot{x} = f(x, d, u)$ becomes ISS (or iISS, etc) with respect to d, once that $u = k(x)$ is substituted. Notice that for systems with disturbances typically $f(0, d, 0)$ need not vanish (example: additive disturbances for linear systems), so this problem is qualitatively different from the robust-clf problem since uniform stabilization is not possible. There has been substantial work by many authors in this area; let us single out among them the work (Teel and Praly, 2000), which deals primarily with systems of the form $\dot{x} = f(x, d) + g(x)u$ (affine in control, and control vector fields are independent of disturbances) and with assigning precise upper bounds to the "nonlinear gain" obtained in terms of d, and (Deng and Krstić, 2000), which, for the class of systems that can be put in output-feedback form (controller canonical form with an added stochastic output injection term), produces, via appropriate clf's, stochastic ISS behavior ("NSS" = noise to state stability, meaning that solutions converge in probability to a residual set whose radius is proportional to bounds on covariances).

In connection with our example from tracking design for a robot, we mention here that the paper (Marino and Tomei, 1999) proposed the reformulation of tracking problems by means of the notion of input to state stability. The goal was to strengthen the robustness properties of tracking designs, and the notion of ISS was instrumental in the precise characterization of performance. Incidentally, the same example was used, for a different purpose—namely, to illustrate a different nonlinear tracking design which produces ISS, as opposed to merely iISS, behavior—in the paper (Angeli, 1999a).

Neural-net control techniques using ISS are mentioned in (Sanchez and Perez, 1999).

A problem of decentralized robust output-feedback control with disturbance attenuation for a class of large-scale dynamic systems, achieving ISS and iISS properties, is studied in (Jiang et al., 1999).

Incremental ISS is the notion that estimates differences $|x_1(t) - x_2(t)|$ in terms of $\mathcal{KL}$ decay of differences of initial states, and differences of norms of inputs. It provides a way to formulate notions of sensitivity to initial conditions and controls (not local like Lyapunov exponents or as in (Lohmiller and Slotine, 1998), but of a more global character, see (Angeli, Angeli)); in particular when there are no inputs one obtains "incremental GAS", which can be completely characterized in Lyapunov terms using the result in (Lin et al., 1996), since it coincides with stability with respect to the diagonal of the system consisting of two parallel copies of the same system. This area is of interest, among other reasons, because of the possibility of its use in information transmission by synchronization of diffusively coupled dynamical systems ((Pogromsky et al., 1999)) in which the stability of the diagonal is indeed the behavior of interest.

Small-gain theorems for ISS and IOS notions originated with (Jiang et al., 1994); a purely operator version (cf. (Ingalls et al., 1999)) of the IOS small-gain theorem holds as well. There are ISS-small gain theorems for certain infinite dimensional classes of systems such as delay systems, see (Teel, 1998).

The notion of IOSS is called "detectability" in (Sontag, 1989b) (where it is phrased in input/output, as opposed to state space, terms, and applied to questions of parameterization of controllers) and was called "strong unboundedness observability" in (Jiang et al., 1994). IOSS and its incremental variant are very closely related to the OSS-type detectability notions pursued in (Krener, 1999); see also the emphasis on ISS guarantees for observers in (Marino et al., 1999). The use of ISS-like formalism for studying observers, and hence implicitly the IOSS property, has also appeared several times in other authors' work, such as the papers (Hu, 1991; Lu, 1995a; Pan et al., 1993).

It is worth pointing out that several authors had independently suggested that one should *define* "detectability" in dissipation terms. For example, in (Lu, 1995b), Equation 15, one finds detectability defined by the requirement that there should exist a differentiable storage function V satisfying our dissipation inequality but with the special choice $\alpha_3(r) := r^2$ (there were no inputs in the class of systems considered there). A variation of this is to weaken the dissipation inequality, to require merely

$$x \neq 0 \Rightarrow \nabla V(x)\, f(x,u) < \alpha_3(|y|)$$

(again, with no inputs), as done for instance in the definition of detectability given in (Morse, 1995). Observe that this represents a slight weakening of the ISS property, in so far as there is no "margin" of stability $-\alpha_1(|x|)$.

Norm-estimators are motivated by developments appeared in (Jiang and Praly, 1992) and (Praly and Wang, 1996).

The notion studied in (Shiriaev, 1998) is very close to the combination of IOSS and IOS being pursued in (Ingalls, Sontag, and Wang, Ingalls et al.).

Partial asymptotic stability for differential equations is a particular case of output stability (IOS when there are no inputs) in our sense; see (Vorotnikov, 1993) for a survey of the area, as well as the book (Rumyantsev and Oziraner, 1987), which contains a converse theorem for a restricted type of output stability. (We thank Anton Shiriaev for bringing this latter reference to our attention.)

References

Angeli, D., A Lyapunov approach to incremental stability properties. to appear.

Angeli, D., B. Ingalls, E. D. Sontag, and Y. Wang, Asymptotic Characterizations of IOSS. submitted.

Angeli, D., E. D. Sontag, and Y. Wang, "A characterization of integral input to state stability," *IEEE Trans. Auto. Cont.*, **45**, 1082–1097 (2000a).

Angeli, D., E. D. Sontag, and Y. Wang, "Further equivalences and semiglobal versions of integral input to state stability," *Dynamics and Control*, **10**, 127–149 (2000b).

Angeli, D., E. D. Sontag, and Y. Wang, A Note on input-to-state stability with input derivatives, In *Proc. Nonlinear Control System Design Symposium* (2001). to appear.

Angeli, D., "Input-to-state stability of PD-controlled robotic systems," *Automatica*, **35**, 1285–1290 (1999a).

Angeli, D., "Intrinsic robustness of global asymptotic stability," *Sys. Cont. Let.*, **38**, 297–307 (1999b).

Arcak, M., D. Angeli, and E. D. Sontag, Stabilization of cascades using integral input-to-state stability. submitted.

Byrnes, C. I. and A. Isidori, "Local stabilization of minimum phase nonlinear systems," *Sys. Cont. Let.*, **11**, 9–17 (1988).

Chaves, M. and E. D. Sontag, State-Estimators for chemical reaction networks of Feinberg-Horn-Jackson zero deficiency type. submitted.

Christofides, P. D. and A. R. Teel, "Singular perturbations and input-to-state stability," *IEEE Trans. Auto. Cont.*, **41**, 1645–1650 (1996).

Coron, J. M., L. Praly, and A. Teel, Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and input-output techniques, In Isidori, A., editor, *Trends in Control*. Springer-Verlag, London (1995).

Deng, H. and M. Krstić, "Output-feedback stablization of stochastic nonlinear systems driven by noise of unknown covariance," *Sys. Cont. Let.*, **39**, 173–182 (2000).

Edwards, H., Y. Lin, and Y. Wang, On input-to-state stability for time varying nonlinear systems, In *Proc. 39th IEEE Conf. Decision and Control, Sydney, Dec. 2000*, pages 3501–3506. IEEE Publications (2000).

Fah, N. C. S., "Input-to-state stability with respect to measurement disturbances for one-dimensional systems," *Control, Optimisation and Calculus of Variations*, **4**, 99–121 (1999).

Feinberg, M., "Chemical reaction network structure and the stabiliy of complex isothermal reactors—I. The deficiency zero and deficiency one theorems," *Chem. Eng. Sci.*, **42**, 2229–2268 (1987). Review Article 25.

Feinberg, M., "The existence and uniqueness of steady states for a class of chemical reaction networks," *Archive for Rational Mechanics and Analysis*, **132**, 311–370 (1995).

Freeman, R. A. and P. V. Kokotovi'c, *Robust Nonlinear Control Design, State-Space and Lyapunov Techniques*. Birkhäuser, Boston (1996).

Freeman, R. A., "Global internal stabilizability does not imply global external stabilizability for small sensor disturbances," *IEEE Trans. Auto. Cont.*, **40**, 2119–2122 (1996).

Fujimoto, K. and T. Sugie, State-space characterization of Youla parametrization for nonlinear systems based on input-to-state stability, In *Proc. 37th IEEE Conf. Decision and Control*, pages 2479–2484, Tampa (1998).

Grune, L., F. R. Wirth, and E. D. Sontag, "Asymptotic stability equals exponential stability and ISS equals finite energy gain—if you twist your eyes," *Sys. Cont. Let.*, **38**, 127–134 (1999).

Hespanha, J. P. and A. S. Morse, "Certainty equivalence implies detectability," *Sys. Cont. Let.*, **36**, 1–13 (1999a).

Hespanha, J. P. and A. S. Morse, Supervisory control of integral-input-to-state stabilizing controllers, In *Proc. of the* 5^{th} *European Control Conference*, Karlsruhe (1999b).

Hu, X. M., "On state observers for nonlinear systems," *Sys. Cont. Let.*, **17**, 645–473 (1991).

Ingalls, B. and Y. Wang, On input-to-output stability for systems not uniformly bounded, In *Proc. IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, St. Petersburg (2001). to appear.

Ingalls, B., E. D. Sontag, and Y. Wang. In preparation.

Ingalls, B., E. D. Sontag, and Y. Wang, Remarks on input to output stability, In *Proc. IEEE Conf. Decision and Control*, pages 1226–1231, Phoenix. IEEE Publications (1999).

Ingalls, B., E. D. Sontag, and Y. Wang, Generalizations of asymptotic gain characterizations of ISS to input-to-output stability, In *Proc. American Control Conf.* (2001). to appear.

Isidori, A., *Nonlinear Control Systems*. Springer-Verlag, London, third edition (1995).

Isidori, A., *Nonlinear Control Systems II*. Springer-Verlag, London (1999).

Jiang, Z.-P. and L. Praly, "Preliminary results about robust Lagrange stability in adaptive nonlinear regulation," *Int. J. of Adaptive Control and Signal Processing*, **6**, 285–307 (1992).

Jiang, Z.-P. and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*. to appear.

Jiang, Z.-P., A. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals and Systems*, **7**, 95–120 (1994).

Jiang, Z.-P., F. Khorrami, and D. J. Hill, Decentralized output-feedback control with disturbance attenuation for large-scale nonlinear systems, In *Proc. 38th IEEE Conf. Decision and Control*, pages 3271–3276, Phoenix (1999).

Jiang, Z.-P., E. D. Sontag, and Y. Wang, Input-to-state stability for discrete-time nonlinear systems, In *Proc. 14th IFAC World Congress*, volume E, pages 277–282, Beijing (1999).

Kazakos, D. and J. Tsinias, "The input-to-state stability condition and global stabilization of discrete-time systems," *IEEE Trans. Auto. Cont.*, **39**, 2111–13 (1994).

Khalil, H. K., *Nonlinear Systems, Second Edition*. Prentice-Hall, Upper Saddle River, NJ (1996).

Krener, A. J., A Lyapunov theory of nonlinear observers, In *Stochastic analysis, control, optimization and applications*, pages 409–420. Birkhäuser, Boston (1999).

Krichman, M., E. D. Sontag, and Y. Wang, Input-output-to-state stability. submitted.

Krichman, M., *A Lyapunov approach to detectability of nonlinear systems*, PhD thesis, Rutgers University (2000). E. Sontag, advisor.

Krstić, M. and H. Deng, *Stabilization of Uncertain Nonlinear Systems*. Springer-Verlag, London (1998).

Krstić, M., I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York (1995).

Liberzon, D., A. S. Morse, and E. D. Sontag, "Output-input stability and minimum-phase nonlinear systems," *IEEE Trans. Auto. Cont.*, pages 2106–2111 (2000). Preliminary version appeared as "A new definition of the minimum-phase property for nonlinear systems, with an application to adaptive control", in *Proc. IEEE Conf. Decision and Control, Sydney, Dec. 2000.*

Liberzon, D., Nonlinear stabilization by hybrid quantized feedback, In Lynch, N. and B. H. Krogh, editors, *Proc. Third International Workshop on Hybrid Systems: Computation and Control*, volume 1790 of *Lecture Notes in Computer Science*, pages 243–257 (2000).

Lin, Y., E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Cont. Opt.*, **34**, 124–160 (1996).

Lohmiller, W. and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, **34**, 683–696 (1998).

Lu, W. M., "A class of globally stabilizing controllers for nonlinear systems," *Sys. Cont. Let.*, **25**, 13–19 (1995a).

Lu, W. M., "A state-space approach to parameterization of stabilizing controllers for nonlinear systems," *IEEE Trans. Auto. Cont.*, **40**, 1576–1588 (1995b).

Marino, R. and P. Tomei, "Nonlinear output feedback tracking with almost disturbance decoupling," *IEEE Trans. Auto. Cont.*, **44**, 18–28 (1999).

Marino, R., G. Santosuosso, and P. Tomei, Robust adaptive observers for nonlinear systems with bounded disturbances, In *Proc. 38th IEEE Conf. Decision and Control*, pages 5200–5205, Phoenix (1999).

McKeithan, T. W., "Kinetic proofreading in T-cell receptor signal transduction," *Proc. Natl. Acad. Sci. USA*, **92**, 5042–5046 (1995).

Morse, A. S., Control using logic-based switching, In Isidori, A., editor, *Trends in Control: A European Perspective*, pages 69–114. Springer, London (1995).

Nešić, D. and E. D. Sontag, "Input-to-state stabilization of linear systems with positive outputs," *Sys. Cont. Let.*, **35**, 245–255 (1998).

Nešić, D. and A. R. Teel, "Input-to-state stability for nonlinear time-varying systems via averaging," *Mathematics of Control, Signals and Systems*. to appear.

Nešić, D., A. R. Teel, and E. D. Sontag, "Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems," *Sys. Cont. Let.*, **38**, 49–60 (1999).

Pan, D. J., Z. Z. Han, and Z. J. Zhang, "Bounded-input-bounded-output stabilization of nonlinear systems using state detectors," *Sys. Cont. Let.*, **21**, 189–198 (1993).

Pogromsky, A. Y., T. Glad, and H. Nijmeije, "On diffusion driven oscillations in coupled dynamical systems," *Int. J. Bifurcation Chaos*, **9**, 629–644 (1999).

Praly, P. and Z. P. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics," *Sys. Cont. Let.*, **21**, 19–33 (1993).

Praly, L. and Y. Wang, "Stabilization in spite of matched unmodelled dynamics and an equivalent definition of input-to-state stability," *Mathematics of Control, Signals and Systems*, **9**, 1–33 (1996).

Rumyantsev, V. V. and A. S. Oziraner, *Stability and Stabilization of Motion with Respect to Part of the Variables (in Russian)*. Nauka, Moscow (1987).

Sanchez, E. N. and J. P. Perez, "Input-to-state stability (ISS) analysis for dynamic neural networks," *IEEE Trans. Circ. and Sys. I: Fund. Theory and Appl.*, **46**, 1395–1398 (1999).

Sepulchre, R., M. Jankovic, and P. V. Kokotović, *Constructive Nonlinear Control*. Springer (1997).

Shiriaev, A. S., The notion of V-detectability and stabilization of invariant sets of nonlinear systems, In *Proc. 37th IEEE Conf. Decision and Control*, pages 2509–2514, Tampa (1998).

Sontag, E. D., "Structure and stability of certain chemical networks and applications to the kinetic proofreading model of T-cell receptor signal transduction," *IEEE Trans. Auto. Cont.* Preprints available as math.DS/9912237 (1999) revised as math.DS/0002113 (2000) "in Los Alamos Archive, `http://arXiv.org`".

Sontag, E. D. and Y. Wang, "Lyapunov characterizations of input to output stability," *SIAM J. Cont. Opt.* to appear.

Sontag, E. D. and Y. Wang, "On characterizations of the input-to-state stability property," *Sys. Cont. Let.*, **24**, 351–359 (1995a).

Sontag, E. D. and Y. Wang, Various results concerning set input-to-state stability, In *Proc. IEEE Conf. Decision and Control, New Orleans, Dec. 1995*, pages 1330–1335. IEEE Publications (1995b).

Sontag, E. D. and Y. Wang, "New characterizations of input to state stability," *IEEE Trans. Auto. Cont.*, **41**, 1283–1294 (1996).

Sontag, E. D., "Smooth stabilization implies coprime factorization," *IEEE Trans. Auto. Cont.*, **34**, 435–443 (1989a).

Sontag, E. D., Some connections between stabilization and factorization, In *Proc. IEEE Conf. Decision and Control*, pages 990–995, Tampa. IEEE Publications (1989b).

Sontag, E. D., "Comments on integral variants of ISS," *Sys. Cont. Let.*, **34**, 93–100 (1998a).

Sontag, E. D., *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer-Verlag, New York, second edition (1998b).

Sontag, E. D., Stability and stabilization: Discontinuities and the effect of disturbances, In Clarke, F. H. and R. J. Stern, editors, *Nonlinear Analysis, Differential Equations and Control (Proc. NATO Advanced Study Institute, Montreal, Jul/Aug 1998)*, pages 551–598, Dordrecht. Kluwer (1999).

Sussmann, H. J., E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Auto. Cont.*, **39**, 2411–2425 (1994).

Teel, A. R. and L. Praly, "On assigning the derivative of a disturbance attenuation control," *Mathematics of Control, Signals and Systems*, **13**, 95–124 (2000).

Teel, A. and E. D. Sontag, "Changing supply functions in input/state stable systems," *IEEE Trans. Auto. Cont.*, **40**, 1476–1478 (1995).

Teel, A. R., D. Nešić, and P. V. Kokotović, A note on input-to-state stability of sampled-data nonlinear systems, In *Proc. 37th IEEE Conf. Decision and Control*, pages 2473–2479, Tampa (1998).

Teel, A. R., "Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem," *IEEE Trans. Auto. Cont.*, **43**, 960–964 (1998).

Tsinias, J. and I. Karafyllis, "ISS property for time-varying systems and application to partial-static feedback stabilization and asymptotic tracking," *IEEE Trans. Auto. Cont.*, **44**, 2173–2184 (1999).

Tsinias, J., "Stochastic input-to-state stability and applications to global feedback stabilization," *Int. J. Control*, **71**, 907–930 (1998).

Vorotnikov, V. I., "Stability and stabilization of motion: Research approaches, results, distinctive characteristics," *Automation and Remote Control*, **54**, 339–397 (1993).