Efficient Nonlinear Model Predictive Control: Exploiting the Volterra-Laguerre Model Structure

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Abstract

An analytical solution to the nonlinear model predictive control (NMPC) optimization problem is derived for single–input single–output (SISO) systems modeled by second–order Volterra–Laguerre models. All input moves except the current move ($m > 1$ in the NMPC framework) are approximated by solving an unconstrained linear MPC problem which utilizes a locally accurate linear model of the process. This linear MPC problem has an analytical solution; this is substituted into a nonlinear equation which is solved exactly for the current input move, $\Delta u(k|k)$. Results using this multi–m NMPC formulation are superior to a previously developed analytical NMPC controller that required $m = 1$ (Parker and Doyle III, 1998).

Keywords

Bioreactor control, Nonlinear model predictive control, Volterra-Laguerre models

Introduction

Model predictive control (MPC) is a control algorithm of industrial and academic interest (Allgöwer et al., 1999; Biegler, 1998) that solves an optimization problem on-line at each time step. For processes that display highly nonlinear behavior, whether due to the operating conditions or nonlinear dynamics (*e.g.* input multiplicative processes and high-purity distillation), performance degradation or instability can result when linear control algorithms are utilized. The use of nonlinear MPC (NMPC) can diminish this performance loss while retaining the multivariable and constraint handling capabilities of MPC.

A high-fidelity nonlinear process model and an optimization routine capable of solving the on-line optimization problem in real-time are required to reap the benefits of the NMPC algorithm. The use of a fundamental process model is conceptually appealing in that the process physics can be explicitly incorporated. Unfortunately, these models require significant time (often measured in man-months or more) and effort to construct, and the resulting NMPC optimization problems are nonconvex and computationally unattractive for most realistic systems (Zheng, 1997; Mayne, 1996).

In place of a fundamental model one can substitute a nonlinear empirical model identified from process data. These data-driven models capture only the input-output behavior of the process, thereby sacrificing physical understanding for rapid model development. One popular model structure, and the model form used in this work, is the second-order Volterra model (Doyle III et al., 1995; Zheng and Zafiriou, 1995) given by the equation:

$$\hat{y}(k) = \sum_{i=1}^{M} h_1(i)u(k-i) + \sum_{i=1}^{M}\sum_{j=1}^{M} h_2(i,j)u(k-i)u(k-j) \quad (1)$$

This model structure can capture the behavior of fading memory nonlinear processes, such as the bioreactor case study examined below. Highly parameterized Volterra models can be efficiently projected onto the Laguerre basis to produce a Volterra-Laguerre model (Schetzen, 1980; Dumont et al., 1994; Zheng and Zafiriou, 1995):

$$\ell(k+1) = A(\alpha)\ell(k) + B(\alpha)u(k) \quad (2)$$

$$\hat{y}(k) = C^T\ell(k) + \ell^T(k)D\ell(k) \quad (3)$$

Nonconvex NMPC optimization problems result from the use of this model form (the objective function is 4^{th}-order in u). If only a single input move is of interest, the optimization problem can be solved analytically (Dumont et al., 1994; Parker and Doyle III, 1998). Another approach to this NMPC problem is to solve for only the current input ($u(k|k)$) exactly, and solve for any future moves approximately using linear MPC and a locally accurate process model (Zheng, 1997) because these future moves are never actually implemented. This paper addresses the synthesis of an NMPC controller which analytically calculates an input profile by combining elements of Parker and Doyle III (1998) and Zheng (1997). Although a benefit of the current work is its computational efficiency, the concept of approximating future manipulated variable moves is employed here because it facilitates the analytical solution to the $m > 1$ problem.

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Input-Output Model Identification

Volterra model identification is accomplished using Algorithm 3 from (Parker et al., 2001). This involves a decomposition of (1) as follows:

$$y(k) = h_0 + \mathcal{L}(k) + \mathcal{D}(k) + \mathcal{O}(k) \tag{4}$$

$$\mathcal{L}(k) = \sum_{i=1}^{M} h_1(i)u(k-i)$$

$$\mathcal{D}(k) = \sum_{i=1}^{M} h_2(i,i)u^2(k-i)$$

$$\mathcal{O}(k) = 2\sum_{i=1}^{M}\sum_{j=1}^{i-1} h_2(i,j)u(k-i)u(k-j)$$

Here the linear, second-order diagonal, and off-diagonal coefficient contributions are given by $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{O}$, respectively. Tailored input sequences, which excite specific contributions and minimize others, provide superior coefficient identification compared to cross-correlation techniques (Parker et al., 2001). The identified Volterra model is then projected onto the Laguerre basis to address the noise sensitivity of Volterra models and simultaneously reduce the model parameterization (9 vs. 860 unique parameters for the Volterra-Laguerre and Volterra models, respectively). This projection yields a Volterra-Laguerre model which optimally approximates the identified Volterra model in the mean-squared error sense. The C vector and D matrix are calculated via least-squares from the identified Volterra kernels, and the Laguerre time-scale, $0 < \alpha \leq 1$, is selected to minimize the error between the identified Volterra model and the expanded Volterra-Laguerre model (where expansion is the inverse of the projection operation). The resulting process model is given by (2) and (3).

Controller Synthesis

The NMPC controller utilizes the standard squared 2-norm objective function given by:

$$\min_{\Delta\mathcal{U}(k|k)} \|\Gamma_y \left[\mathcal{R}(k+1) - \mathcal{Y}(k+1|k)\right]\|_2^2 + \|\Gamma_u \Delta\mathcal{U}(k|k)\|_2^2 \tag{5}$$

The solution developed below employs the formalism of solving for $\Delta\mathcal{U}$ as opposed to absolute $\mathcal{U}$. Matrices Γ_y and Γ_u are used to trade off setpoint tracking versus manipulated variable movement, respectively. The minimization problem in (5) is solved at each sample time for a series of m manipulated variable moves which minimize the objective over a prediction horizon of length p. An analytical solution to the $m = 1$ problem has been developed for SISO problems modeled using the Volterra-Laguerre structure (2) and (3) (Parker and Doyle III, 1998). Limits on the input-output dimension and m resulted from an inability to solve a third-order vector or matrix polynomial.

Utilizing the linear approximation of future manipulated variable moves (Zheng, 1997), the setpoint tracking term of (5) can be decomposed according to the following equation:

$$\min_{\Delta\mathcal{U}(k|k)} \|\Gamma_y \left[\mathcal{R}(k+1) - \mathcal{Y}_N(k+1|k) - \mathcal{Y}_L(k+1|k)\right]\|_2^2 + \|\Gamma_u \Delta\mathcal{U}(k|k)\|_2^2 \tag{6}$$

Here the terms $\mathcal{Y}_N$ and $\mathcal{Y}_L$ represent the contributions of the first calculated input move (nonlinear, exact) and the remaining $m-1$ future input moves (linear, approximate), respectively. The linear component of the problem can be formulated as the solution to a modified reference signal, $\mathcal{R}_L(k+1|k) = \mathcal{R}(k+1) - \mathcal{Y}_N(k+1|k)$:

$$\min_{\Delta\mathcal{U}_L(k+1|k)} \|\Gamma_{yL} \left[\mathcal{R}_L(k+1|k) - \mathcal{Y}_L(k+1|k)\right]\|_2^2 + \|\Gamma_{uL} \Delta\mathcal{U}_L(k+1|k)\|_2^2 \tag{7}$$

An analytical solution to this problem exists (García et al., 1989). The controller model is developed by combining the linear process dynamics (2) with the linearization of the process output (3). The resulting linear controller model is given by (8) and (9).

$$x_L(k+i|k) = \begin{cases} 0 & i = 1 \\ \sum_{j=1}^{i} \bar{A}_{m-j-1} B \Delta u(k+j|k) & 2 \leq i \leq m-1 \\ A^{i-m+1} \sum_{j=1}^{m-1} \bar{A}_{m-j-1} B \Delta u(k+j|k) & i \geq m \end{cases} = G\Delta\mathcal{U}_L(k+1|k) \tag{8}$$

$$y_L(k+i) = \left[C^T + 2x_{lin}^T D\right] x_L(k+i|k) = H x_L(k+i|k) \tag{9}$$

The matrix $\bar{A}_i = (A^{i-1} + A^{i-2} + \ldots + I)$, and the $m-1$ future input moves are given by $\Delta\mathcal{U}_L$. For input multiplicative processes, the x_{lin} vector must change with operating point because no linear integrating controller can stabilize an input multiplicative process at the optimum (Morari, 1983). In this work the matrix H was recalculated at each time step using a local linearization about the current process state, $x_L(k)$ (García, 1984). The G matrix is static, and hence calculated off-line in this work.

The solution to the linear problem is given by the following equation:

$$\Delta\mathcal{U}_L(k+1|k) = \mathcal{K}(\mathcal{R}(k+2) - \mathcal{Y}_N(k+2|k)) \tag{10}$$

where

$$\mathcal{K} = \left(G^T H^T \Gamma_{yL}^T \Gamma_{yL} H G + \Gamma_{uL}^T \Gamma_{uL}\right)^{-1} G^T H^T \Gamma_{yL}^T \Gamma_{yL} \tag{11}$$

The weighting matrices $\Gamma_{yL} = \Gamma_y(2:p, 2:p)$ and $\Gamma_{uL} = \Gamma_u(2:m, 2:m)$ are of dimension $p-1$ and $m-1$, respectively. Equation (10) represents a solution for $\Delta\mathcal{U}_L(k+1|k)$ in terms of only the first input move, $\Delta u(k|k)$. The remainder of the derivation, given below, conceptually follows Parker and Doyle III (1998).

Substituting (10) into (6), yields the objective function

$$\min_{\Delta u(k|k)} \mathcal{E}^T \left[\mathcal{I}^T\Gamma_y^T\Gamma_y\mathcal{I} + \mathcal{K}^T\Gamma_{uL}^T\Gamma_{uL}\mathcal{K}\right]\mathcal{E} + \Delta u^2(k|k)\Gamma_u^2(1,1) \tag{12}$$

where

$$\mathcal{E} = (\mathcal{R}(k+1) - \mathcal{Y}_N(k+1|k)) \tag{13}$$

$$\mathcal{I} = I_{p-1\times p-1} - HG\mathcal{K} \tag{14}$$

$$\mathcal{Y}_N(k+i|k) = \epsilon_0 + \epsilon_1\Delta u(k|k) + \epsilon_2\Delta u^2(k|k) \tag{15}$$

$$\epsilon_2(k+i|k) = B^T\bar{A}_i^T D\bar{A}_i B \tag{16}$$

$$\epsilon_1(k+i|k) = C^T\bar{A}_i B + 2\ell^T(k)\left(A^i\right)^T D\bar{A}_i B + 2\epsilon_2(k+i|k)u(k-1) \tag{17}$$

$$\epsilon_0(k+i|k) = C^T\left(A^i\right)\ell(k) + \ell^T(k)\left(A^i\right)^T D\left(A^i\right)\ell(k) + \epsilon_1(k+i|k)u(k-1) \tag{18}$$

The matrices $\mathcal{K}$ and $\bar{A}_i$ are defined as above, and I is an appropriately sized identity matrix. This objective function is a function of the current Volterra-Laguerre state, $\ell(k)$, the immediate past input, $u(k-1)$, and $\Delta u(k|k)$. If E_i is defined as

$$E_i = \begin{bmatrix} \epsilon_i(k+1|k) \\ \epsilon_i(k+2|k) \\ \vdots \\ \epsilon_i(k+p|k) \end{bmatrix} \tag{19}$$

then the optimal $\Delta u(k|k)$ can be calculated by differentiating equation (12) with respect to $\Delta u(k|k)$ and setting the result equal to zero:

$$0 = \xi_3\Delta u^3(k|k) + \xi_2\Delta u^2(k|k) + \xi_1\Delta u(k|k) + \xi_0 \tag{20}$$

where

$$\xi_3 = 2E_2^T\Gamma_{yN}^T\Gamma_{yN}E_2 \tag{21}$$

$$\xi_2 = 3E_1^T\Gamma_{yN}^T\Gamma_{yN}E_2 \tag{22}$$

$$\xi_1 = 2E_2^T\Gamma_{yN}^T\Gamma_{yN}E_0 + E_1^T\Gamma_{yN}^T\Gamma_{yN}E_1 - 2E_2^T\Gamma_{yN}^T\Gamma_{yN}\mathcal{R}(k+1) + \Gamma_u^2(1,1) \tag{23}$$

$$\xi_0 = E_1^T\Gamma_{yN}^T\Gamma_{yN}E_0 - E_1^T\Gamma_{yN}^T\Gamma_{yN}\mathcal{R}(k+1) \tag{24}$$

$$\Gamma_{yN} = \begin{bmatrix} \Gamma_y(1,1) & 0 \\ 0 & \mathcal{I}^T\Gamma_{yL}^T\Gamma_{yL}\mathcal{I} + \mathcal{K}^T\Gamma_{uL}^T\Gamma_{uL}\mathcal{K} \end{bmatrix} \tag{25}$$

Provided that the ξ_i's are real, a solution to (20) exists (Tuma, 1987). The solution method involves a substitution for $\Delta u(k|k)$, and interested readers are referred to (Tuma, 1987) as the complete derivation is omitted due to space limitations. The cubic equation solution involves a term, $\mathcal{D}$, which lies beneath a radical similar to $\sqrt{b^2-4ac}$ in the quadratic equation. Roots of (20) are determined by the value of $\mathcal{D}$ as follows:

$$\begin{array}{ll} \mathcal{D} > 0 & \text{one real root, 2 complex roots} \\ \mathcal{D} = 0 & \text{3 real roots, at least 2 equal} \\ \mathcal{D} < 0 & \text{3 real unequal roots} \end{array} \tag{26}$$

All real roots are transformed back to the initial problem space and are analyzed for optimality by simulating the Volterra-Laguerre model over the prediction horizon to calculate an objective function value (OFV). The $\Delta u(k|k)$ which minimizes the OFV is implemented.

The analytical NMPC algorithm developed in this work retains the constraint handling capabilities of the algorithm developed previously (Parker and Doyle III, 1998). However, only the first (nonlinear) input move is constrained. In this formulation, solutions which cause the manipulated variable to violate imposed magnitude constraints are replaced by the constraint itself, and the OFV is recalculated. The unconstrained solution of the linear controller subproblem is used. This is done to maintain the existence of an analytical solution to the control problem, (5). Constraint handling on the future moves can be implemented in a "soft" framework, where Γ_{uL} can be tuned independently from $\Gamma_u(1,1)$, so that the linear moves are penalized for large magnitude changes. Although the algorithm developed in the current work has relaxed the limitation of $m = 1$ imposed on the analytical NMPC algorithm in Parker and Doyle III (1998), it is still limited to SISO problems. Formulation of the multivariable problem with $m > 1$ is feasible in the context of this algorithm. However, equation (20) would be a third-order vector polynomial, and an analytical solution for this problem requires further work. Rate constraints are not included in this formulation because they are not relevant to the case study below, but incorporation of rate constraints into the nonlinear problem would be straightforward.

Case Study: Continuous Bioreactor

A model for the growth of *Klebsiella pneumoniae* on glucose in a continuous-flow bioreactor was developed by Baloo and Ramkrishna (1991). Cell biomass exit concentration (g/L) was the output of interest, and dilution rate (hr^{-1}) was the manipulated variable. The nominal operating condition used in this study was 0.97 hr^{-1} yielding a biomass concentration of 0.2373 g/L.

From this system, a second-order Volterra-Laguerre model was identified using Algorithm 3 from Parker et al. (2001). The sample time was 15 minutes, and the Volterra model memory was $M = 40$. The inputs and outputs were scaled according to $u = \frac{u_{actual}-u_{nominal}}{0.08}$ and $y = \frac{y_{actual}-y_{nominal}}{0.01}$. The input sequence amplitude for identification was 2.375. After projection, the result-

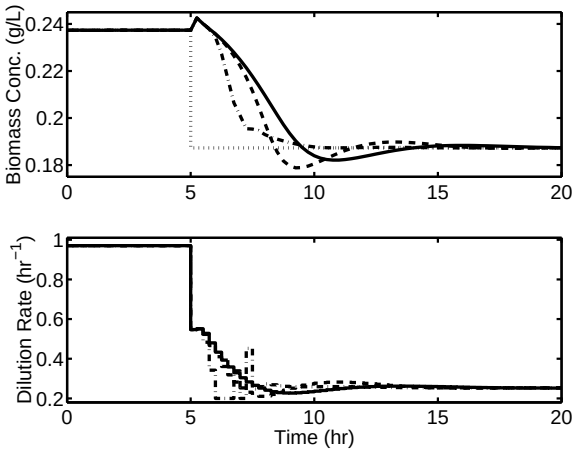

Figure 1: Response to a -0.05 g/L change in the reference (dotted) at $t = 5$ hr for various move horizon lengths: $m = 1$ (solid), $m = 14$, (dashed), and $m = 14$ w/ RLS (dash-dot). Other tuning parameters: $p = 16, \Gamma_y = I_{p\times p}, \Gamma_u(1,1) = 0, \Gamma_{uL} = I_{m-1\times m-1}$.

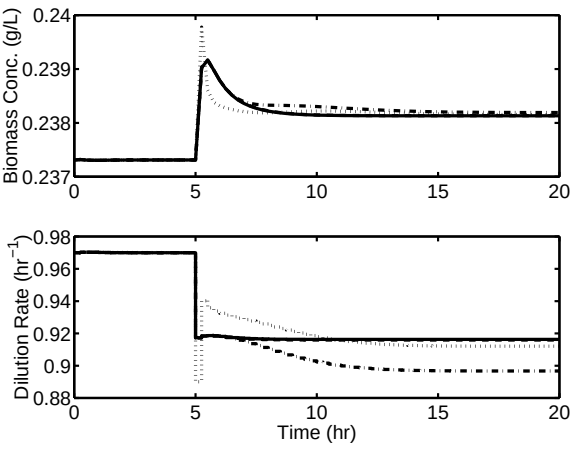

Figure 2: Response to a +0.025 g/L change in the reference at $t = 5$ hr for various move horizon lengths: $m = 1$ (solid), $m = 14$, (dashed), and $m = 14$ w/ RLS (dash-dot), and nonlinear programming MPC with $m = 14$ and RLS (dotted). Other tuning parameters: $p = 16, \Gamma_y = I_{p\times p}, \Gamma_u(1,1) = 0, \Gamma_{uL} = 0.1 I_{m-1\times m-1}$.

ing Volterra-Laguerre model ((2), (3)) with $\alpha = 0.59$ had the following matrices:

$$A = \begin{bmatrix} 0.59 & 0 & 0 \\ 0.652 & 0.59 & 0 \\ -0.385 & 0.652 & 0.59 \end{bmatrix} \tag{27}$$

$$B = [0.807 \quad -0.476 \quad 0.281]^T \tag{28}$$

$$C = [-0.249 \quad 0.128 \quad 0.002]^T \tag{29}$$

$$D = \begin{bmatrix} -0.050 & -0.006 & 0.010 \\ -0.006 & -0.023 & 0.008 \\ 0.010 & 0.008 & -0.013 \end{bmatrix} \tag{30}$$

Validation resulted in unbiased residuals of less than 0.5 mg/L, such that in the region of identification, the identified model and the actual differ by less than 2%.

Partial motivation for developing the analytical NMPC controller capable of handling $m > 1$ is the expectation that the use of larger move horizons would result in more aggressive controller response and therefore improved performance. Magnitude constraints of $0.2 \leq u(k) \leq 1.1$ hr^{-1} were imposed on the dilution rate, so that the cells did not starve or get washed out of the reactor, respectively. The response of the system under analytical NMPC control to a step change of -0.05 g/L in the reference signal is shown in Figure 1. The nonlinear programming MPC solution to this problem is not shown, because it is trapped in a local minimum at the high dilution rate constraint, and is therefore unable to track the setpoint change. The increase in move horizon improves tracking performance by 5%. A more aggressive response to the offset observed before $t = 10$ hr can be seen in the manipulated variable. Typical of more aggressive controllers, greater undershoot is also observed. The addition of a recursive least-squares (RLS) algorithm to update the Volterra-Laguerre model on-line is straightforward, and the C and D matrices are updated using a standard algorithm (Ljung, 1987). The improvement in the model accuracy is evidenced by the lack of undershoot and reduced oscillation around the reference. The more accurate model improves the controller prediction, thereby improving performance by 14% versus $m = 14$ without RLS and by 18% versus the $m = 1$ analytical NMPC controller. It should be noted that changes in the tuning weights, Γ_y and Γ_u, will change the degree of performance improvement.

In the case of an unreachable setpoint, the analytical NMPC algorithm performs as shown in Figure 2. All controllers remained stable. There was little observable difference between the $m = 1$ and $m = 14$ controllers. The full nonlinear solution implemented with RLS (nonlinear programming MPC, the dotted line) outperformed the controllers without RLS, but the analytical solution controller with $m = 14$ and RLS reached a steady state closest to the actual process optimum (dilution rate= 0.89hr^{-1}). It is possible that the full nonlinear solution could approach the analytical solution if the measurement signal were not noise-free, which limited the ability of the algorithm to update the model after $t = 10$ hr.

Summary

Given the nonlinear nature of many processes (*e.g.* bioreactors and CSTRs) and the nonlinear behaviors which result from operating in certain regimes (*e.g.* high purity

distillation), nonlinear MPC can offer significant performance improvements. The use of data-driven models in NMPC facilitates model development, and the choice of a particular structure can further simplify controller synthesis. By exploiting the structure of the Volterra-Laguerre (or equivalently, Volterra) model, an analytical solution to the NMPC problem was derived. Although this solution is not exact for $m > 1$, as it includes an approximation for the future input moves, significant performance improvement was observed in comparison with controllers synthesized using $m = 1$. A standard recursive least-squares algorithm, used in conjunction with the analytical NMPC controller, led to superior performance due to the ability of the algorithm to further update the model based on the current operating point.

Acknowledgments

This work was initiated while the author was at the University of Delaware and was partially funded by the University of Delaware Process Control and Monitoring Consortium. Additional funding has been provided by the Lai Family Foundation. The author would like to thank F. J. Doyle III for his insightful comments on this work.

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