Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Optimal Consumption-Investment Problems in Incomplete Markets with Random Coefficients

Netzahualcóyotl Castañeda-Leyva*
Universidad Autónoma de Aguascalientes
Departamento de Estadística
Av. Universidad 940,
Ciudad Universitaria 20100
Aguascalientes, Ags., MÉXICO
ncastane@uaa.mx

Daniel Henández-Hernández
Centro de Investigación en Matemáticas
Callejón de Jalisco S/N, Valenciana
Guanajuato, Gto. 36240 MÉXICO
dher@cimat.mx

Abstract

In this work we present the explicit solution of an optimal investment problem in an incomplete financial market, for HARA and logarithmic utility functions. The market follows a generalization of the Black and Scholes diffusion model, which consists of a bank account, a risky asset, and an economic external factor. The coefficients of the underlying diffusion processes are random and depend on the economic external factor. This market includes more realistic financial scenarios where the martingale methodology and stochastic control techniques, established in Castañeda-Leyva and Hernández-Hernández [2], are applied.

Keywords: Optimal investment and consumption, incomplete markets, stochastic volatility, martingale method, optimal control, Black-Scholes model.

1 Introduction

We consider the problem of maximizing the expected utility of final wealth and consumption in a finite horizon $T > 0$, when the financial market is incomplete. This problem is solved using the martingale method, consisting in translating the original investor's problem, called *primal* problem into a convex optimization problem, and then solve the associated dual problem; see Karatzas and Shreve [9]. In Section 2 the model is presented, whereas the martingale method is developed in Section 3. Finally, an explicit solution is given for the particular examples of the hyperbolic absolute risk aversion (HARA) and the logarithmic utility functions.

2 The model

In this section we shall introduce a factor model for the financial market based on the model presented by Bielecki and Pliska [1]; we also state the problem we want to solve.

The financial market is governed by a standard two-dimensional Brownian motion (BM) $\{(W_{1t}, W_{2t}), \mathcal{F}_t\}_{0\leq t\leq T}$, defined in a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmentation of the filtration $\left\{\mathcal{F}_t^{(W_1,W_2)}\right\}_{0\leq t\leq T}$. This market is composed by a bank account, a risky asset, and a correlated external factor, such that:

1. The *bank account* process is given by

$$S_t^0 \circeq \exp\left(\int_0^t r(Y_u)\,du\right); \quad t\in[0,T],$$

where $r(\cdot)$ is the *interest rate* function.

2. The *asset price* process S satisfies the stochastic differential equation

$$dS_t = S_t\left[\mu(Y_t)\,dt + \sigma(Y_t)\,dW_{1t}\right], \tag{1}$$

with $S_0 = 1$. Here $\mu(\cdot)$ and $\sigma(\cdot)$ are the *rate return* and the *volatility* functions, respectively.

3. The dynamics of the *external factor* Y is give by the stochastic differential equation

$$\begin{aligned} &dY_t \\ &= g(Y_t)\,dt + \rho(Y_t)\,dW_{1t} + \varepsilon(Y_t)\,dW_{2t}, \end{aligned} \tag{2}$$

*The research of this author was supported by grants PROMEP-UAAGS-EXB-69 and UAA-PIEST-04-01.

with $Y_0 = y \in \mathbf{R}$.

All the coefficients may be random and depend on the external economic factor Y. In particular, when ρ and ε are constants with $\rho^2 + \varepsilon^2 = 1$, then ρ is the correlation coefficient between the underlying BM's of the asset price and the external factor. This is the case studied in Castañeda-Leyva and Hernández-Hernández [2]. In particular, when $\rho = \pm 1$ the market is complete. Otherwise, it becomes incomplete, since the external factor cannot be traded.

CONDITION 2.1. *1. $\mu(\cdot), \sigma(\cdot)$, and $r(\cdot)$ belong to $C_b^2(\mathbf{R})$.*
2. $r(\cdot) \geq 0$ and $\sigma(\cdot) > \sigma_0$, for some $\sigma_0 > 0$.
3. $g(\cdot) \in C^1(\mathbf{R})$ such that $g'(\cdot) \in C_b(\mathbf{R})$.
4. $\rho(\cdot), \varepsilon(\cdot)$ belong to $C_b^1(\mathbf{R})$ such that $\rho(\cdot) \geq \rho_0$ and $\varepsilon(\cdot) \geq \varepsilon_0$, for some $\rho_0, \varepsilon_0 > 0$. Without loss of generality assume that $\sigma_0 \leq \rho^2(\cdot) + \varepsilon^2(\cdot) \leq 1$.

An important example is when the external factor Y is modeled as an Ornstein-Uhlenbeck process, where in this case, $g(y) = -\beta(y - \bar{y})$ and $\rho(\cdot)$ and $\varepsilon(\cdot)$ are constant functions with $\rho^2 + \varepsilon^2 = 1$; for some fixed parameters $\beta, \bar{y} > 0$.

Now, consider a single investor with *initial capital* x, who generates a *wealth* process

$$X_t^{\pi,c} \doteq X_t^{x,y,\pi,c}; \quad t \in [0,T],$$

following the integral equation

$$\begin{aligned} & X_t^{\pi,c} + \int_0^t c_u du \\ \doteq \; & x + \int_0^t \left(r(Y_u) X_u^{\pi,c} + [\mu(Y_u) - r(Y_u)] \pi_u\right) du \\ & + \int_0^t \pi_u \sigma(Y_u) dW_{1u}. \end{aligned}$$

Here $\{\pi_t, \mathcal{F}_t\}_{0 \leq t \leq T}$ and $\{c_t, \mathcal{F}_t\}_{0 \leq t \leq T}$ are the trading *portfolio* process and *consumption* process respectively, such that are progressively measurable with

$$\int_0^T \pi_u^2 du, \; \int_0^T c_t dt < \infty,$$

and $c \geq 0$ a.s. The trading *strategy* (π, c) is *admissible* if $X^{\pi,c} \geq 0$ a.s. and the set of such trading strategies is denoted as $\mathcal{A}(x,y)$. Finally, assume that $x, T \in \mathbf{R}_+ \doteq (0,\infty)$, and $y \in \mathbf{R}$.

Given U_1, $U_2 : \mathbf{R}_+ \to \mathbf{R}$ *utility* functions, the *investor's problem* consists in to

$$\text{(3)} \qquad \text{maximize } E\left\{U_1\left(X_T^{\pi,c}\right) + \int_0^T U_2(c_t)\, dt\right\},$$

over $(\pi, c) \in \mathcal{A}(x,y)$, as well as to provide an *optimal* trading strategy $(\hat{\pi}, \hat{c})$. Here $U(\cdot) = U_1(\cdot), U_2(\cdot)$ is strictly increasing, strictly concave, and differentiable, such that $U'(\infty) \doteq \lim_{b \uparrow \infty} U'(b) = 0$ and $U'(0+) \doteq \lim_{b \downarrow 0} U'(b) = \infty$. To obtain the solution of this optimization problem, we will use the martingale approach and stochastic control techniques; see Castañeda-Leyva and Hernández-Hernández [2] and Harrison and Pliska [7].

Define the *equivalent local martingale measures* family as

$$\begin{aligned} \mathcal{P} \doteq \; & \{Q \mid P \prec Q \prec P, \; S/S^0 \text{ is a } Q\text{-local} \\ & \text{martingale, for all } y \in \mathbf{R}\}, \end{aligned}$$

and $\mathcal{M}$ as the set of all progressively measurable processes $\{\nu_t, \mathcal{F}_t\}_{t \in [0,T]}$, with $E \int_0^T \nu_u^2 du < \infty$, such that the local martingale

$$Z_t^\nu \doteq e^{-\int_0^t [\theta(Y_u) dW_{1u} + \nu_u dW_{2u}] - \frac{1}{2}\int_0^t [\theta^2(Y_u) + \nu_u^2] du}$$

is a martingale, for all $y \in \mathbf{R}$. Here

$$\theta(\cdot) \doteq \frac{\mu(\cdot) - r(\cdot)}{\sigma(\cdot)}.$$

Note that $\theta(\cdot) \in C_b^2(\mathbf{R})$, since $\sigma(\cdot) > \sigma_0 > 0$.

The process ν in $\mathcal{M}$ plays the role of control process in the associate *dual* problem. Moreover, for each $\nu \in \mathcal{M}$, a probability measure $P^\nu \in \mathcal{P}$ can be defined in $(\Omega, \mathcal{F}_T)$ as

$$P^\nu(A) \doteq \int_A Z_T^\nu dP; \quad A \in \mathcal{F}_T.$$

In this sense, $\mathcal{M}$ is a subset of $\mathcal{P}$. Under the measure P^ν, $\{(W_{1t}^\nu, W_{2t}^\nu), \mathcal{F}_t\}_{0 \leq t \leq T}$ is a BM, where

$$\text{(4)} \qquad \begin{aligned} W_{1t}^\nu & \doteq W_{1t} + \int_0^t \theta(Y_u)\, du, \\ W_{2t}^\nu & \doteq W_{2t} + \int_0^t \nu_u du. \end{aligned}$$

In addition, S/S^0 is a continuous P^ν-martingale and $X_\cdot^{\pi,c}/S_\cdot^0 + \int_0^\cdot \left(c_t/S_t^0\right) dt$ is a nonnegative continuous

P^ν-local martingale and a P^ν-supermartingale; see Karatzas and Shreve [8].

Finally, note that our market is incomplete and is free of arbitrage opportunities, since $\mathcal{P}$ is non empty and non trivial; see Delbaen and Schachermayer [4].

3 Martingale method

In this section the investor's problem is formulated as a convex optimization problem and, using the martingale method, the associated dual problem is obtained. Furthermore, a relevant relationship between the optimal solutions of both problems is presented.

The following lemma allows us to obtain a useful characterization of the set of admissible trading strategies $\mathcal{A}(x,y)$. It is analogous to Theorem 1 in Cuoco [3] and Theorem 5.6.2 in Karatzas and Shreve [9]; see also Lemma 2.2 in Castañeda-Leyva and Hernández-Hernández [2].

LEMMA 3.1. *Let $B \geq 0$ be a $\mathcal{F}_T$-measurable random variable and c a consumption process such that*

$$\sup_{\nu\in\mathcal{M}} E^\nu\left\{\frac{B}{S_T^0}+\int_0^T \frac{c_t}{S_t^0}dt\right\}\leq x. \tag{5}$$

Then, there exists a trading portfolio π such that $(\pi,c)\in\mathcal{A}(x,y)$ and $X_T^{\pi,c}\geq B$ a.s. Conversely, if $(\pi,c)\in\mathcal{A}(x,y)$, then $B\doteq X_T^{\pi,c}$ satisfies the budget constraint *(5).*

Therefore, the investor's problem (3) can be written as

$$\text{maximize } E\left\{U_1(B)+\int_0^T U_2(c_t)\,dt\right\}, \tag{6}$$

over $(B,c)\in\mathcal{B}(x,y)$, where

$$\mathcal{B}(x,y) \;\doteq\; \{(B,c)\mid B\geq 0 \text{ and } c\geq 0 \text{ such that satisfy (5)}\}.$$

Here the argument "$B\geq 0$ and $c\geq 0$" means that B is a nonnegative $\mathcal{F}_T$ measurable random variable and c is a consumption process. This problem will be referred to as the *primal* problem; see Section 8.6 in Luenberger [10].

For a utility function $U(\cdot)$, define the conjugate function

$$\tilde{U}(z)\doteq\sup_{b>0}\{U(b)-zb\};\; z>0. \tag{7}$$

From elementary calculus, it follows that

$$\tilde{U}(z)=U(I(z))-zI(z);\; z>0, \tag{8}$$

where $I(\cdot)$ is the inverse function of $U'(\cdot)$.

The associated *dual functional* to the primal problem (6) is defined as

$$\begin{aligned} &L(\nu,\lambda)\\ \doteq\;& \lambda x+\sup_{B\geq 0,c\geq 0}\left\{E\left[U_1(B)+\int_0^T U_2(c_t)\,dt\right]\right.\\ &\left.-\lambda E^\nu\left[\frac{B}{S_T^0}+\int_0^T\frac{c_t}{S_t^0}dt\right]\right\}; \end{aligned}$$

for $\nu\in\mathcal{M}$ and $\lambda\geq 0$. The *dual* problem is to

$$\text{minimize } L(\nu,\lambda),\text{ over } \nu\in\mathcal{M},\ \lambda>0. \tag{9}$$

Note that the dual functional $L(\nu,\lambda)$ is given by

$$\begin{aligned} L(\nu,\lambda) \;=\;& E\left[\tilde{U}_1\left(\lambda\frac{Z_T^\nu}{S_T^0}\right)\right.\\ &\left.+\int_0^T\tilde{U}_2\left(\lambda\frac{Z_t^\nu}{S_t^0}\right)dt\right]+\lambda x; \end{aligned} \tag{10}$$

for $\nu\in\mathcal{M}$, $\lambda>0$. It is easy to see that

$$\begin{aligned} &\sup_{(B,c)\in\mathcal{B}(x,y)} E\left\{U_1(B)+\int_0^T U_2(c_t)\,dt\right\}\\ &\leq \inf_{\nu\in\mathcal{M},\lambda>0} L(\nu,\lambda). \end{aligned} \tag{11}$$

When the equality holds in (11), we say that there is no duality *gap*. This is true for logarithmic and HARA utility functions.

The next proposition shows the relationship between the optimal solutions of the problems (6) and (9); compare with Proposition 6.3.8 in Karatzas and Shreve [9].

PROPOSITION 3.1. *Assume that for some $\left(\hat{\nu},\hat{\lambda}\right)\in\mathcal{M}\times\mathbf{R}_+$ the pair $\left(\hat{B},\hat{c}\right)$, defined as*

$$\hat{B}\doteq I_1\left(\hat{\lambda}\frac{Z_T^{\hat{\nu}}}{S_T^0}\right),\quad \hat{c}_t\doteq I_2\left(\hat{\lambda}\frac{Z_t^{\hat{\nu}}}{S_t^0}\right); \tag{12}$$

belongs to $\mathcal{B}(x,y)$ and satisfies

$$E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0}+\int_0^T\frac{\hat{c}_t}{S_t^0}dt\right]=x. \tag{13}$$

Then, $\left(\hat{B},\hat{c}\right)$ is the optimal solution to the primal problem (6), whereas $\left(\hat{\nu},\hat{\lambda}\right)$ is the optimal solution to the dual problem (9). In particular, there is no duality gap.

4 HARA and logarithmic utility functions

In this section the solution to the consumption-investment problem shall be given when the utility function is HARA and logarithmic. The martingale method allows us to obtain an explicit form for the optimal process $\hat{\nu}$ and the optimal trading strategy $(\hat{\pi}, \hat{c})$.

Here we assume that $U(\cdot) = U_1(\cdot) = U_2(\cdot)$, where

$$U(b) \doteq \frac{b^\gamma}{\gamma}; \quad b > 0,$$

with $0 \neq \gamma < 1$. From (9) and (10), we wish to

$$\text{minimize } J(T, y, \nu), \text{ over } \nu \in \mathcal{M}, \tag{14}$$

where

$$J(T, y, \nu) \doteq E\left[\left(\frac{Z_T^\nu}{S_T^0}\right)^\alpha + \int_0^T \left(\frac{Z_t^\nu}{S_t^0}\right)^\alpha dt\right]$$

and $\alpha \doteq -\gamma/(1-\gamma)$. Note that $0 \neq \alpha < 1$. We present only the solution to the case when $\gamma > 0$, since the solution for $\gamma < 0$ is similar. An advantage of the martingale method applied in this case is the reduction to just one control variable ($\nu \in \mathcal{M}$). On the other hand, observe that, for $t \in [0, T]$

$$(Z_t^\nu)^\alpha = Z_t^{\alpha,\nu} e^{-\frac{1}{2}\alpha(1-\alpha)\int_0^t [\theta^2(Y_s)+\nu_s^2]ds},$$

where

$$\begin{aligned} Z_t^{\alpha,\nu} &\doteq \exp\left(-\alpha \int_0^t [\theta(Y_u) dW_{1u} + \nu_u dW_{2u}]\right. \\ &\quad \left. -\frac{\alpha^2}{2}\int_0^t \left[\theta^2(Y_u) + \nu_u^2\right] du\right). \end{aligned}$$

Analogous to the measure P^ν and the BM on (4), using the process $Z^{\alpha,\nu}$, define the measure $P^{\alpha,\nu}$ in $\mathcal{F}_T$ and the BM $(W_1^{\alpha,\nu}, W_2^{\alpha,\nu})$; see Fleming and Hernández-Hernández [5]. Under $P^{\alpha,\nu}$, the dynamics of the process Y satisfies

$$\begin{aligned} dY_t &= [g(Y_t) - \alpha\rho(Y_t)\theta(Y_t) - \alpha\varepsilon(Y_t)\nu_t] dt \\ &\quad + \rho(Y_t) dW_{1t}^{\alpha,\nu} + \varepsilon(Y_t) dW_{2t}^{\alpha,\nu}. \end{aligned} \tag{15}$$

Therefore

$$\begin{aligned} & J(T, y, \nu) \\ = \; & E^{\alpha,\nu}\left[e^{\int_0^T q(Y_t,\nu_t)dt} + \int_0^T e^{\int_0^t q(Y_s,\nu_s)ds} dt\right], \end{aligned}$$

for $\nu \in \mathcal{M}$, where

$$q(y, v) \doteq -\alpha\left[r(y) + \frac{1}{2}(1-\alpha)\left(\theta^2(y) + v^2\right)\right];$$

for $y, v \in \mathbf{R}$. Note that $q(y, v)$ is bounded, provided the control space is compact.

Define the *value function* of (14) as

$$W(T, y) \doteq \inf_{\nu \in \mathcal{M}} J(T, y, \nu). \tag{16}$$

Now, temporarily, consider $[-M, M]$ as the bounded control space, that is $\nu \in \mathcal{M}^M$; for $M > 0$ given. The verification theorem below states that

$$w(T, y) = W^M(T, y);$$

where $w(T, y)$ is the unique smooth function in $C^{1,2}\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right) \cap C_p\left(\bar{\mathbf{R}}_+ \times \mathbf{R}\right)$ satisfying the associated Hamilton Jacobi Bellman equation

$$\begin{aligned} w_T &= 1 + \frac{1}{2}\left(\rho^2 + \varepsilon^2\right) w_{yy} + (g - \alpha\rho\theta) w_y \\ &\quad -\alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right] w + \alpha \times \\ &\quad \sup_{v \in [-M,M]}\left\{-\varepsilon w_y v - \frac{1}{2}(1-\alpha) w v^2\right\}, \end{aligned} \tag{17}$$

with $w(0, y) = 1$; see Theorem IV.4.3 and Remark IV.4.1 in Fleming and Soner [6]. The Markov policy induced by the previous equation is

$$\nu^*(t, y) = -\frac{\varepsilon(y)}{1-\alpha}\frac{w_y(t, y)}{w(t, y)}, \tag{18}$$

if

$$|\varepsilon(y) w_y(t, y)| \leq M(1-\alpha) w(t, y), \; w(t, y) \neq 0,$$

and

$$\nu^*(t, y) = -M \operatorname{sgn}(\varepsilon(y) w_y(t, y)),$$

otherwise. We will prove below that $W^M(T, y)$ does not depend on M.

Theorem 4.1. (Verification) *For $M > 0$, let $w(T, y)$ be the unique solution to (17). Then*

$$\begin{aligned} w(T, y) &\leq J(T, y, \nu), \\ w(T, y) &= W^M(T, y) = J(T, y, \hat{\nu}), \end{aligned}$$

for $\nu \in \mathcal{M}^M$, where

$$\hat{\nu}_t = \nu^*(T - t, Y_t) \in \mathcal{M}^M$$

and $\nu^*(t, y)$ is the Markov policy defined in (18). That is, $\hat{\nu}$ is the optimal control process for the constrained auxiliary problem relative to (16).

THEOREM 4.2. *Let $W(T,y)$ be the unconstrained value function (16). There exists a constant $K>0$ such that for $M > K/(1-\alpha)$, implies that*

$$w(T,y) = W(T,y)$$

and, the optimal control process is give by

$$\hat{\nu}_t = \nu^*(T-t, Y_t); \tag{19}$$

Furthermore, $W \in C^{1,2}(\bar{\mathbf{R}}_+ \times \mathbf{R}) \cap C_b^{0,1}(\bar{\mathbf{R}}_+ \times \mathbf{R})$ and solves the differential equation

$$\begin{aligned} W_T &= 1 + \frac{1}{2}W_{yy} + (g - \alpha\rho\theta) W_y \\ &\quad -\alpha\left[r + \frac{1}{2}(1-\alpha)\theta^2\right] W - \frac{\gamma\varepsilon^2}{2}\frac{W_y^2}{W}, \end{aligned} \tag{20}$$

with $W(0,y) = 1$.

Now we give the explicit optimal trading strategy for the investor's problem.

According to (12), we consider

$$\begin{aligned} \hat{B} &\doteq I\left(\hat{\lambda}\frac{Z_T^{\hat{\nu}}}{S_T^0}\right) = \frac{x}{\Lambda_{\hat{\nu}}}\left(\frac{S_T^0}{Z_T^{\hat{\nu}}}\right)^{1-\alpha}, \\ \hat{c}_t &\doteq I\left(\hat{\lambda}\frac{Z_t^{\hat{\nu}}}{S_t^0}\right) = \frac{x}{\Lambda_{\hat{\nu}}}\left(\frac{S_t^0}{Z_t^{\hat{\nu}}}\right)^{1-\alpha}; \end{aligned}$$

where

$$\begin{aligned} \Lambda_{\hat{\nu}} &= J(T,y,\hat{\nu}) = W(T,y), \\ \hat{\lambda} &= \left(\frac{x}{\Lambda_{\hat{\nu}}}\right)^{1-\alpha}, \end{aligned}$$

and $\hat{\nu}$ is given by (19). Now, define

$$\frac{\hat{X}_t}{S_t^0} \doteq E^{\hat{\nu}}\left[\frac{\hat{B}}{S_T^0} + \int_t^T \frac{\hat{c}_u}{S_u^0}du \mid \mathcal{F}_t\right];\ t \in [0,T].$$

We get

$$\frac{\hat{X}_t}{S_t^0} = x\frac{\left[Z_t^{\hat{\nu}}\right]^{\alpha-1}}{\left[S_t^0\right]^{\alpha}}\frac{W(T-t,Y_t)}{W(T,y)}$$

and, by Ito's formula,

$$d\frac{\hat{X}}{S^0} + \frac{\hat{c}}{S^0}dt = \frac{\hat{\pi}}{S^0}\sigma dW_1^{\hat{\nu}};$$

where

$$\begin{aligned} \hat{\pi}_t &\doteq \pi^*\left(T-t, \hat{X}_t, Y_t\right), \\ \hat{c}_t &= c^*\left(T-t, \hat{X}_t, Y_t\right), \end{aligned}$$

with

$$\begin{aligned} &\pi^*(t,x,y) \\ &\doteq \frac{x}{\sigma(y)}\left[(1-\alpha)\theta(y) + \rho(y)\frac{W_y(t,y)}{W(t,y)}\right] \end{aligned}$$

and

$$c^*(t,x,y) \doteq \frac{x}{W(t,y)}.$$

Logarithmic example. In (3), suppose that

$$U_1(b) = U_2(b) \doteq \log b; \quad b > 0.$$

Then, from (9) and (10), the dual problem is to

$$\text{minimize} \quad E\left\{\int_0^T \nu_t^2 dt + \int_0^T \int_0^t \nu_s^2 ds dt\right\},$$

over $\nu \in \mathcal{M}$.

The optimal values are given by

$$\hat{\nu} \equiv 0 \quad \text{and} \quad \hat{\lambda} = \frac{1+T}{x}.$$

Finally, similarly to the HARA case, the optimal trading strategy $(\hat{\pi}, \hat{c})$ is given by the feedback form

$$\begin{aligned} \hat{\pi}_t &= \pi^*\left(t, X_t^{\hat{\pi},\hat{c}}, Y_t\right), \\ \hat{c}_t &= c^*\left(t, X_t^{\hat{\pi},\hat{c}}, Y_t\right); \end{aligned}$$

where

$$\begin{aligned} \pi^*(t,x,y) &\doteq x\frac{\mu(y) - r(y)}{\sigma^2(y)} \\ &= x\frac{\theta(y)}{\sigma(y)} \end{aligned}$$

and

$$c^*(t,x,y) \doteq \frac{x}{1+T-t};$$

for $(t,x,y) \in [0,T] \times \mathbf{R}_+ \times \mathbf{R}$.

References

[1] Bielecki, T.R. and S.R. Pliska. Risk Sensitive dynamic asset management, *Appl. Math. Optim.* **39** (1999), 337-360.

[2] Castañeda-Leyva, N. and D. Hernández-Hernández. Optimal consumption-investment problems in incomplete financial markets with stochastic coefficients, *SIAM J. Control Optim.* (2005).

[3] Cuoco, D. Optimal consumption and equilibrium prices with trading portfolio constraints and stochastic income, *J. Econom. Theory.* **72** (1997), 33-73.

[4] Delbaen, F. and W. Schachermayer. A general version of the fundamental theorem of asset pricing, *Math. Ann.* **300** (1994), 463-520.

[5] Fleming, W.H. and D. Hernández-Hernández. An optimal consumption model with stochastic volatility, *Finan. Stoch.* **7** (2003), 245-262.

[6] Fleming, W.H. and H.M. Soner. *Controlled Markov processes and viscosity solutions,* Springer-Verlag New York, 1993.

[7] Harrison, J.M. and S.R. Pliska. Martingales and stochastic integrals in the theory of continuous trading, *Stoch. Proc. and Appl.* **11** (1981), 215-260.

[8] Karatzas, I. and S.E. Shreve. *Brownian Motion and Stochastic Calculus*, Springer-Verlag New York, 1991.

[9] Karatzas, I. and S.E. Shreve. *Methods of Mathematical Finance,* Springer-Verlag New York, 1998.

[10] Luenberger, D.G. *Optimization by Vector Space Methods*, John Wiley & Sons New York, 1969.