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Disturbance Tolerance and Rejection of Linear Systems with Imprecise Knowledge of Actuator Input Output Characteristics

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Abstract— **In this paper, we study the robustness of linear systems with respect to the disturbances and the uncertainties in the actuator input output characteristics. Disturbances either bounded in energy or bounded in magnitude are considered. The actuator input output characteristics are assumed to reside in a so-called generalized sector bounded by piecewise linear curves. Robust bounded state stability of the closed-loop system is first defined and characterized in terms of linear matrix inequalities (LMIs). Based on this characterization, the evaluation of the disturbance tolerance and disturbance rejection capabilities of the closed-loop system under a given feedback law is formulated into and solved as optimization problems with LMI constraints. The maximal tolerable disturbance is then determined by optimizing the disturbance tolerance capability of the closed-loop system over the choice of feedback gains. Similarly, the design of feedback gain that maximizes the disturbance rejection capability of can be carried out by viewing the feedback gain as an additional free parameter in the optimization problem for the evaluation of the disturbance rejection capability under a given feedback gain.**

I. INTRODUCTION

In this paper, we consider the robustness analysis and state feedback design for an uncertain nonlinear system,

$$\left\{ \begin{array}{rcl} \dot{x} &=& Ax + B\psi(u,t) + E\omega, \\ u &=& Fx, \\ z &=& Cx, \end{array} \right. \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the control input, $\omega \in \mathbf{R}^p$ is the disturbance, $z \in \mathbf{R}^q$ is the controlled output of the system, and the function $\psi(u,t)$ represents the actuator input output characteristics, which is a nonlinear function, such as a saturation like function, and is not precisely known. We also assume that ω belongs to one of the following two classes of disturbances whose energy or magnitudes are bounded by a given number $\alpha > 0$,

$$\begin{array}{rcl} \mathcal{W}_\alpha^1 &:=& \left\{ \omega : \mathbf{R}_+ \to \mathbf{R}^q : \int_0^\infty \omega^{\mathrm{T}}(t)\omega(t)dt \le \alpha \right\}, \\ \mathcal{W}_\alpha^2 &:=& \left\{ \omega : \mathbf{R}_+ \to \mathbf{R}^q : \omega^{\mathrm{T}}(t)\omega(t) \le \alpha, \ \forall t \ge 0 \right\}. \end{array}$$

The analysis and design of control systems in the presence of saturation or saturation type nonlinearities and external disturbances have been studied by many authors. A small sample of their works include [1], [2], [4], [5], [7], [9], [10], [11], [12], [13], [14], [15], [16], [18], [19].

More recently, we considered in [3] the situation where no boundedness assumption is made on the magnitude of the disturbances and the system initial conditions are not necessarily zero, and proposed an LMI based approach to the analysis and design of closed-loop system under linear state feedback laws. Both the questions of disturbance tolerance and disturbance rejection were addressed. The approach proposed in [3] has also been extended to the output feedback setting [20] and the discrete-time setting [18], [19].

The objective of this paper is to revisit the problems posed in [3] in a broader setting so that robustness to the uncertainties in the actuator input output characteristics can be addressed simultaneously. This is made possible by our recent work on stabilization of linear systems with actuator nonlinearities [6], in which uncertainties in the actuator nonlinearities are assumed to reside in a so-called generalized sector bounded by two convex or concave curves.

We will first establish sufficient conditions, in terms of linear matrix inequalities (LMIs), under which trajectories of the given system (1) starting from within a bounded set of initial conditions remain bounded. Such a property of the system can be called *robust bounded state stability*. Based on these conditions, we will be able to formulate and solve the problems of assessing disturbance tolerance and disturbance rejection capabilities as optimization problems with LMI constraints. The design of state feedback laws to enhance the disturbance tolerance/rejection capabilities of the closed-loop system can then be carried out by viewing the state feedback gain as an additional free parameter in these optimization problems.

The disturbance tolerance capability is measured by the maximal energy/magnitude bound α, say α^*, under which any system trajectory starting from the given bounded set of initial conditions remains bounded. For an $\alpha \le \alpha^*$, the size of the set of initial conditions from which any trajectory remains bounded can also effectively indicate the disturbance tolerance capability.

One way to measure the disturbance rejection capability is to estimate the restricted $\mathcal{L}_2$ gain over $\mathcal{W}_\alpha^1$, or the maximum of the l_∞ norm of the output with zero initial condition over $\mathcal{W}_\alpha^2$. As trajectories starting from a given bounded set may be driven out of the set by tolerable energy bounded disturbances, but will remain within a larger bounded set, the gap between the two sets is also an indication of the disturbance rejection capability.

Notation: For a vector $F \in \mathbf{R}^{1\times n}$, denote $\mathcal{L}(F) := \{x \in \mathbf{R}^n : |Fx| \le 1\}$. For a positive definite matrix $P \in \mathbf{R}^{n\times n}$, and a positive scalar ρ, denote $\varepsilon(P,\rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \le \rho\}$. We use $\mathrm{sat}(u)$ to denote the standard saturation function, *i.e.*, $\mathrm{sat}(u) = \mathrm{sign}(u)\min\{1, |u|\}$. Given a set of vectors $x_1, x_2, \cdots, x_N$, we use $\mathrm{co}\{x_1, x_2, \cdots, x_N\}$

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to denote the convex hull of these vectors.

II. Robust Bounded State Stability

We will first consider actuator nonlinearities that reside in a generalized sector bounded by two concave piecewise linear functions. The results for generalized sectors where one or both boundaries are convex can be established in the same manner.

We will need to recall the following lemma from [5].

Lemma 1: Given an $F \in \mathbf{R}^{1\times n}$ and a positive definite matrix $P \in \mathbf{R}^{n\times n}$. Then, for any $H \in \mathbf{R}^{1\times n}$ such that $|Hx| \le 1$, then $\mathrm{sat}(Fx) \in \mathrm{co}\,\{Fx, Hx\}$.

A. Piecewise linear functions with one bend

Consider the system (1) with

$$\psi(u,t) \in \mathrm{co}\{\psi_1(u), \psi_2(u)\}, \tag{2}$$

where $\psi_i(u), i \in [1,2]$, are odd symmetric and

$$\psi_i(u) = \begin{cases} k_{i0}u, & \text{if } u \in [0, b_{i1}], \\ k_{i1}u + c_{i1}, & \text{if } u \in (b_{i1}, \infty), \end{cases} \tag{3}$$

where $b_{i1} = \frac{c_{i1}}{k_{i0}-k_{i1}} > 0$, $k_{i0} > k_{i1}$, $k_{i0} > 0$ (see Fig. 1).

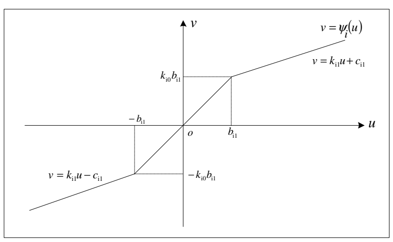

Fig. 1. A concave piecewise linear function with one bend, $\psi_i(u)$.

Theorem 1: Consider system (1) with $\psi(u,t)$ given by (2) and $\omega \in \mathcal{W}_\alpha^1$. Let the positive definite matrix $P \in \mathbf{R}^{n\times n}$ be given.

(a) If there exist $H_1, H_2 \in \mathbf{R}^{1\times n}$ and a positive number η such that, for $i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm T}P+P(A+k_{i0}BF)+\dfrac{1}{\eta}PEE^{\mathrm T}P\le 0, \\ (A+BH_i)^{\mathrm T}P+P(A+BH_i)+\dfrac{1}{\eta}PEE^{\mathrm T}P\le 0, \end{cases} \tag{4}$$

and $\varepsilon(P, 1+\alpha\eta) \subset \mathcal{L}\left(\frac{H_i - k_{i1}F}{c_{i1}}\right)$, then any trajectory of the closed-loop system (1) that starts from inside of $\varepsilon(P,1)$ will remain inside of $\varepsilon(P, 1+\alpha\eta)$.

(b) If there exist $H_1, H_2 \in R^{1\times n}$ such that, for $i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm T}P+P(A+k_{i0}BF)+PEE^{\mathrm T}P \le 0, \\ (A+BH_i)^{\mathrm T}P+P(A+BH_i)+PEE^{\mathrm T}P \le 0, \end{cases} \tag{5}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_i - k_{i1}F}{c_{i1}}\right)$, then the trajectory of the closed-loop system that starts from the origin will remain inside the ellipsoid $\varepsilon(P,\alpha)$.

Theorem 2: Consider system (1) with $\psi(u,t)$ given by (2) and $\omega \in \mathcal{W}_\alpha^2$. Let the positive definite matrix $P \in \mathbf{R}^{n\times n}$ be given. If there exist an $H \in \mathbf{R}^{1\times n}$ and a positive number η such that, for $i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm T}P+P(A+k_{i0}BF)+\dfrac{1}{\eta}PEE^{\mathrm T}P+\eta\alpha P \le 0, \\ (A+BH_i)^{\mathrm T}P+P(A+BH_i)+\dfrac{1}{\eta}PEE^{\mathrm T}P+\eta\alpha P \le 0, \end{cases} \tag{6}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_i - k_{i1}F}{c_{i1}}\right)$, then $\varepsilon(P,\alpha)$ is an invariant set.

B. Piecewise linear function with multiple bends

Consider system (1) with

$$\psi(u,t) \in \{\psi_1(u), \psi_2(u)\}, \tag{7}$$

where $\psi_i(u), i \in [1,2]$, are odd symmetric and

$$\psi_i(u) = \begin{cases} k_{i0}u, & \text{if } u \in [0, b_{i1}], \\ k_{i1}u + c_{i1}, & \text{if } u \in (b_{i1}, b_{i2}), \\ \quad\vdots & \\ k_{iN_i}u + c_{iN_i}, & \text{if } u \in (b_{iN_i}, \infty), \end{cases} \tag{8}$$

where $k_{i0} > k_{i1} > k_{i2} > \cdots > k_{iN_i}$, $k_{i(N_i-1)} > 0$ (see Fig. 2).

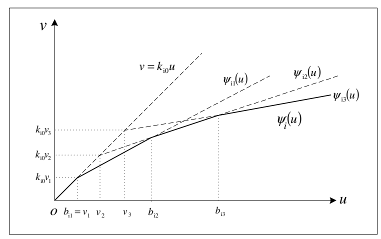

Fig. 2. Concave piecewise linear function with multiple bends, $\psi_i(u)$.

Clearly, we can define (see Fig. 2), for $j \in [1, N_i]$,

$$\psi_{ij}(u) := \begin{cases} k_{i0}u, & \text{if } u \in [0, v_{ij}], \\ k_{ij}u + c_{ij}, & \text{if } u \in (v_{ij}, \infty). \end{cases} \tag{9}$$

Then, we have [6],

$$\psi_i(u) \quad \in \quad \mathrm{co}\,\{\psi_{i1}(u), \psi_{i2}(u), \cdots, \psi_{iN_i}(u)\}\,. \tag{10}$$

Since $\psi_{ij}(u)$ is a concave piecewise linear function with one bend, by Theorems 1 and 2, we have the following theorems.

Theorem 3: Consider the system (1) with $\psi(u,t)$ defined by (7) and with $\omega \in \mathcal{W}_\alpha^1$. Let the positive definite matrix $P \in \mathbf{R}^{n\times n}$ be given.

(a) If there exist $H_{ij} \in \mathbf{R}^{1\times n}, j \in [1, N_i], i \in [1,2]$, and a positive number η such that, for $j \in [1,N_i], i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm T}P+P(A+k_{i0}BF)+\dfrac{1}{\eta}PEE^{\mathrm T}P\le 0, \\ (A+BH_{ij})^{\mathrm T}P+P(A+BH_{ij})+\dfrac{1}{\eta}PEE^{\mathrm T}P\le 0, \end{cases} \tag{11}$$

and $\varepsilon(P, 1+\alpha\eta) \subset \mathcal{L}\left(\frac{H_{ij} - k_{ij}F}{c_{ij}}\right)$, then all trajectories starting from $\varepsilon(P,1)$ remain inside $\varepsilon(P, 1+\alpha\eta)$.

(b) If there exist $H_{ij} \in R^{1\times n}, j \in [1,N_i], i \in [1,2]$, such that, for $j \in [1,N_i], i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm T}P+P(A+k_{i0}BF)+PEE^{\mathrm T}P\le 0, \\ (A+BH_{ij})^{\mathrm T}P+P(A+BH_{ij})+PEE^{\mathrm T}P\le 0, \end{cases} \tag{12}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij} - k_{ij}F}{c_{ij}}\right)$, then all trajectories starting from the origin remain inside $\varepsilon(P,\alpha)$.

Theorem 4: Consider system (1) with $\psi(u,t)$ defined by (7) and $\omega \in \mathcal{W}_\alpha^2$. Let the positive definite matrix $P \in \mathbf{R}^{n\times n}$ be given. If there exist $H_{ij} \in \mathbf{R}^{1\times n}, j \in [1, N_i], i \in [1,2]$, and a positive number η such that, for $j \in [1, N_i], i \in [1,2]$,

$$\begin{cases}(A+k_{i0}BF)^{\mathrm{T}}P+P(A+k_{i0}BF)+\dfrac{1}{\eta}PEE^{\mathrm{T}}P+\eta\alpha P\le 0,\\ (A+BH_{ij})^{\mathrm{T}}P+P(A+BH_{ij})+\dfrac{1}{\eta}PEE^{\mathrm{T}}P+\eta\alpha P\le 0,\end{cases} \tag{13}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right)$, then $\varepsilon(P,\alpha)$ is an invariant set.

III. Disturbance Tolerance

Consider system (1) under a given feedback gain F with a given set of initial conditions. The disturbance tolerance capability of the system can be measured by the largest α, say α_F^*, such that any trajectory of system (1) remains bounded. The maximum level of disturbance that the system (1) can tolerate by an appropriate design of F is then given by $\alpha^* = \sup_F \alpha_F^*$.

We will consider system (1) with initial conditions from, say $\varepsilon(S,1), S>0$. We will also consider the case with zero initial condition, which will arise in the estimation of the restricted $\mathcal{L}_2$ gain.

Disturbance tolerance with $\omega \in \mathcal{W}_\alpha^1$ and non-zero i.c.

By Item (a) of Theorem 3, this problem can be formulated into the following optimization problem,

$$\sup_{P>0,\eta>0,H_{ij},j\in[1,N_i],i\in[1,2]} \alpha, \tag{14}$$
$$\text{s.t.}\quad \text{a) } \varepsilon(S,1)\subset\varepsilon(P,1),$$
$$\text{b) Inequalities (11)},$$
$$\text{c) } \varepsilon(P,1+\alpha\eta)\subset\mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right).$$

To transform the optimization problem (14) into an LMI problem, let $\bar{\alpha} = \sqrt{\alpha}$, $Q = P^{-1}$, $Y_{ij} = H_{ij}Q, j \in [1,N_i], i = [1,2]$, and $\mu = \frac{1}{1+\alpha\eta} \in (0,1)$. Then constraints in (14) are equivalent to the following constraints,

$$\begin{bmatrix} S & I \\ I & Q \end{bmatrix} \ge 0, \tag{15}$$

$$\begin{cases}\begin{bmatrix} Q(A+k_{i0}BF)^{\mathrm{T}}+(A+k_{i0}BF)Q & \bar{\alpha}E \\ \bar{\alpha}E^{\mathrm{T}} & \dfrac{\mu-1}{\mu}I\end{bmatrix}\le 0,\\ \begin{bmatrix} QA^{\mathrm{T}}+AQ+(BY_{ij})^{\mathrm{T}}+BY_{ij} & \bar{\alpha}E \\ \bar{\alpha}E^{\mathrm{T}} & \dfrac{\mu-1}{\mu}I\end{bmatrix}\le 0,\\ j\in[1,N_i], i\in[1,2],\end{cases} \tag{16}$$

$$\begin{bmatrix} \mu & \dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}} \\ \left(\dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,\quad j\in[1,N_i], i\in[1,2]. \tag{17}$$

The optimization problem (14) is equivalent to

$$\sup_{Q>0,\mu\in(0,1),Y_{ij},j\in[1,N_i],i\in[1,2]} \bar{\alpha}, \tag{18}$$
$$\text{s.t.}\quad (15),(16),(17).$$

Obviously, all constraints in (18) are LMIs for a fixed value of μ. Thus, by sweeping μ over the interval $(0,1)$, the global maximum of $\bar{\alpha}$, and thus α_F^*, can be obtained.

To find the maximum disturbance tolerance capability α^*, we will view F as a free parameter. By an additional change of variable $Z = FQ$, (16) and (17) become

$$\begin{cases}\begin{bmatrix} QA^{\mathrm{T}}+AQ+k_{i0}BZ+k_{i0}(BZ)^{\mathrm{T}} & \bar{\alpha}E \\ \bar{\alpha}E^{\mathrm{T}} & \dfrac{\mu-1}{\mu}I\end{bmatrix}\le 0,\\ \begin{bmatrix} QA^{\mathrm{T}}+AQ+(BY_{ij})^{\mathrm{T}}+BY_{ij} & \bar{\alpha}E \\ \bar{\alpha}E^{\mathrm{T}} & \dfrac{\mu-1}{\mu}I\end{bmatrix}\le 0,\\ j\in[1,N_i], i\in[1,2],\end{cases} \tag{19}$$

$$\begin{bmatrix} \mu & \dfrac{Y_{ij}-k_{ij}Z}{c_{ij}} \\ \left(\dfrac{Y_{ij}-k_{ij}Z}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,\quad j\in[1,N_i], i\in[1,2], \tag{20}$$

respectively. Hence, we can formulate the problem of finding α^* into the following LMI problem,

$$\sup_{Q>0,\mu\in(0,1),Z,Y_{ij},j\in[1,N_i],i\in[1,2]} \bar{\alpha}, \tag{21}$$
$$\text{s.t.}\quad (15),(19),(20).$$

Disturbance tolerance with $\omega \in \mathcal{W}_\alpha^1$ and zero i.c.:

By Theorem 3, Item (b), this problem can be formulated into the following optimization problem,

$$\sup_{P>0,\eta>0,H_{ij},j\in[1,N_i],i\in[1,2]} \alpha, \tag{22}$$
$$\text{s.t.}\quad \text{a) Inequalities (12)},$$
$$\text{b) } \varepsilon(P,\alpha)\subset\mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right).$$

By the change of variable, $\nu = \frac{1}{\alpha}$, $Q = P^{-1}$ and $Y_{ij} = H_{ij}Q, j \in [1,N_i], i \in [1,2]$, constraint a) in (22) is equivalent to

$$\begin{cases} Q(A+k_{i0}BF)^{\mathrm{T}}+(A+k_{i0}BF)Q+EE^{\mathrm{T}}\le 0,\\ QA^{\mathrm{T}}+AQ+(BY_{ij})^{\mathrm{T}}+BY_{ij}+EE^{\mathrm{T}}\le 0,\\ j\in[1,N_i], i\in[1,2].\end{cases} \tag{23}$$

Constraint b) is equivalent to

$$\begin{bmatrix} \nu & \dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}} \\ \left(\dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,\quad j\in[1,N_i], i\in[1,2]. \tag{24}$$

Hence the optimization problem (22) is equivalent to the following LMI optimization problem,

$$\inf_{Q>0,Y_{ij},j\in[1,N_i],i\in[1,2]} \nu, \tag{25}$$
$$\text{s.t.}\quad (23),(24).$$

With the solution ν^*, the disturbance tolerance capability of the system (1) under a given F is then given by $\alpha_F^* = \frac{1}{\nu^*}$. To determine the maximum disturbance tolerance α^*, we solve the optimization problem (25) by an additional change of variable $Z = FQ$.

Disturbance tolerance with $\omega \in \mathcal{W}_\alpha^2$:
By Theorem 4, the problem is equivalent to

$$\sup_{P>0,\eta>0,H_{ij},j\in[1,N_i],i\in[1,2]} \alpha, \tag{26}$$
$$\text{s.t.} \quad \text{a) } \varepsilon(S,1) \subset \varepsilon(P,1),$$
$$\text{b) Inequalities (13)},$$
$$\text{c) } \varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right), j\in[1,N_i], i\in[1,2].$$

Let $Q = \alpha P^{-1}$, $Y_{ij} = \alpha H_{ij} Q, j \in [1, N_i], i \in [1,2]$ and $\hat{\eta} = \eta\alpha$, then constraint a) in (26) is equivalent to

$$\begin{bmatrix} \alpha S & \alpha I \\ \alpha I & Q \end{bmatrix} > 0, \tag{27}$$

constraint b) is equivalent to

$$\left\{ \begin{array}{l} \begin{bmatrix} Q(A+k_{i0}BF)^{\mathrm{T}}+(A+k_{i0}BF)Q+\hat{\eta}Q & \alpha E \\ \alpha E^{\mathrm{T}} & -\hat{\eta} \end{bmatrix} \le 0, \\ \begin{bmatrix} QA^{\mathrm{T}}+AQ+BY_{ij}+(BY_{ij})^{\mathrm{T}}+\hat{\eta}Q & \alpha E \\ \alpha E^{\mathrm{T}} & -\hat{\eta} \end{bmatrix} \le 0, \\ j \in [1, Ni], i \in [1,2], \end{array} \right. \tag{28}$$

and constraint c) is equivalent to

$$\begin{bmatrix} 1 & \dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}} \\ \left(\dfrac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,$$
$$j \in [1, N_i], i \in [1,2]. \tag{29}$$

Hence, the optimization problem (26) can be transformed into the following optimization problem,

$$\inf_{Q>0,\hat{\eta}>0,Y_{ij},j\in[1,N_i],i\in[1,2]} \alpha, \tag{30}$$
$$\text{s.t.} \quad (27), (28), (29)$$

in which all constraints are LMIs for a fixed value of $\hat{\eta}$. The global maximum of α, α_F^*, can be obtained by sweeping $\hat{\eta}$ over the interval $(0, \infty)$. In the case of zero initial condition, constraint a) in (26) is automatically satisfied and thus can be left out. As before, α^* can be determined by solving the optimization problem (30) with an additional change of variable $Z = FQ$.

Example. Consider the system (1) with

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$F = \begin{bmatrix} 1.2231 & -2.2486 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Let $\psi(u,t)$ reside in a generalized sector [6] bounded by $\psi_1(u) = u$ and $\psi_2(u)$ of the form (7) with $N = 5$ and $(k_{20}, k_{21}, k_{22}, k_{23}, k_{24}, k_{25}) = (0.7845, 0.3218, 0.1419, 0.0622, 0.0243, 0)$, $(c_{21}, c_{22}, c_{23}, c_{24}, c_{25}) = (0.4636, 0.8234, 1.0625, 1.2517, 1.4464)$, (see the solid straight line and the solid piecewise linear curves in Fig. 3).

Let the set $\varepsilon(S,1)$ be defined by $S = I$. Then, the disturbance tolerance capability can be assessed as follows.
For $\omega \in \mathcal{W}_\alpha^1$ with $x(0) \in \varepsilon(S,1)$:

- By solving the optimization problem (18), we obtain $\alpha_F^* = 543.9380$, with $\eta_F^* = 0.36$. In this simulation, as well as the simulation throughout the paper, we use $\psi(u,t) = \tan^{-1}(u)$, which resides in the generalized sector defined by $\psi_1(u)$ and $\psi_2(u)$.
- By solving the optimization problem (21), we obtain $\alpha^* = 846.9855$, with $F^* = \begin{bmatrix} 9574 & -31417 \end{bmatrix}, \eta^* = 0.25$. Shown in Fig. 4 are some simulation results.

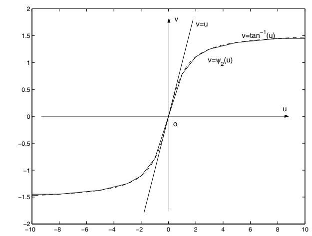

Fig. 3. The generalized sector defined by $\psi_1(u)$ and $\psi_2(u)$ in the example.

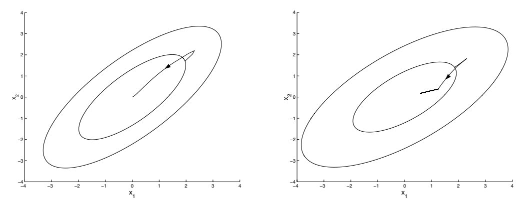

Fig. 4. $\omega \in \mathcal{W}_{\alpha_F^*}^1$: $\varepsilon(P_F^*, 1)$, $\varepsilon(P_F^*, 1+\alpha_F^*\eta_F^*)$ and a trajectory(left); $\omega \in \mathcal{W}_{\alpha^*}^1$: $\varepsilon(P^*, 1)$, $\varepsilon(P^*, 1+\alpha^*\eta^*)$ and a trajectory(right).

For $\omega \in \mathcal{W}_\alpha^1$ with $x(0) = 0$:

- By solving the optimization problem (25), we obtain $\alpha_F^* = 1007.5$.
- Furthermore, we can also obtain α^* and F^* as, $\alpha^* = 1315.8$, $F^* = \begin{bmatrix} 2.0679\times 10^6 & -2.9446\times 10^6 \end{bmatrix}$.

For $\omega \in \mathcal{W}_\alpha^2$:

- In this case, we can solve the optimization problem (30) to obtain $\alpha_F^* = 4.5294$.
- Furthermore, we can obtain $\alpha^* = 7.0822, F^* = \begin{bmatrix} 11768 & -46909 \end{bmatrix}$. Shown in Fig. 5 are some simulation results.

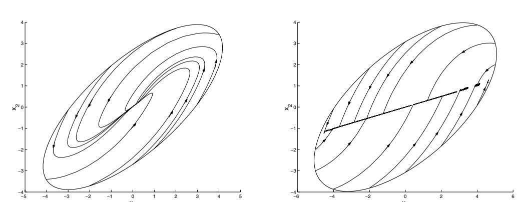

Fig. 5. $\omega \in \mathcal{W}_{\alpha_F^*}^2$: $\varepsilon(P_F^*, \alpha_F^*)$ and some trajectories(left); $\omega \in \mathcal{W}_{\alpha^*}^2$: $\varepsilon(P^*, \alpha^*)$ and some trajectories(right).

IV. Disturbance Rejection

A. Energy bounded disturbances

In the case that $\omega \in \mathcal{W}_\alpha^1$, the disturbance rejection capability can be measured by the gap between the two nested ellipsoids $\varepsilon(P,1)$ and $\varepsilon(P, 1+\alpha\eta)$. The smaller the gap between the two nested ellipsoids is, the stronger is the disturbance rejection capability. For a given $\alpha \le \alpha_F^*$ (or

$\alpha \le \alpha^*$), the gap between the two ellipsoids is measured by the value of η. Another way to assess the disturbance rejection capability is to estimate the restricted $\mathcal{L}_2$ gain.

The gap between $\varepsilon(P,1)$ and $\varepsilon(P,1+\alpha\eta)$:

Under a given F, the level of disturbance rejection η_F^* can be determined by solving the optimization problem,

$$\inf_{P>0,H_{ij}} \eta, \tag{31}$$

s.t. a) $\varepsilon(S,1) \subset \varepsilon(P,1)$,

b) Inequalities (11),

c) $\varepsilon(P,1+\alpha\eta) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right), j \in [1,N_i], i \in [1,2]$.

Let $Q = P^{-1}$, $\tilde{\eta} = \frac{1}{\eta}$, and $Y_{ij} = H_{ij}Q, j \in [1,N_i], i \in [1,2]$, then constraint b) in (31) is equivalent to

$$\begin{cases} Q(A+k_{i0}BF)^{\mathrm{T}} + (A+k_{i0}BF)Q + \tilde{\eta}EE^{\mathrm{T}} \le 0, \\ QA^{\mathrm{T}} + AQ + BY_{ij} + (BY_{ij})^{\mathrm{T}} + \tilde{\eta}EE^{\mathrm{T}} \le 0. \end{cases} \tag{32}$$

By using Schur complement, constraint c) is equivalent to

$$\begin{bmatrix} Q & \left(\frac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & \left(\frac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} \\ \frac{Y_{ij}-k_{ij}FQ}{c_{ij}} & 1 & 0 \\ \frac{Y_{ij}-k_{ij}FQ}{c_{ij}} & 0 & \frac{\tilde{\eta}}{\alpha} \end{bmatrix} \ge 0,$$

$$j \in [1,N_i], i \in [1,2]. \tag{33}$$

Hence problem (31) is equivalent to the LMI problem,

$$\sup_{Q>0,Y_{ij},j\in[1,N_i],i\in[1,2]} \tilde{\eta}, \tag{34}$$

$$\text{s.t.} \quad (15),(32),(33).$$

The design of feedback gain F to achieve a higher level of disturbance rejection η^* can be carried out by viewing F as an additional free parameter and using an additional change of variable $Z = FQ$ in the above optimization problem.

The restricted $\mathcal{L}_2$ gain:

Theorem 5: Consider system (1) with $\psi(u,t)$ defined by (7) and with $\omega \in \mathcal{W}_\alpha^1$. Let F and $\alpha \le \alpha_F^*$ be given. For a given $\gamma > 0$, if there exist a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and matrices $H_{ij} \in \mathbf{R}^{1\times n}, j \in [1,N_i], i \in [1,2]$, such that, for $j \in [1,N_i], i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm{T}}P + P(A+k_{i0}BF) + PEE^{\mathrm{T}}P + \frac{1}{\gamma^2}C^{\mathrm{T}}C \le 0, \\ (A+BH_{ij})^{\mathrm{T}}P + P(A+BH_{ij}) + PEE^{\mathrm{T}}P + \frac{1}{\gamma^2}C^{\mathrm{T}}C \le 0, \end{cases} \tag{35}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F_{ij}}{c_{ij}}\right)$, then, the restricted $\mathcal{L}_2$ gain from ω to z, with $x(0) = 0$, is less than or equal to γ.

By Theorem 5, the problem of determining the restricted $\mathcal{L}_2$ gain, γ_F^*, can be formulated into and solved as the following optimization problem,

$$\inf_{P>0,H_{ij},j\in[1,N_i],i\in[1,2]} \gamma^2, \tag{36}$$

a) Inequalities (35),

b) $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right), j \in [1,N_i], i \in [1,2]$.

By the change of variable, $Q = P^{-1}$ and $Y_{ij} = H_{ij}Q, j \in [1,N_i], i \in [1,2]$, and application of the Schur complement, constraint a) in (36) is equivalent to

$$\begin{cases} \begin{bmatrix} Q(A+k_{i0}BF)^{\mathrm{T}} + (A+k_{i0}BF)Q & E & QC^{\mathrm{T}} \\ E^{\mathrm{T}} & -I & 0 \\ CQ & 0 & -\gamma^2 I \end{bmatrix} \le 0, \\ \begin{bmatrix} QA^{\mathrm{T}} + AQ + BY_{ij} + (BY_{ij})^{\mathrm{T}} & E & QC^{\mathrm{T}} \\ E^{\mathrm{T}} & -I & 0 \\ CQ & 0 & -\gamma^2 I \end{bmatrix} \le 0, \\ j \in [1,N_i], i \in [1,2]. \end{cases} \tag{37}$$

Constrain b) is equivalent to

$$\begin{bmatrix} \frac{1}{\alpha} & \frac{Y_{ij}-k_{ij}FQ}{c_{ij}} \\ \left(\frac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,$$

$$j \in [1,N_i], i \in [1,2]. \tag{38}$$

Then, problem (36) is equivalent to the LMI problem,

$$\inf_{Q>0,Y_{ij},j\in[1,N_i],i\in[1,2]} \gamma^2, \tag{39}$$

$$\text{s.t.} \quad (37),(38).$$

As before, the determination of F that minimizes γ_F^* can be carried out by viewing F as a free parameter and by using an additional change of variable $Z = FQ$. The resulting minimal γ_F^* will be denoted as γ^*.

B. Magnitude bounded disturbances

For system (1) with $\omega \in \mathcal{W}_\alpha^2$, we will use the maximum l_∞ norm of the output with zero initial condition to indicate the disturbance rejection capability.

Theorem 6: Consider system (1) with $\psi(u,t)$ defined by (7) and with $\omega \in \mathcal{W}_\alpha^2$. Let the feedback gain F and $\alpha \le \alpha_F^*$ be given. For a given positive constant ζ, the maximum l_∞ norm of the output of the system (1) is less than or equal to ζ if there exist a positive definite matrix $P \in \mathbf{R}^{n\times n}$, matrices $H_{ij} \in \mathbf{R}^{1\times n}, j \in [1,N_i], i \in [1,2]$, and a positive scalar η, such that, for $j \in [1,N_i], i \in [1,2]$,

$$\begin{cases} (A+k_{i0}BF)^{\mathrm{T}}P + P(A+k_{i0}BF) + \frac{1}{\eta}PEE^{\mathrm{T}}P + \eta\alpha P \le 0, \\ (A+BH_{ij})^{\mathrm{T}}P + P(A+BH_{ij}) + \frac{1}{\eta}PEE^{\mathrm{T}}P + \eta\alpha P \le 0, \\ C^{\mathrm{T}}C \le \frac{\zeta^2}{\alpha}P, \end{cases} \tag{40}$$

and $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right)$.

By Theorem 6, we have the optimization problem,

$$\inf_{P>0,\eta>0,H_{ij},j\in[1,N_i],i\in[1,2]} \zeta^2, \tag{41}$$

s.t. a) Inequalities (40),

b) $\varepsilon(P,\alpha) \subset \mathcal{L}\left(\frac{H_{ij}-k_{ij}F}{c_{ij}}\right)$.

By the change of variable, $Q = P^{-1}$ and $Y_{ij} = H_{ij}Q, j \in [1,N_i], i \in [1,2]$, the problem (41) is equivalent to

$$\inf_{Q>0,\eta>0,Y_{ij},j\in[1,N_i],i\in[1,2]} \zeta^2, \quad (42)$$

$$\text{s.t. a) } Q(A+k_{i0}BF)^{\mathrm{T}}+(A+k_{i0}BF)Q+\frac{1}{\eta}EE^{\mathrm{T}}+\eta\alpha Q \le 0,$$

$$\text{b) } QA+A^{\mathrm{T}}Q+BY_{ij}+(BY_{ij})^{\mathrm{T}}+\frac{1}{\eta}EE^{\mathrm{T}}+\eta\alpha Q \le 0,$$

$$\text{c) } CQC^{\mathrm{T}} \le \frac{\zeta^2}{\alpha}I,$$

$$\text{d) } \begin{bmatrix} \frac{1}{\alpha} & \frac{Y_{ij}-k_{ij}FQ}{c_{ij}} \\ \left(\frac{Y_{ij}-k_{ij}FQ}{c_{ij}}\right)^{\mathrm{T}} & Q \end{bmatrix} \ge 0,$$

$$j \in [1, N_i], i \in [1, 2],$$

where all constraints in (42) are LMIs. The global minimum of ζ can be obtained by sweeping η over the interval $(0, \infty)$. We note that a small ζ implies a small invariant set $\varepsilon(P, \alpha)$. Thus, the obtained ζ_F^* is a good approximation to the ζ_F^* for zero initial condition.

Once again, the above optimization problem can be easily adapted for the design of F. We will denote $\zeta^* = \inf_F \zeta_F^*$.

Example (continued).

For $\omega \in \mathcal{W}_\alpha^1$*:*

- We recall from Section III that, for the given F, $\alpha_F^* = 543.9380$. Let $\alpha = 200$. Solving optimization problem (34), we obtain that $\eta_F^* = 0.0012$. Similarly, the level of disturbance rejection can be optimized over the choice of F, leading to $\eta^* = 9.0190 \times 10^{-4}$, with $F^* = \left[0.7243 \times 10^6 \quad -1.2701 \times 10^6 \right]$. Shown in Fig. 6 are some simulation results.

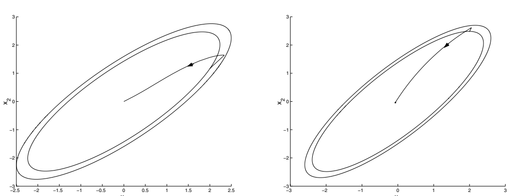

Fig. 6. $\omega \in \mathcal{W}_{200}^1$: $\varepsilon(P_F^*, 1)$, $\varepsilon(P_F^*, 1 + 200\eta_F^*)$ and a trajectory(left); $\varepsilon(P^*, 1)$, $\varepsilon(P^*, 1 + 200\eta^*)$ and a trajectory(right).

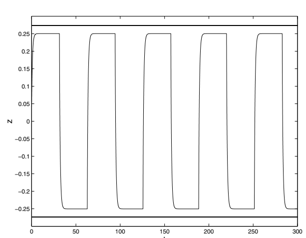

Fig. 7. $\omega \in \mathcal{W}_\alpha^2$: The output of the system (1).

- The restricted $\mathcal{L}_2$ gain within $\mathcal{W}_{200}^1$ is $\gamma_F^* = 0.1119$. By minimizing γ_F^* over the choice of F, we obtain, $\gamma^* = \inf_F \gamma_F^* = 0.0929$, with $F^* = \left[2.2018 \times 10^6 \quad -1.6670 \times 10^6 \right]$.

For $\omega \in \mathcal{W}_\alpha^2$*:*

- We recall from Section III that, for the given F, $\alpha_F^* = 4.5294$. Let $\alpha = 2$. Solving the optimization problem (42), we obtain $\zeta_F^* = 0.2735$. Fig. 7 shows the output of system (1) with zero initial condition and in the presence of the disturbance $\omega = 2\mathrm{sign}(\sin(0.1t))$. In the figure, the bounds on the output, $z = \pm 0.2735$, are shown as two straight lines. By minimizing ζ_F^* over the choice of F, we obtain $\zeta^* = 0.2052$, with $F^* = \left[1.6552 \times 10^8 \quad -1.3679 \times 10^8 \right]$.

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