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Adaptive Control of a Class of Non-Affine Systems using Neural Networks

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***Abstract*— A neural control synthesis method is considered for a class of non-affine uncertain single-input, single-output systems. The method eliminates a fixed-point assumption and does not assume boundedness on the time derivative of a control effectiveness term. One or the other of these assumptions exist in earlier papers on this subject. Using Lyapunov's direct method, it is shown that all the signals of the closed-loop system are uniformly ultimately bounded, and that the tracking error converges to an adjustable neighborhood of the origin. Simulation with a Van Der Pol equation with non-affine control terms illustrates the approach.**

I. INTRODUCTION

In recent years, there have been a number of efforts on developing systematic design tools for adaptive control of uncertain nonlinear systems. The common assumptions in these efforts were that the system to be controlled is affine, i.e. the plant is linear in the input variables, and the nonlinearities are linearly parameterized by unknown parameters [1]. Employing a neural network (NN) in adaptive control has relaxed the assumption on linear parameterized nonlinearities and greatly broadened the class of systems that can be treated by adaptive control [2], [3]. However, developing a systematic synthesis method for general non-affine systems still remains as a challenging problem.

The difficulty associated with a control design for non-affine control systems is that an explicit inverting control design is in general not possible even if the inverse exists [4]. One may avoid this problem by introducing an integrator before control signal, i.e., $u = \int_0^t \dot{u}dt$, and treating the augmented system as an affine system with respect to the new input $\dot{u}$ as in [4], [5]. However, the augmented system may have undesirable properties such as: stabilizability of the new system with a static feedback implies only stabilizability of the original system by dynamic feedback, the relative degree of the new system is one higher than that of the original system, and the transformation into normal form may introduce singularities [6]. In [7], [8], an inverting controller is designed with a NN employed for compensating modelling errors due to inexact inversion. The main feature of an inverting approach is that the uncertainty to be approximated by an adaptive signal contains the adaptive signal itself as a part of uncertainty, thus constituting a fixed-point problem. In [7], [8] this problem is addressed by assuming the uncertainty is a contraction mapping with respect to the adaptive control signal. This introduces two conditions:1) the control effectiveness of the design model should have known sign, and 2) the magnitude of it should be greater than half the actual value. Nevertheless, the stability analysis presented in [7], [8] did not explicitly use the assumption and the requirement for the assumption was not clear. In [9], [10], a NN was employed to approximate an ideal control signal the existence of which is guaranteed by the implicit function theorem. However, the stability proof in [9], [10] requires that the time derivative of the control effectiveness term should be bounded *apriori* . Motivated by [9], Ref.s [11], [12] clarified the role of known sign in the stability analysis, but required a similar assumption on the time derivative of the control effectiveness term as in [9]. This assumption raises the question of circularity with respect to proof of boundedness.

In this paper, we relax the assumption on the boundedness of the time-derivative of the control effectiveness term. In addition, we show that the known sign condition for the control effectiveness term and its boundedness on the domain of interest are sufficient to achieve an adjustable ultimate bound for a class of non-affine systems with internal dynamics, which previously could only be achieved for affine systems without internal dynamics [2], [3]. As is common in the literature [2], [3], [9], we introduce an error signal the convergence of which guarantees convergence of the output tracking error. By this, we prove that the output tracking error converges to the adjustable neighborhood of the origin. We also clarifies the relations between initial conditions, adaptation gains, and the size of the ultimate bound.

The paper is organized as follows. The problem is formulated in Section II. Control design based on input-output linearization for systems with internal dynamics is presented in Section III. Uncertainty approximation using NNs and the NN weights update law are described in Section IV. A stability analysis is given in Section V and the introduction of a robustifying signal is described in Section VI. Simulation results are provided in Section VII. We summarize the paper in Section VIII. Throughout the manuscript, $\|\cdot\|$ means Euclidean norm, and $\mathbb{B}_a$ represents the set $\{x \, : \, \|x\| \leq a, \; a > 0\}$ in a Euclidean space with a compatible dimension.

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II. Problem Formulation

Consider a single-input single output (SISO) non-affine system described by the following normal form [6]

$$\begin{aligned}\dot{\xi}_i &= \xi_{i+1},\ i=1,\dots,r-1\\ \dot{\xi}_r &= h(\xi,\eta,u)\\ \dot{\eta} &= f_o(\eta,\xi,u)\\ y &= \xi_1,\end{aligned} \tag{1}$$

where $\xi = [\xi_1,\dots,\xi_r]^T \in \mathbb{B}_{R_\xi} \subset \mathbb{R}^r$, $\eta \in \mathbb{B}_{R_\eta} \subset \mathbb{R}^{n-r}$, $u \in D_u \subset \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $h(\xi,\eta)$ is a smooth partially known function ($h(0,0,0)=0$), and $f_o(\eta,\xi,u)$ is a smooth partially known vector field ($f_o(0,0,0)=0$).

Assumption 1: The relative degree r is known, and $h_u(\xi,\eta,u) \triangleq \frac{\partial h(\xi,\eta,u)}{\partial u} \neq 0$ on $\mathbb{B}_{R_\xi}\times\mathbb{B}_{R_\eta}\times D_u$ has a known sign. Without loss of generality, we assume $h_u > 0$.

Assumption 2: A desired trajectory $y_d(t)$ is bounded and r-times differentiable with respect to time. Let $\xi_d = [y_d,\dots,y_d^{(r-1)}(t)]^T$, $\bar{\xi}_d = [\xi_d^T\ y_d^{(r)}]^T$. Then, $\xi_d \in \mathbb{B}_a \subset \mathbb{R}^r$, and $\bar{\xi}_d \in \mathbb{B}_b \subset R^{r+1}$ with known bounds $a,b>0$.

The control objective is to design a control law for u so that the system output y tracks the desired trajectory y_d with bounded error.

III. Error System and Input-Output Linearization

A. Error System

Let $e = y_d - y$. Following [3], [9], we define the tracking error as

$$r = e^{(r-1)} + \lambda_{r-1}e^{(r-2)} + \dots + \lambda_1 e = [\Lambda^T 1]\tilde{\xi}, \tag{2}$$

where $\tilde{\xi} = \xi_d - \xi$, and λ_i's are chosen such that $s^{(r-1)} + \lambda_{r-1}s^{(r-2)} + \dots + \lambda_1$ is Hurwitz. Define $\zeta = [\tilde{\xi}_1,\tilde{\xi}_2,\dots,\tilde{\xi}_{r-1}]^T$. The following properties will be used through the manuscript, the proof of which is given in [10, Lemma 2.1].

Proposition 1: Define $r_M(t) = \sup_{0\le s\le t}|r(s)|$ and $R_{T_1} = \sup_{s\ge T_1}|r(s)|$. The following inequalities hold:

$$\|\zeta(t)\| \le k_0\|\zeta(0)\| + \tfrac{k_0}{\lambda_0}r_M(t) \tag{3}$$

$$\|\zeta(t)\| \le k_0 e^{-\lambda_0 t}\left\{\|\zeta(0)\| + \tfrac{e^{\lambda_0 T_1}}{\lambda_0}r_M(T_1)\right\} + \tfrac{k_0}{\lambda_0}R_{T_1} \tag{4}$$

$$\left\|\tilde{\xi}_r\right\| \le r_M(t) + d_1\|\zeta\| \tag{5}$$

$$\left\|\tilde{\xi}\right\| \le d_2 r_M(t) + d_3 \tag{6}$$

where $k_0>0$, $\lambda_0>0$, $d_1 = \|\Lambda\|$, $d_2 = k_0/\lambda_0(1+d_1)+1$, and $d_3 = k_0(1+d_1)\|\zeta(0)\|$.

With the definition of the tracking error in (2), the following error dynamics are immediate

$$\dot{r} = y_d^{(r)} - [0\ \Lambda^T]\tilde{\xi} - h(\xi,\eta,u). \tag{7}$$

B. Input-Output Linearization and Inversion Error

Input-output linearization is carried out by introducing an *invertible* (with respect to u) model

$$\nu = \hat{h}(\xi,\eta,u), \tag{8}$$

which is the best available model, and ν is the so-called *pseudo-control* [7].

Assumption 3: $\hat{h}_u \triangleq \frac{\partial}{\partial u}[\hat{h}(\xi,\eta,u)] > 0$, and there exists $b_l,b_u>0$ such that $b_l \le h_u/\hat{h}_u \le b_u$ on $\mathbb{B}_{R_\xi}\times\mathbb{B}_{R_\eta}\times D_u$.

The inversion process involves designing a pseudo-control signal:

$$\nu = \nu_{rm} + \nu_{dc} - \nu_{ad}, \tag{9}$$

where $\nu_{rm} = -y_d^{(r)} + [0\ \Lambda^T]\tilde{\xi}$, ν_{dc} is a linear control signal that is to stabilize the error dynamics in the absence of modelling error, ν_{ad} is an adaptive signal used to approximately cancel the modelling error. Inverting (8) leads to the following control law:

$$u = \hat{h}^{-1}(\xi,\eta,\nu). \tag{10}$$

Let $\nu_{dc} = Kr$. Then, applying (10) to (7) leads to the following error dynamics:

$$\dot{r} = -Kr + \nu_{ad} - \Delta(\xi,\eta,u), \tag{11}$$

where the modelling error Δ is defined as:

$$\Delta(\xi,\eta,u) \triangleq h(\xi,\eta,u) - \hat{h}(\xi,\eta,u). \tag{12}$$

Ref. [7] points out that finding ν_{ad} to cancel $\Delta(\xi,\eta,u)$ that includes ν_{ad} through u constitutes a fixed-point problem and introduces a contraction mapping assumption on Δ with respect to its argument ν_{ad}. To avoid this, we follow [9], [11], [12]. The term $\nu_{ad}-\Delta$ can be expressed as

$$\begin{aligned}\nu_{ad}-\Delta =& -h(\xi,\eta,\hat{h}^{-1}(\xi,\eta,\nu_{rm}+Kr-\nu_{ad}))\\ &+\nu_{rm}+Kr\end{aligned} \tag{13}$$

Note that $r = r(\xi,\xi_d)$ and $\nu_{rm} = \nu_{rm}(\xi,\bar{\xi}_d)$. Since $\frac{\partial h}{\partial u}\frac{\partial \hat{h}^{-1}}{\partial \nu_{ad}} \neq 0$ by Assumptions 1 and 3 , applying the implicit function theorem as in [9] guarantees that there exists a smooth function $\nu_{ad}^* = \nu_{ad}^*(\xi,\eta,\bar{\xi}_d)$ such that

$$-h(\xi,\eta,\hat{h}^{-1}(\xi,\eta,\nu_{rm}+kr-\nu_{ad}^*)) + \nu_{rm} + Kr = 0 \tag{14}$$

for every $(\xi,\eta,\bar{\xi}_d) \in \mathbb{B}_{R_\xi}\times\mathbb{B}_{R_\eta}\times\mathbb{B}_b$. With the definition of ν_{ad}^*, and using the mean value theorem [13], (14) can be expressed as

$$\begin{aligned}\nu_{ad}-\Delta =& -h(\xi,\eta,\hat{h}^{-1}(\xi,\eta,\nu_{rm}+Kr-\nu_{ad}))\\ &+h(\xi,\eta,\hat{h}^{-1}(\xi,\eta,\nu_{rm}+kr-\nu_{ad}^*))\\ =& h_\nu(\bar{\nu})[\nu_{ad}-\nu_{ad}^*]\end{aligned} \tag{15}$$

where $h_\nu(\bar{\nu}) \triangleq \frac{\partial h}{\partial u}\frac{\partial \hat{h}^{-1}}{\partial \nu}\Big|_{\nu=\bar{\nu}} = \frac{\partial h}{\partial u}\Big/\frac{\partial \hat{h}}{\partial u}\Big|_{u=\hat{h}^{-1}(\xi,\eta,\bar{\nu})}$, and $\bar{\nu} = \nu_{rm} + Kr - \theta\nu_{ad} - (1-\theta)\nu_{ad}^*$, $\theta\in[0,1]$. Using (15), Eq. (11) can be expressed as

$$\dot{r} = -Kr + h_\nu(\bar{\nu})[\nu_{ad}-\nu_{ad}^*]. \tag{16}$$

C. Internal Dynamics

While the notion of minimum phase is well established for affine control systems [14], a corresponding notion for non-affine systems is still an open problem, because for non-affine systems the zero dynamics may not be uniquely defined. In other words, different control laws may yield either stable or unstable zero dynamics [6]. To avoid this situation, we introduce assumptions similar to those in [12], [15].

Assumption 4: Let $q_c(\xi,\eta,\bar{\xi}_d) \triangleq \hat{h}^{-1}(\nu_{rm}(\xi,\bar{\xi}_d) + Kr(\xi,\xi_d) - \nu^*_{ad}(\xi,\eta,\xi_d))$. The system $\dot{\eta} = f_0(\eta,\xi_d,q_c(\xi_d,\eta,\xi_d))$ has a unique steady state solution $\eta_c(t) \in D_\eta$ and $\|\eta_c(t)\| \le c_\eta$. Furthermore, with $\tilde{\eta} = \eta_c(t) - \eta(t)$, the system

$$\begin{aligned}\dot{\tilde{\eta}} &= f_0(\eta,\xi_d,q_c(\xi_d,\eta,\bar{\xi}_d)) - f_0(\eta,\xi,q_c(\xi,\eta,\bar{\xi}_d)) \\ &= \tilde{f}_0(\tilde{\eta},\tilde{\xi},\eta_c,\bar{\xi}_d)\end{aligned} \tag{17}$$

has a continuously differentiable function $V_\eta(t,\tilde{\eta})$ satisfying

$$\begin{aligned} c_1\|\tilde{\eta}\|^2 &\le V_\eta(t,\tilde{\eta}) \le c_2\|\tilde{\eta}\|^2 \\ \dot{V}_\eta &\le -c_3\|\tilde{\eta}\|^2 + c_4\|\tilde{\eta}\|\left\|\tilde{\xi}\right\|. \end{aligned} \tag{18}$$

This assumption implies that with $\tilde{\xi}$ as an input, the internal dynamics in (17) are input-to-state stable (ISS) [16]. Substituting the bound for $\tilde{\xi}$ in (6) into (18) leads to

$$\dot{V}_\eta \le -c_3\|\tilde{\eta}\|\left[\|\tilde{\eta}\| - \frac{c_4}{c_3}(d_2 r_M(t) + d_3)\right]. \tag{19}$$

Whenever $\|\tilde{\eta}\| \ge \frac{c_4}{c_3}(d_2 r_M(t)+d_3)$, $\dot{V}_\eta \le 0$. This leads to the following bound for $\|\tilde{\eta}\|$ (See [3, Theorem 6.4])

$$\|\tilde{\eta}\| \le \sqrt{\frac{c_2}{c_1}}\max(\|\tilde{\eta}(0)\|, \frac{c_4}{c_3}(d_2 r_M(t)+d_3)). \tag{20}$$

Then the internal state η is bounded by

$$\|\eta\| \le c_\eta + \sqrt{\frac{c_2}{c_1}}\max(\|\tilde{\eta}(0)\|, \frac{c_4}{c_3}(d_2 r_M(t)+d_3)). \tag{21}$$

IV. NN Approximation and Adaptive Law

A radial basis function NN (RBF NN) is used to approximate ν^*_{ad} in (14). It is a universal approximator if a set of basis functions can be selected over a compact domain of approximation. Following the approach in [17], given $\varepsilon^* > 0$, the continuous ideal signal ν^*_{ad} can be parameterized via a suitably chosen set of Gaussian basis functions on the compact set $\Omega_{\bar{x}} = B_{R_\xi} \times B_{R_\eta} \times B_b$

$$\nu^*_{ad}(\xi,\eta,\bar{\xi}_d) = W^T\phi(\bar{x}) + \varepsilon(\bar{x}), \quad \|\varepsilon\| \le \varepsilon^*, \tag{22}$$

where $\bar{x} = (\xi,\eta,\bar{\xi}_d)$, $W \in \mathbb{R}^N$ is a vector of unknown constants, $\phi(\bar{x}) \in R^N$ is a vector of basis functions, and $\varepsilon(\bar{x})$ is the function reconstruction error.

Assumption 5: On the compact set $\Omega_{\bar{x}}$, the ideal NN weight vector W is bounded, i.e., $\|W\| \le W^*$.

The adaptive signal ν_{ad} is designed as

$$\nu_{ad} = \hat{W}^T\phi(\bar{x}) \tag{23}$$

where $\hat{W}(t)$ are the weight estimates for W. The NN weights are updated using

$$\dot{\hat{W}} = -F[\phi(\bar{x})r + \sigma(\hat{W} - \hat{W}_0)] \tag{24}$$

where $F > 0$ is a learning rate, and $\sigma > 0$ is a $\sigma-$modification factor.

V. Stability Analysis

Substituting (22) and (23) into (16), we have

$$\dot{r} = -Kr + b_u\tilde{W}^T\phi + \{h_\nu(\bar{\nu}) - b_u\}\tilde{W}^T\phi - h_\nu(\bar{\nu})\varepsilon, \tag{25}$$

where $\tilde{W} = \hat{W} - W$. With Assumption 3, the following is immediate

$$\left|(h_\nu(\bar{\nu}) - b_u)\tilde{W}^T\phi\right| \le b^*\left\|\tilde{W}\right\|\|\phi\|, \tag{26}$$

where $b^* = b_u - b_l > 0$.

Theorem 1: Consider the system in (1) regulated by the control law in (10). Suppose that Assumptions 1-5 hold, and that the NN weights are updated according to (24). Then, all the closed loop signals are uniformly ultimately bounded, and the tracking error r is attracted to an adjustable neighborhood of the origin if the following conditions are satisfied:

i): $\sigma > \dfrac{2b^{*2}\|\phi\|^2}{b_u K}$

ii): $\hat{W}(0)$ is set such that with a constant $w > 0$

$$\tilde{W}(0) \in \mathbb{B}_w \triangleq \left\{\tilde{W} \in \mathbb{R}^N \ : \ \left\|\tilde{W}\right\| \le w\right\} \tag{27}$$

iii): $\|\xi(0)\| \le p_0(\alpha_M)$ and $\|\eta(0)\| \le q_\eta$, where

$$\begin{aligned} \alpha_M &= \max\Big\{\alpha > 0 \ : p_0(\alpha) \le R_\xi, \ p_1(\alpha) \le R_\xi, \\ &\qquad\qquad p_2(\alpha) \le R_\eta - c_\eta\Big\}, \\ q_\eta &= \sqrt{\frac{c_1}{c_2}}(R_\eta - c_\eta) - c_\eta, \end{aligned} \tag{28}$$

and

$$\begin{aligned} p_0(\alpha) &= \frac{\sqrt{2\alpha - \frac{b_u}{\lambda_{\min}(F)}w^2} - (1+d_1)a}{1+d_1} \\ p_1(\alpha) &= [1 + k_0(1+d_1)]a + d_2\sqrt{2\alpha} + k_0(1+d_1)p_0(\alpha), \\ p_2(\alpha) &= \sqrt{\frac{c_2}{c_1}}\frac{c_4}{c_3}\left[d_2\sqrt{2\alpha} + k_0(1+d_1)[a + p_0(\alpha)]\right] \end{aligned}$$

Proof: The proof is carried out in two steps.

Step 1. Since the NN approximation in (22) holds when $(\xi,\eta,\xi_d) \in \Omega_{\bar{x}}$, we suppose that $(\xi,\eta) \in B_{R_\xi} \times B_{R_\eta}$ for all $t \ge 0$, which is proven in Step 2. Let us consider the following Lyapunov candidate function

$$V(r,\tilde{W}) = \frac{1}{2}r^2 + \frac{b_u}{2}\tilde{W}^T F^{-1}\tilde{W}. \tag{29}$$

Then the following is immediate.

$$V_1(|r|, \left\|\tilde{W}\right\|) \le V(r,\tilde{W}) \le V_2(|r|, \left\|\tilde{W}\right\|), \tag{30}$$

where $V_1(|r|,\|\tilde{W}\|)=\frac{1}{2}r^2+\frac{b_u}{2\lambda_{\max}(F)}\|\tilde{W}\|^2$, and $V_2(|r|,\|\tilde{W}\|)=\frac{1}{2}r^2+\frac{b_u}{2\lambda_{\min}(F)}\|\tilde{W}\|^2$. Along with (25), we have

$$\begin{aligned}\dot{V}=&-Kr^2-rh_v(\bar{v})\varepsilon+r\{h_v(\bar{v})-b_u\}\tilde{W}^T\phi\\&+b_u\tilde{W}^T\left\{\phi r+F^{-1}\dot{\hat{W}}\right\}.\end{aligned}\tag{31}$$

After applying the update law in (24), using (26), $\dot{V}$ is upper bounded by

$$\dot{V}\le -Kr^2+\beta_1|r|+\beta_2|r|\|\tilde{W}\|-b_u\sigma\tilde{W}^T(\hat{W}-\hat{W}_0),\tag{32}$$

where $\beta_1=b_u\varepsilon^*$ and $\beta_2=b^*\|\phi\|$. With the following property and bounds on product terms:

$$\tilde{W}^T(\hat{W}-\hat{W}_0)=\frac{1}{2}\|\tilde{W}\|^2+\frac{1}{2}\|\hat{W}-\hat{W}_0\|^2-\frac{1}{2}\|W-\hat{W}_0\|^2,$$

$$\beta_1|r|\le\frac{K}{4}|r|^2+\frac{\beta_1^2}{K},\ \beta_2|r|\|\tilde{W}\|\le\frac{K}{4}|r|^2+\frac{\beta_2^2}{K}\|\tilde{W}\|^2,$$

rearranging terms leads to

$$\dot{V}\le-\frac{K}{2}r^2-\frac{b_u}{2}(\sigma-\frac{2\beta_2^2}{b_uK})\|\tilde{W}\|^2+\beta\tag{33}$$

where

$$\beta=\frac{\beta_1^2}{K}+\frac{b_u}{2}\sigma\|W-\hat{W}_0\|^2.\tag{34}$$

Let $\theta_r\triangleq\sqrt{\frac{2\beta}{K}}$, $\theta_{\tilde{W}}\triangleq\sqrt{\frac{2\beta}{b_u(\sigma-\frac{2\beta_2^2}{b_uK})}}$. Then, either $|r|>\theta_r$ or $\|\tilde{W}\|>\theta_{\tilde{W}}$ renders $\dot{V}<0$. Thus, we conclude that $r(t)$ and $\tilde{W}$ are bounded. Moreover, there exists a $T_1>0$ such that $V(r,\tilde{W})\le V_2(\theta_r,\theta_{\tilde{W}}),\ \forall t\ge T_1$, where

$$V_2(\theta_r,\theta_{\tilde{W}})=\gamma\triangleq\frac{\beta}{K}+\frac{1}{\lambda_{\min}(F)}\frac{\beta}{\sigma-\frac{2\beta_2^2}{b_uK}}\tag{35}$$

which can be made small by properly choosing K,σ,F. This leads to:

$$|r(t)|\le\sqrt{2\gamma},\ \forall t\ge T_1\Longrightarrow R_{T_1}\le\sqrt{2\gamma}.\tag{36}$$

From (4), it is clear that $\|\zeta(t)\|$ converges to arbitrarily close to $\frac{k_0}{\lambda_0}\sqrt{2\gamma}$. This implies that the output tracking error $y_d(t)-y(t)$ can be regulated within a adjustable bound for $t\ge T_1$. The boundedness of η is immediate, using (21), from that $r_M(t)$ is bounded for all $t\ge0$.

Step 2. To complete the proof, we need to show that given initial conditions, $(\xi(t),\eta(t))\in B_{R_\xi}\times B_{R_\eta}$ for all $t\ge0$. Towards this end, we define a Lyapunov level set

$$L_\alpha\triangleq\left\{(r,\tilde{W})\ :\ V(r,\tilde{W})\le\alpha\right\},\tag{37}$$

where α can be varied. By Eq.(2), $r(t)$ is bounded as $|r(t)|\le(1+d_1)a+(1+d_1)\|\xi(t)\|$. Therefore, the conditions $\|\xi(0)\|\le p_0(\alpha_M)\le R_\xi$ and $\tilde{W}(0)\in\mathbb{B}_w$ mean that $(r(0),\tilde{W}(0))\in L_{\alpha_M}$. If $V(r(0),\tilde{W}(0))\le\gamma$, L_γ becomes an positively invariant set. Otherwise, $\dot{V}<0$ whenever $L_{\alpha_M}\setminus L_\gamma$, therefore L_{α_M} becomes an positively invariant set. Now, in order to prove that $(\xi(t),\eta(t))\in\mathbb{B}_{R_\xi}\times\mathbb{B}_{R_\eta}$ for all $t\ge0$, it suffices to show that $(r(t),\tilde{W}(t))\in L_{\alpha_M}$ implies $(\xi(t),\eta(t))\in\mathbb{B}_{R_\xi}\times\mathbb{B}_{R_\eta}$. If $(r,\tilde{W})\in L_{\alpha_M}$, $|r|\le\sqrt{2\alpha_M}$. Then, by Eq.(6), $\|\xi\|\le\|\xi_d\|+\left\|\tilde{\xi}\right\|\le a+d_2\sqrt{2\alpha_M}+k_0(1+d_1)\|\zeta(0)\|\le a+d_2\sqrt{2\alpha_M}+k_0(1+d_1)[a+p_0(\alpha_M)]\le p_1(\alpha_M)\le R_\xi$. For η, from Eq.(21), we proceed as follows. If $\|\tilde{\eta}(0)\|>\frac{c_4}{c_3}(d_2\sqrt{2\alpha_M}+k_0(1+d_1)[a+p_0(\alpha_M)]$, then $\|\eta\|\le\|\eta_c(t)\|+\|\tilde{\eta}\|\le c_\eta+\sqrt{\frac{c_2}{c_1}}\|\tilde{\eta}(0)\|\le c_\eta+\sqrt{\frac{c_2}{c_1}}[c_\eta+q_\eta]\le R_\eta$. Otherwise, $\|\eta\|\le c_\eta+\sqrt{\frac{c_2}{c_1}}\frac{c_4}{c_3}(d_2\sqrt{2\alpha_M}+k_0(1+d_1)[a+p_0(\alpha_M)]\le c_\eta+p_2(\alpha_M)\le R_\eta$. Therefore we conclude that $(r,\tilde{W})\in L_{\alpha_M}$ guarantees that $(\xi,\eta)\in B_{R_\xi}\times B_{R_\eta}$. This completes the proof. ■

VI. Alternative Design with a Robustifying Signal

When the parameters b_l,b_u and W^* in Assumptions 3 and 5 are known, a robustifying signal can be introduced in the form

$$v_{ad}=v_{nn}+v_{rb},\tag{38}$$

where v_{nn} is the same as in (23) and v_{rb} is designed as

$$v_{rb}=-\frac{b_r}{1-b_r}[\|\hat{W}\|+W^*]\|\phi(\bar{x})\|\operatorname{sgn}(r),\tag{39}$$

where $b_r=1-\frac{b_l}{b_u}<1$.

Theorem 2: Consider the system in (1) regulated by the control law in (10) with the NN weights updated by (24) and v_{ad} designed as in (38). Suppose that Assumptions 1-5 hold, and that the bounds b_l,b_u and W^* are known. Then, all the closed loop signals are uniformly ultimately bounded and the tracking error r is attracted to an adjustable neighborhood of the origin under the same initial conditions as in Theorem 1.

Note that condition i) in Theorem 1 is not required. The significance of this theorem is explained in the remark 2.

Proof: Most of the stability proof is the same as that for Theorem 1. However, with the robustifying signal in (39), the bound $|\left\{\frac{h_v(\bar{v})}{b_u}-1\right\}\left[v_{rb}+\tilde{W}^T\phi\right]|\le b_r|v_{rb}|+b_r\|\tilde{W}\|\|\phi\|$ is required. Consider the Lyapunov candidate function given in (29). Then, similarly as (25), we have

$$\begin{aligned}\dot{V}\le&-Kr^2+rb_u\left[v_{rb}+\{\frac{h_v(\bar{v})}{b_u}-1\}(v_{rb}+\tilde{W}^T\phi)\right]\\&-rh_v(\bar{v})\varepsilon-b_u\sigma\tilde{W}^T(\hat{W}-\hat{W}_0).\end{aligned}\tag{40}$$

Applying the robustifying term in (39) leads to

$$\dot{V}\le-Kr^2+\beta_1|r|-b_u\sigma\tilde{W}^T(\hat{W}-\hat{W}_0),\tag{41}$$

which is the same as that in (32) with $\beta_2=0$. Thus, following the same line, it is straightforward to obtain the following.

$$\dot{V}\le-\frac{K}{2}r^2-\frac{b_u}{2}\sigma\|\tilde{W}\|^2+\beta_n\tag{42}$$

where

$$\beta_n=\frac{\beta_1^2}{2K}+\frac{b_u}{2}\sigma\|W-\hat{W}_0\|^2.\tag{43}$$

Let $\theta_{n_r}\triangleq\sqrt{\frac{2\beta_n}{K}}$, $\theta_{n_{\tilde{W}}}\triangleq\sqrt{\frac{2\beta_n}{b_u\sigma}}$. Then, either $|r|>\theta_{n_r}$ or $\|\tilde{W}\|>\theta_{n_{\tilde{W}}}$ renders $\dot{V}<0$. Thus, we conclude that $r(t)$ and

$\tilde{W}$ are bounded. Moreover, there exists a $T_2 > 0$ such that $V(r,\tilde{W}) \leq V_2(\theta_{n_r}, \theta_{n_{\tilde{W}}})$, $\forall t \geq T_2$, where

$$V_2(\theta_{n_r}, \theta_{n_{\tilde{W}}}) = \gamma_n \triangleq \frac{\beta_n}{K} + \frac{1}{\lambda_{\min}(F)} \frac{\beta_n}{\sigma} \tag{44}$$

which can be made small by properly choosing K, σ, F. The rest of the stability proof is the same as in the proof for Theorem 1. ■

Remark 1: The discontinuity in (39) raises the question of the existence of a solution in the sense of Filippov [18], and raises an implementation issue. Therefore, a stability proof with the sign function replaced by a hyperbolic tangent function is under investigation.

Remark 2: How the sizes of the ultimate bounds in Theorems 1 and 2 determine the sizes of ultimate bounds on r and $\tilde{W}$ can be considered as follows. Since $V_1(r,\tilde{W}) \leq V(r,\tilde{W}) \leq V_2(r,\tilde{W})$, $|r|$ and $\tilde{W}$ are ultimately bounded by

$$r^2 + \frac{b_u}{\lambda_{\max}(F)} \left\|\tilde{W}\right\|^2 \leq C, \tag{45}$$

where C can be either 2γ in (35) for Theorem 1 or $2\gamma_n$ in (44) for Theorem 2. Figure 1 depicts the sets in (45) for Theorems 1 and 2 when the same parameters for K,σ, and F are used. It can be seen that the robustifying term further reduces the size of the ultimate bound and thus further reduces the tracking error. Also note that the adaptation gain F determines the "shape" of the ultimate bound set in (45). Let us consider the set in (45) for Theorem 1. The set is

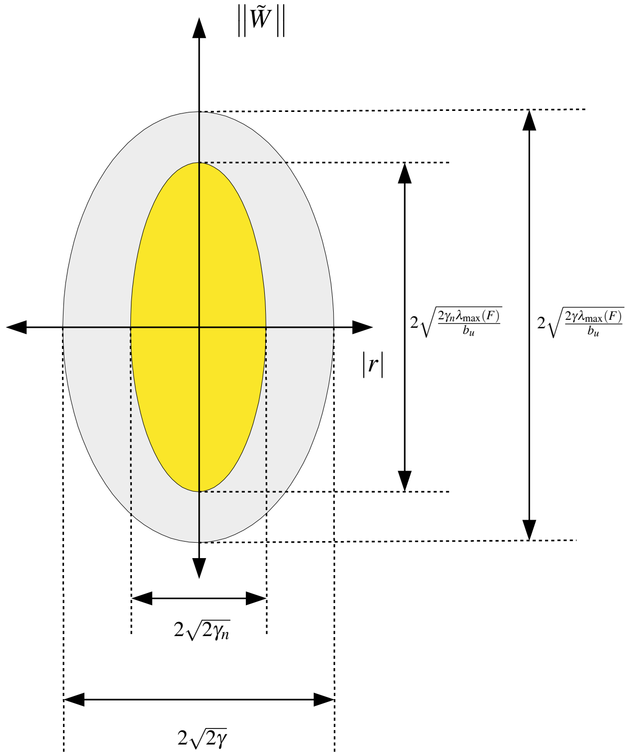


Fig. 1. Diagram for Ultimate Bounds

determined by β in (34) and γ in (35) that depend on the parameters K, σ, and F. Reducing β requires small NN reconstruction error (β_1), a good initial guess for the ideal NN weights ($\left\|\tilde{W}(0)\right\|^2$), and a high gain for K.

Once the control gain K is fixed, σ is set to satisfy the first condition in Theorem 1. Then, β is fixed. In this case, the size of the tracking error is determined by $\sqrt{2\gamma}$, which can be decreased by increasing the adaptive gain F. For the case of Theorem 2, the same arguments as the above holds with the only difference being the size of the ultimate bound as depicted in Figure 1.

Remark 3: We can show that the ultimate bound on the tracking error r (thus e) can be made arbitrarily small by choosing K and F as follows. First, it is straightforward to show that for any given $\varepsilon_0 > 0$ there exists $K_0 \in \mathbb{R}$ such that $\beta / \left(\sigma - \frac{2\beta_2^2}{b_u K}\right) \leq \left(\frac{b_u}{2}\left\|\tilde{W}(0)\right\|^2 + \varepsilon_0\right)$ for $\forall K \geq K_0$. Now, for given $\varepsilon_* > 0$, let λ_0, ε_0, and K_3 be such that $\lambda_0 > \frac{2b_u w^2}{\varepsilon_*}$, $\varepsilon_0 \leq \frac{\lambda_0}{2}\varepsilon_*$, and $2\beta/K \leq \frac{\varepsilon_*}{4}$, $\forall K \geq K_3$ respectively. By selecting F such that $F > \lambda_0 I$, we can ensure that $2\gamma \leq \varepsilon_*$ for all $K \geq K^* \triangleq \max(K_3, K_0)$. This can be seen by

$$2\gamma = \frac{2\beta}{K} + \frac{2}{\lambda_{\min}(F)} \frac{\beta}{\sigma - \frac{2\beta_2^2}{b_u K}} \leq \frac{\varepsilon_*}{4} + \frac{2}{\lambda_0}\left[\frac{b_u}{2}\left\|\tilde{W}(0)\right\|^2 + \varepsilon_0\right]$$
$$\leq \frac{\varepsilon_*}{4} + \frac{\varepsilon_*}{2} + \frac{\varepsilon_*}{4} = \varepsilon_*.$$

Since $|r(t)| \leq \sqrt{2\gamma}$, $t \geq T_1$, $|r| \leq \sqrt{\varepsilon_*}$, $\forall t \geq T_1$. In other words, the ultimate bound for r can be made arbitrarily small.

Remark 4: The sizes of the sets $\mathbb{B}_{R_\xi}$ and $\mathbb{B}_{R_\eta}$ are determined by the number of neurons employed for NN approximation. Once R_ξ and R_η are fixed, this determines the size α_M in (28), which is the size of the largest positively invariant set in $(r,\tilde{W})$ space. Then, increasing w in (27) leads to decrease in $p_0(\alpha_M)$ in (1), which restricts $\|\xi(0)\|$.

VII. Simulation Results

We illustrate the approach using a Van Der Pol equation described by

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -2(x_1^2 - 1)x_2 - x_1 + (2 + \sin(x_1 x_2))[u + \frac{1}{3}u^3 + \sin(u)] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_3 - 0.2x_4 + x_1 \\
y &= x_1 + x_3,
\end{aligned}$$

with initial conditions: $x_1(0) = 0.5$, $x_2(0) = 2.5$, $x_3(0) = 0$, and $x_4(0) = 0.2$. With the transformation $[\xi_1\ \xi_2\ \eta_1\ \eta_2]^T = [y\ \dot{y}\ x_3\ x_4]^T$, the system is put into a normal form:

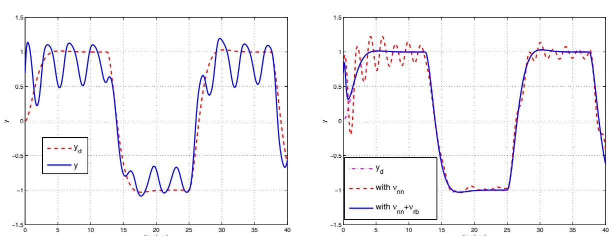
$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -2((\xi_1 - \eta_1)^2 - 1)(\xi_2 - \eta_2) - \eta_1 - 0.2\eta_2 \\
&\quad + (2 + \sin([\xi_1 - \eta_1][\xi_2 - \eta_2]))[u + \frac{1}{3}u^3 + \sin(u)] \\
\dot{\eta}_1 &= \eta_2 \\
\dot{\eta}_2 &= -2\eta_1 - 0.2\eta_2 + \xi_1 \\
y &= \xi_1.
\end{aligned}$$

The desired trajectory y_d is constructed by a command filter driven by a square wave reference command $y_c(t)$ as follows: $\ddot{y}_d = -2\omega_c\zeta_c\dot{y}_d + \omega_c^2 y_d + \omega_c^2 y_c(t)$, where $\omega_c = 1$[rad/sec] and $\zeta_c = 0.8$. The error system is designed such that $r = \dot{e} + \lambda e$, where $\lambda = 3$. The invertible design model is chosen as $\ddot{y} = 10u$ with the pseudo-control design as $\nu_{rm} = \ddot{y}_d + \lambda(\dot{y}_d - y)$, $\nu_{dc} = Kr$, $K = 2$. When $u \in [-2,2]$, the above design model leads to $0.1 \le \frac{\partial h}{\partial u} / \frac{\partial \hat{h}}{\partial u} \le 2$.

Figure 2(a) shows the output response when the inverting controller is applied without the adaptive signal. Note oscillatory behavior caused by the nonlinear elements in the Van Der Pol equation. To compensate for the uncertainties, a RBF NN consisting of 6 neurons are employed. The basis functions of the NN are given by

$$\phi_i(\bar{x}) = e^{-(\bar{x}-\bar{x}_i)^T(\bar{x}-\bar{x}_i)/R_\phi^2}, \; i = 1, \ldots, 6,$$

where $\bar{x}_{c_i}$ are randomly selected from a grid of points on the input domain, and R_ϕ is set as 1. In simulation, the robustifying term in (39) is realized as $\nu_{rb} = -\frac{b_r}{1-b_r}[\|\hat{W}\| + W^*]\|\phi(\bar{x})\| \tanh(\frac{r}{\varepsilon_r})$, where $b_r = 0.95$, $W^* = 4$, and $\varepsilon_r = 0.01$, and $\tanh(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}}$. Figure 2(b) shows output responses when the adaptive signals ν_{ad} in (23) and (38) are applied with $F = 10I$ and $\sigma = 0.1$. Even though the adaptive signal greatly suppresses the oscillations compared to Figure 2(a), there still remain oscillations. With the robustifying signal ν_{rb} added, the tracking error is regulated close to zero, illustrating its smaller ultimate bound as explained in Figure 1.



(a) Time Response of y without Adaptive Signal

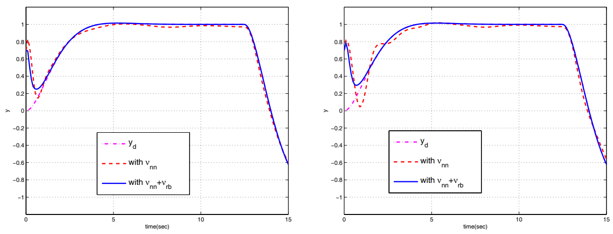
(b) Output Responses with and without the Robustifying Signal with $F = 10I$ and $K = 2$.

Fig. 2. Output responses with and without Adaptive Signal

Figures 3 illustrates that the ultimate bound on the tracking error can be reduced by either increasing the adaptation gain F (Figure 3(a)) or increasing the gain K (Figure 3(b)). Note that increasing the adaptation gain is more effective than increasing the control gain for suppressing the initial oscillations in this example.

VIII. Summary

A synthesis method that uses a neural network to compensate for inexact inversion error is considered for a class of non-affine systems. The adaptive signal is obtained without introducing a contraction mapping assumption, and without assuming a bound on the time derivative of the control effectiveness. The only requirement is that the control effectiveness has a known sign and is bounded. The stability proof shows that all the signals in the closed-loop system are bounded, and the tracking error can be adjusted using the control parameters. Simulation results with a Van Der Pol equation illustrate the approach for varying sets of gains used in the control design.



(a) Output Responses with $F = 100I$ and $K = 2$.

(b) Output Responses with $F = 10I$ and $K = 5$.

Fig. 3. Output Responses with varying F and K.

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