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Stochastic Stability of a Class of Distributed Delay Systems

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Abstract— **Criteria for the stability of a class of distributed delay systems are given via the stochastic version of the Lyapunov-Krasovskii theory. The new results obtained follow from a state augmentation technique which reduces the distributed delay system to one with a single crisp (point-) delay.**

I. Background: Stochastic Retarded Systems

Systems with delays are ubiquitous in nature and man made systems. Uncertainty can be incorporated either as an expression of our lack of precise knowledge or as a true driving force. In the latter case it is useful to model the system by a stochastic or noise driven model, as is familiar in the finite dimensional case. This leads to the study of stochastic functional differential equations. (SFDE) In this section we quickly introduce some background material on solutions of SFDE's and recall some definitions and results in stochastic stability.

A. Solutions of SFDE

A standard notation in the theory of delay systems is to denote

$$x_t \stackrel{\text{def}}{=} \{\, x(t+s) \mid s \in [-\tau, 0] \stackrel{\text{def}}{=} \mathcal{I}\,\}, \tag{1}$$

where the delay τ is finite, and x is n-dimensional. A functional differential system is modelled by a functional differential equation

$$\dot{x} = f(t, x_t). \tag{2}$$

The state space is the Banach space $X_{\text{det}} = C(\mathcal{I}, \mathbb{R}^n)$ equipped with the sup-norm [3]:

$$\forall \phi \in X_{\text{det}}: \quad \|\phi\|_C = \sup_{-\tau \le s \le 0} |\phi(s)|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

In the stochastic sense then, the state space is $X_{\text{stoch}} = \mathcal{L}^2(\Omega, C(\mathcal{I}, \mathbb{R}^n))$, and $\theta \in X_{\text{stoch}}$ implies $\|\theta(\omega)\|_C$ is in $\mathcal{L}^2$ with respect to Ω, i.e., $\|\theta\|_{X_{\text{stoch}}} = \left[\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)\right]^{1/2}$, with trajectories being elements of $C\left([0,T], \mathcal{L}^2(\Omega, X_{\text{det}})\right)$. The general stochastic functional differential equation (SFDE) is studied by Mohammed in his monograph [14]. It is shown that under a suitable Lipschitz condition (albeit in a 'stochasticized' form) and other technical conditions, a solution, i.e., a trajectory, exists. It is unique up to equivalence of stochastic processes, and exhibits interesting contnuity and differentiability properties.

B. Stochastic Stability

Substantial work in the stability of stochastic time delay systems originated in the works of Kolmanovskii, Myshkis, Nosov and Shaikhet [7], [6], [8], Mohammed [14] and Mao [12], [13], and many others. As in the deterministic case, the main approach is Lyapunov based, and centers around the martingale convergence theorem and Itô's formula [5]. In the deterministic case, two main methods are used: Lyapunov-Krasovskii based methods (based on Lyapunov functionals) and Razumikhin methods, based on Lyapunov functions. Their difference is elaborated in [3]. In this paper we shall work uniquely with the Lyapunov-Krasovskii type theorems extended to the stochastic case.

Let the distributed delay system be modeled by the Itô stochastic differential equation (where w is a standard Wiener process)

$$dx(t) = f(t, x_t)\, dt + g(t, x_t)\, dw(t), \tag{3}$$

We recall the standard definitions and results:

Definition 1. If $f(t,0) = 0$ and $g(t,0) = 0$, then the equilibrium solution $x_t \equiv 0$ of (3) is *globally asymptotically stable in probability* if $\forall\, s \ge 0$, and $\forall\, \epsilon \ge 0$,

$$\lim_{x \to 0} \mathbf{P}\{\sup_{s<t} |x_t^{s,\phi}| > \epsilon\} = 0$$
$$\mathbf{P}\{\lim_{t\to\infty} |x_t^{s,\phi}| = 0\} = 1.$$

Here $x_t^{s,x}$ is the solution at time t for the system with initial condition $x_t = \phi$ when $t = s$.

Definition 2: The stochastic delay-differential equation (3) is *robustly (asymptotically) stochastically stable* (R(A)SS) if it is globally (asymptotically) stable for all values of the delay(s).

For a linear delay systems with multiplicative noise,

$$dx(t) = [Ax(t) + B(x(t-\tau)]\, dt + [Cx(t) + Dx(t-\tau)]\, dw(t), \tag{4}$$

the following Riccati type conditions for stability were obtained in [2], [20]:

Theorem 1: *The system (4) is RSS if either of the following hold:*

i) *There exist symmetric positive definite matrices* P, Q, R *such that*

$$W(P)+Q+(PB+C'PD)Q^{-1}(B'P+D'PC)+R=0$$

ii) *There exist symmetric positive definite matrices* P, R, S *such that*

$$W(P)+S+(B'P+D'PC)S^{-1}(PB+C'PD)+R=0,$$

where we defined the linear form

$$W(X) = A'X + XA + C'XC + D'XD. \tag{5}$$

Proof: Use the Lyapunov-Krasovskii functional

$$V(x) = x'Px + \int_{t-\tau}^{t} x'(\sigma)Qx(\sigma)\, d\sigma.$$

and apply the Itô-differential rule:

$$dV = dx'\, Px + x'P\, dx + [x'Qx - x_\tau Qx_\tau]\, dt + dx'\, P\, dx.$$

Substituting (4) and taking expectations yields a quadratic form, $\mathcal{L}V$, in $[\,x'\ x_\tau'\,]'$ with weight matrix

$$\begin{bmatrix} A'P+PA+Q+C'PC & PB+C'PB \\ B'P+D'PC & D'PD-Q \end{bmatrix}. \tag{6}$$

From this, an LMI-condition (negative definiteness of the above weight matrix) follows. The two Riccati equations follow by setting either $D'PD - Q = -R < 0$ or $S = -(A'P + PA + Q + C'PC) > 0$. ∎

Remarks

1. The Lyapunov–Krasovskii functional, V, used in the theorem fails to be quasitame (see [14]), due to the presence of the (unbounded) quadratic forms, but the direct application of Itô's formula and super-martingale estimates justify its use in stability analysis.
2. Various straightforward generalizations of the above result are easily obtained with more general Lyapunov-Krasovskii functionals. For instance, Kolmanovskii and Shaikhet considered also sufficient conditions for time-varying and distributed delays [8]. It should be noted that, as in the deterministic case, the bound $\dot{\tau}(t) \leq 1$ on all delays is allowed [17]. A simple explanation is that as time proceeds, the lower boundary of the interval for the definition of the state, i.e., $t-\tau(t)$, keeps moving forward. 'The system never has to remember what it already forgot.' Hence the idea that the state is a sufficient statistic (Markovian character in the stochastic case) is conserved.
3. Korenevskii [9] and Zelentsovskii [22] used a special choice of the Lyapunov functional for which a *linear* sufficient condition LMI was obtained.

In many practical situations some information about the delay is available, and can be incorporated in the criteria. A useful definition for delay dependent stability is:

Definition 3. The stochastic delay-differential equation is *delay-dependent robustly stochastically stable* ($\overline{\tau}$-RSS) if it is globally asymptotically stable for all values of the delay(s) in $[0, \overline{\tau})$.

II. A DISTRIBUTED DELAY SYSTEM

In this section we consider a distributed stochastic delay system

$$dx(t) = \left(Ax(t) + \int_{t-\tau}^{t} B(t-s)x(s)\, ds\right) dt + Cx(t)\, dw(t), \tag{7}$$

where $B(\theta)$ satisfies

$$\dot{B}(\theta) = MB(\theta), \tag{8}$$

for some *constant* matrix M. The idea is to reduce this equation to a system with crisp delay, for which theorem 1 is applicable. Thus let for some invertible matrix S

$$y(t) = S^{-1}\int_0^{\tau} B(\theta)x(t-\theta)\, d\theta, \tag{9}$$

and rewrite the distributed system (7) as the coupled set (denoting $B(t)$ by B_t)

$$\begin{bmatrix} dx(t) \\ dy(t) \end{bmatrix} = \begin{bmatrix} A & S \\ S^{-1}B_0 & S^{-1}MS \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \tag{10}$$
$$+ \begin{bmatrix} 0 & 0 \\ -S^{-1}B_\tau & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} dw(t).$$

While (8) may seem like a big restriction, we point out that any higher order matrix differential equation can be handled as well, at the expense of a more encumbering notation. The ideas, using methods from multivariable realization theory, are summarized in the next section and details can be found in [18]. What follows illustrates the general idea. Moreover, any matrix function $B(\theta)$ can be arbitrarily closely approximated in a finite interval by a $\hat{B}(\theta)$ satisfying a matrix differential equation with constant (matrix) coefficients.

The simple criterion (6) for $D = 0$, with $\mathcal{P}$ and $\mathcal{Q}$ in the form

$$\mathcal{P} = \begin{bmatrix} P_1 & P \\ P' & P_2 \end{bmatrix}, \qquad \mathcal{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

can be applied, thus giving the LMI

$$\begin{bmatrix} W(P_1) + B_0'S^{-T}P' + PS^{-1}B_0 + Q_1 + PZ(S,Q_1)P' & : \\ S'P_1 + P'A + P_2S^{-1}B_0 + S'M'S^{-T}P' + P_2Z(S,Q_1)P' & : \end{bmatrix}$$
$$\begin{matrix} : \ P_1S + A'P + B_0'S^{-T}P_2 + PS^{-1}MS + PZ(S,Q_1)P_2 \\ : \ 2(S'P)_s + 2(P_2S^{-1}MS)_s + Q_2 + P_2Z(S,Q_*/1)P_2 \end{matrix} \Bigg] < 0$$

where $Z(S,Q)$ is the nonlinear term $S^{-1}B_\tau Q^{-1}B_\tau' S^{-T}$, and X_s denotes the symmetric part of X.

For instance, setting $P=-pI$, $P_2=qI$ and $Q_1=I$, while $Q_2\to 0$ gives

$$\begin{bmatrix} W(P_1)-p(B_0'S^{-T}+S^{-1}B_0)+I+p^2Z(S,I) & : \\ S'P_1-pA+qS^{-1}B_0-pS'M'S^{-T}-pqZ(S,I) & : \\ : & P_1S-pA'+qB_0'S^{-T}-pS^{-1}MS-pqZ(S,I) \\ : & -pS_s+2q(S^{-1}MS)_s+q^2Z(S,I)\end{bmatrix}<0$$

and $Z(S,I)=S^{-1}B_\tau B_\tau' S^{-T}$. Note that P_1, p and $q>0$ are not completely free. In order to guarantee the positive definiteness of $\mathcal{P}$ the condition

$$P_1 > \frac{p^2}{q}I \qquad (11)$$

needs to be imposed.

We consider now the specific case of invertible B_0. If B_0 is invertible, it follows that B_τ is nonsingular for all τ. Let here $S=B_\tau$, hence $Z(S,I)=I$. The LMI is

$$\begin{bmatrix} W(P_1)-2p(B_0'B_\tau^{-T})_s+(1+p^2)I & : \\ B_\tau'P_1-pA+qB_\tau^{-1}B_0-pB_\tau'MB_\tau^{-T}-pqI & : \\ : & P_1B_\tau-pA'+qB_0'B_\tau^{-T}-pB_\tau^{-1}M'B_\tau-pqI \\ : & -2p(B_\tau)_s+2q(B_\tau^{-1}MB_\tau)_s+q^2I\end{bmatrix}<0. \qquad (12)$$

Consider now a subclass of systems with $B(t)=Be^{\mu t}$, i.e., systems of the form

$$dx(t)=\left(Ax(t)+B\int_{t-\tau}^{t}e^{\mu(t-s)}x(s)\,ds\right)dt+Cx(t)\,dw(t). \qquad (13)$$

This particular delay distribution corresponds to $M=\mu I$. Set $\beta(\tau)=e^{-\mu\tau}$. The LMI (12) reduces to

$$\begin{bmatrix} W(P_1)+[(p-\beta)^2+(1-\beta^2)]I & : \\ \beta^{-1}B'P_1-pA+[q(\beta-p)-p\mu]I & : \\ : & \beta^{-1}P_1B-pA'+[q(\beta-p)-p\mu]I \\ : & -2p\beta^{-1}(B)_s+q(2\mu+q)I\end{bmatrix}<0. \qquad (14)$$

If $\mu\geq 0$, the condition $p(B)_s>0$ is necessary to make the (2,2)-block negative definite. This can only be satisfied if $(B)_s$ is definite. We derive the following simplified sufficient conditions:

Theorem 2: *The stochastic distributed delay system (13) with $\mu>0$ and $B+B'$ positive definite is stochastic stable if a positive definite symmetric matrix P and a positive scalar q can be found such that the following LMI's hold, with $\beta=e^{-\mu\tau}$*

$$qP>\beta^2 I, \qquad (15)$$

$$\begin{bmatrix} W(P)+(1-\beta^2)I & \beta^{-1}PB-\beta A'-\beta\mu I \\ \beta^{-1}B'P-\beta A-\beta\mu I & -2(B)_s+q^2I\end{bmatrix}<0. \qquad (16)$$

Proof: If $B+B'$ is positive definite, the (2,2)-block in the LMI (14) can only be negative definite for positive p. The choice $p=\beta$ yields then the result (16) noting that $\beta(B_\tau)_s=(B)_s$.

■

Note that if instead $(B)_s$ is negative definite, p obviously needs to be negative, but no simple choice stands out.

Theorem 3: *The stochastic distributed delay system (13) with $\mu<0$ is stochastic stable if a positive definite symmetric matrix P can be found such that*

$$\begin{bmatrix} W(P)+I & : & \beta^{-1}PB-\mu\beta I \\ \beta^{-1}B'P-\mu\beta I & : & -\mu^2 I\end{bmatrix}<0. \qquad (17)$$

Proof: If $\mu<0$, then $\beta>1$. With the choice $q=-\mu$ and $p=0$, which by no means is an optimal choice, for the LMI (14), one gets directly the simple criterion (17).

■

Remark. The LMI of Theorem 3 is equivalent to the Riccati inequality (Schur complement property).

$$W(P)+I+\frac{1}{\mu^2}\left(\frac{PB}{\beta}-\mu\beta I\right)\left(\frac{B'P}{\beta}-\mu\beta I\right)<0. \qquad (18)$$

Note that $\mu=0$ yields the system analyzed in [1]. In this case, $\beta=1$ for all τ, and the LMI (16) specializes to the result reported in [19, p. 405].

Corollary: *If the weight B in (7) is constant and positive definite, then if there exists a positive definite matrix P and a positive number q such that*

$$qP>I, \quad \text{and} \quad \begin{bmatrix} W(P) & PB-A' \\ B'P-A & -2(B)_s+q^2I\end{bmatrix}<0,$$

then the stochastic delay system (7) is robustly asymptotically stable (RSS).

Proof: Obviously, this system corresponds with M (or μ) being zero. Hence $\beta=1$, and the resulting LMI is the limit case of Theorem 2.

■

For this system, Florchinger [1] used a Lyapunov-Krasovskii functional of the form

$$V(\phi)=\phi(0)'P\phi(0)+\int_0^\tau\int_{-s}^{0}\phi'(\theta)Q\phi(\theta)\,d\theta\,ds, \qquad (19)$$

and proved that if there exist symmetric $P>0$ and $Q>0$ such that

$$A'P+PA+C'PC+\tau Q+\tau PBQ^{-1}B'P<0, \qquad (20)$$

then the system is τ-RSS in probability.

Comparing this condition with the one from the corollary, which does not depend on the delay τ, one sees that it gets progressively more difficult to satisfy the condition (20) if τ increases, as A must be more and more stable. The corollary expresses stability *independent* of the delay, and therefore also includes arbitrarily large delays.

It can be shown (e.g., take the scalar case for simplicity) that the condition in the corollary is not vacuous, i.e., there exist matrices (A, B, C) for which the criterion can be satisfied.

Differential delay systems driven by multi-dimensional Wiener processes, and having multiple crisp and distributed delay terms can be dealt with in the same way. A technique of reducing a more general distributed delay system with rational kernel to a system with crisp delays is based on multivariable realization theory [4].

III. Rational Distributed Delay

In this section, the reduction of a rational distributed stochastic delay system

$$\begin{aligned} dx(t) &= \left[Ax(t) + \int_0^\tau B(\theta)x(t-\theta)\,d\theta\right] dt \\ &\quad + Cx(t)\,dw(t) \end{aligned} \tag{21}$$

to the lumped delay form is explained. However this straightforward transformation of a distributed delay system may lead to redundant equations.

Definition: *A kernel $B(\cdot)$ is called rational if it can be extended to an $n \times n$ matrix-valued function $\overline{B}(\cdot)$: defined on $[0, \infty)$ for which the Laplace transform is a rational (matrix) function of s.*

Note that what follows can be generalized to include nonrational kernels as well, but at the expense of reducing the system to a time-variant one.

A. Reduction of the Distributed State Equation

First note that if $B(\cdot)$ is rational, there exists a homogeneous differential equation that is solved by $B(\cdot)$. Let N be its degree. The following is immediate:

Lemma 1: For the distributed stochastic delay system (21), define for $i = 1 \ldots N+1$

$$\xi_i(t) = A_i x(t) + \int_0^\tau B_i(\theta)x(t-\theta)\,d\theta \tag{22}$$

$$\xi_{N+1}(t) = x(t). \tag{23}$$

Then for i from $N+1$ down to 1 we get

$$d\xi_i(t) = \xi_{i-1}(t)\,dt - B_i(\tau)x(t-\tau)\,dt + C_i x(t)\,dw(t) \tag{24}$$

where the A_i and $B_i(\cdot)$ in the ξ_i satisfy the recursion ($\mathbf{D}$ be the derivation operator)

$$A_{i-1} = A_i A + B_i(0) \tag{25}$$

$$B_{i-1}(\theta) = (\mathbf{D}B_i + A_i B)(\theta) \tag{26}$$

with the conditions

$$A_{N+1} = I \tag{27}$$

$$B_{N+1}(\cdot) = 0, \tag{28}$$

and

$$C_i = A_i C. \tag{29}$$

As an immediate consequence, we get:

Lemma 2: If $B(\cdot)$ is rational, then there exists a positive integer N and matrices X_i, for $i = 1, \ldots, N+1$, such that

$$\xi_0(t) = -\sum_{i=1}^{N+1} X_i \xi_i(t). \tag{30}$$

Proof: Obviously, N is again the degree of the homogeneous differential equation satisfied by B. By lemma 1, B_{N-k} involves the k-th derivative of B. Hence there exists matrices $X_1, \ldots, X_N$ such that

$$B_0(\theta) + \sum_{i=1}^{N} X_i B_i(\theta) = 0$$

but then, letting $X_0 = I$,

$$\xi_0(t) + \sum_{i=1}^{N} X_i \xi_i(t) = \left(\sum_{i=0}^{N} X_i A_i\right) x(t).$$

or,

$$\xi_0(t) = -\sum_{i=1}^{N} X_i \xi_i(t) + \left(\sum_{i=0}^{N} X_i A_i\right) \xi_{N+1}(t).$$

Set now $X_{N+1} = -\sum_{i=0}^{N} X_i A_i$. ■

Theorem 4: (Reduction)

If $B(\theta)$ is a rational function, then the stochastic distributed delay system (21) can be transformed to a stochastic delay system with lumped delay in an $(N+1)n$-dimensional space. N is the order of a homogenous differential equation that is solved by $B(\cdot)$.

Proof: It follows from lemma 1 that

$$d\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{N+1}(t) \end{bmatrix} = \begin{bmatrix} \xi_0(t) \\ \xi_1(t) \\ \vdots \\ \xi_N(t) \end{bmatrix} dt + \\ - \begin{bmatrix} B_1(\tau) \\ B_2(\tau) \\ \vdots \\ B_N(\tau) \\ 0 \end{bmatrix} \xi_{N+1}(t-\tau)\,dt + \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{N+1} \end{bmatrix} \xi_{N+1}(t)\,dw(t).$$

Finally, invoking lemma 2:

$$\begin{bmatrix} d\xi_1(t) \\ d\xi_2(t) \\ \vdots \\ d\xi_{N+1}(t) \end{bmatrix} = \begin{bmatrix} -X_1 & -X_2 & \cdots & -X_{N+1} \\ I & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{N+1}(t) \end{bmatrix} dt +$$
$$- \begin{bmatrix} 0 & \cdots & 0 & B_1(\tau) \\ 0 & \cdots & \cdots & B_2(\tau) \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & B_N(\tau) \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t-\tau) \\ \xi_2(t-\tau) \\ \vdots \\ \xi_N(t-\tau) \\ \xi_{N+1}(t-\tau) \end{bmatrix} dt +$$
$$+ \begin{bmatrix} 0 & \cdots & 0 & C_1 \\ 0 & \cdots & \cdots & C_2 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & C_{N+1} \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{N+1}(t) \end{bmatrix} dw(t).$$

which is the desired lumped form. ■

This $(N+1)n$-dimensional block-reachable realization may be a redundant representation in the sense that there may be linear combinations of the components of the different ξ_i that are identically zero. In order to obtain a minimal representation, one can proceed as in [4].

B. Realization of Autonomous Systems

In this subsection, the realization of a deterministic autonomous rational distributed delay system using only integrators and fixed delay lines is considered. We proceed from a distributed delay equation to the state space reduction for a delay *system*, viewing it as a two-dimensional system by substituting $\sigma = e^{-\tau s}$ in the transfer function. Such an embedding was used by Morse [15], Sontag [16], Williams and Zakian [21], and Levy [10] in order to study the realization and feedback properties of delay-differential system. A survey of recent applications of the ring of quasipolynomials, i.e., entire functions which are rational in s and $e^{-s\tau}$ is presented in [11].

Since B is assumed rational, there exists an extension $\overline{B}(t)$, defined in $[0,\infty)$, with irreducible rational Laplace transform in left matrix fraction description (MFD)[1] $\hat{B}(s) = \mathcal{D}(s)^{-1}\mathcal{N}(s)$ such that $B(t) = \overline{B}(t)\chi_{[0,\tau]}$ for polynomial matrices $\mathcal{D}(s)$ and $\mathcal{N}(s)$. Without loss of generality, let this matrix fraction description be strictly proper and row reduced. (Nonproperness induces fixed delay terms, and row reduction is accomplished by a unimodular transformation). Apply the operator $\mathcal{D}(\mathbf{D})$ to the autonomous functional equation $\dot{x}(t) = Ax(t) + \int_0^\tau B(\theta)x(t-\theta)\,d\theta$. This yields the dynamical realization (neglecting for now the initial data)

$$\mathbf{D}\mathcal{D}(\mathbf{D})x(t) = [\mathcal{D}(\mathbf{D})A + \mathcal{N}(\mathbf{D})]x(t) - \mathcal{N}(\mathbf{D})x(t-\tau). \tag{31}$$

By assumption of strict properness, there exist a nonsingular matrix D_{hr}, a diagonal matrix $S(\mathbf{D}) = \text{diag}\{\mathbf{D}^{l_i}, i = 1..n\}$, where the l_i are *observability indices* (l_i is the degree of the i-th row of $\mathcal{D}(s)$), and matrices $\Psi(\mathbf{D})$ and coefficient matrices D_{lr} and N_{lr} of appropriate dimension such that

$$\mathcal{D}(\mathbf{D}) = S(\mathbf{D})D_{hr} + \Psi(\mathbf{D})D_{lr} \tag{32}$$
$$\mathcal{N}(\mathbf{D}) = \Psi(\mathbf{D})N_{lr} \tag{33}$$

where $\Psi(s) = \text{Blockdiag}\{[s^{l_i-1}, \dots, s, 1],\ i = 1, \dots, n\}$ Substituting in (31) and defining the *partial state vector*, $\xi(t) = D_{hr}x(t)$, lets one express $\mathbf{D}S(\mathbf{D})\xi(t)$ as

$$\begin{aligned} &\Psi(\mathbf{D})[-D_{lc} + D_{lr}A + N_{lr}]x(t) + \\ &\quad +\Psi(\mathbf{D})_+\overline{D}_{hr}Ax(t) + -\Psi(\mathbf{D})N_{lr}x(t-\tau) \\ &= \Psi_+(\mathbf{D})[-\overline{D}_{lc} + \overline{D}_{lr}A + \overline{N}_{lr} + \\ &\quad +\overline{D}_{hr}A]D_{hr}^{-1}\xi(t) - \Psi(\mathbf{D})N_{lr}D_{hr}^{-1}\xi(t-\tau) \\ &= \Psi_+(\mathbf{D})\hat{K}\xi(t) + \Psi_+(\mathbf{D})\hat{L}\xi(t-\tau). \end{aligned}$$

Here $\Psi_+(s) = \text{Blockdiag}\{[s^{l_i}, \dots, s, 1],\ i = 1, \dots, n\}$. Defining $S_+(s) = sS(s)$, one obtains the realization defining equation,

$$\xi(t) = [S_+(\mathbf{D})]^{-1}\Psi_+(\mathbf{D})\hat{K}\xi(t) + [S_+(\mathbf{D})]^{-1}\Psi_+(\mathbf{D})\hat{L}\xi(t-\tau) \tag{34}$$

which leads to an *observer form* realization (A reachable form realization is obtained by starting from a right fraction description of $\hat{B}(s)$). Finally, the realization is constructed via the *core-observer* realization

$$A_o^{\text{core}} = \text{Blockdiag}\left\{ \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 \\ 0 & \cdots & \cdots & & 0 \end{bmatrix}, \quad l_i \times l_i\ ;\ i = 1\dots, n \right\}$$
$$B_o^{\text{core}} = I_{n_o}; \qquad n_o = \deg\det\mathcal{D}(s) = \sum_{i=1}^{n} l_i$$
$$C_o^{\text{core}} = \text{Blockdiag}\left\{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},\ 1 \times l_i\ ;\ i = 1\dots, n\right\}$$

The full realization

$$\dot{x_o}(t) = A_o x_o(t) + B_o x_o(t-\tau) \tag{35}$$
$$x(t) = C_o^{\text{core}} x_o(t), \tag{36}$$

is obtained by injection of the partial states $\xi(t)$ and the delayed partial states $\xi(t-\tau)$ into the states x_0, more precisely

$$A_o = A_o^{\text{core}} - \hat{K} \tag{37}$$
$$B_o = \hat{L}. \tag{38}$$

The "states" x_o of this realization are the components of $\Psi(D)\,\xi$. Notice that the delayed partial states require the specification of n functions in the interval $[-\tau, 0]$. Further generalizations to nonautonomous systems (input-output systems) with possibly distributed delays in the input and output,

[1]A dual theory starting from a right MFD is easily constructed.

were discussed in [18]. A stochastic version of the above follows along the same lines.

IV. Conclusions

For a class of stochastic distributed delay systems sufficient conditions for the (stochastic) stability were derived via the Lyapunov-Krasovskii theory. It was shown that criteria can be obtained by a state augmentation technique if the distribution of the delay has a 'rational' form, i.e. satisfies a matrix differential equation with constant coefficients.

Some simple sufficient criteria in the form of LMI's or Riccati inequalities have been shown. The class considered is more general than that perviously discussed [1], [19]. Moreover it is shown how the technique of state augmentation may be beneficial to obtain a criterion for stability independent of the delay, where known methods failed.

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