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On Performance Limitation in Tracking Sinusoids

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Abstract— **This paper studies the performance limitation of a feedback system with a given linear time-invariant (LTI) plant in tracking a sinusoidal signal. It continues and goes beyond some recent studies in the same topic in which it is assumed that the controller can access all the past and future values of the reference signal. In this paper, we consider the more realistic (and more difficult) situation where the controller only accesses the current and past values of the reference. An explicit formula of the best attainable performance is obtained for a SISO system which depends on the nonminimum phase zeros of the plant and the frequency of the reference sinusoid. Compared to the previously studied case when the future of the reference is available, this formula clearly shows the extra effort one has to pay to predict the future values of the reference. A partial result for a MIMO system is also given.**

Keywords: Linear system structure, Performance limitation, Optimal control, Tracking, Nonminimum phase.

I. Introduction

In this paper, we further study the performance limitation of a feedback system with a given linear time-invariant plant in tracking a sinusoidal signal. The main issue in such a study is to find the analytical relationship, hopefully simple and insightful, between the best tracking error attainable by designing the controller and the properties of the plant and the reference. In our previous study [12], it was assumed that the dynamic controller not only had the access of the instantaneous values of the reference signal and hence its past history, but also the instantaneous values of all state variables of the exogenous reference generator and hence all the past and future values of the reference. Under this complete or full information assumption, the best attainable tracking error over all possible controller designs was given in terms of the inherent properties, mainly the nonminimum phase zeros, of the plant and the frequency of the reference signal. Although this best attainable performance, called the performance limit, obtained under the complete reference information assumption is more fundamental than that under any other incomplete or partial information assumption where the controller does not have all the past and future values of the reference, it is an ideal case. In real applications, however, it is often the case that the controller can only access the current value of the reference signal. It is then of interest to consider the performance limit under this information constraint. It will be shown that for a SISO plant this performance limit can also be expressed in terms of the nonminimum phase zeros of the plant and the frequency of the reference in a rather simple way. Compared with the performance limit in the complete reference information case, the limiting tracking error contains an extra nonnegative term which is the price we need to pay for the lack of enough information and characterizes the extra effort needed to predict the future values of the reference. For a MIMO plant, the same problem is also addressed with less generality. Only a MIMO system with at most two nonminimum phase zeros will be studied. The performance limit in this special MIMO case exhibits in one hand some interesting insightful features and on the other hand the difficulty in deriving a performance limit for a general MIMO plant.

The studies on performance limitation of feedback systems provide deep understandings to inherent constraints on the best achievable performance of the systems due to the structures and characteristics of the plants. It has been attracting a growing amount of interest in the control community. The type of works related to our study can be traced back to the early 1970s when optimal cheap LQ control was studied by Kwakernaak and Sivan in [8] and later by Francis in [6]. It was shown that perfect regulation can be achieved for right-invertible minimum phase systems but not for general nonminimum phase systems. The performance limitation in tracking/disturbance rejection was first studied by Davison and Scherzinger in [4] where it was shown that perfect tracking/disturbance rejection can be achieve for right-invertible minimum phase systems but not for general nonminimum phase systems. The recent trend is more on the quantitative limits in the achievable performance for nonminimum phase systems. Morari and Zafiriou [9], Qiu and Davison [10] gave simple expressions of the performance limits in tracking step signals for a right invertible plant. A more refined study for multivariable plants was given by Chen, Qiu and Toker in [1]. These works have since been extended to more general references [10], [2], [3], [12], discrete time systems [15], [7], [13], nonlinear systems [11], and systems with uncertainties [5], [14]. This paper follows mostly from [12], where performance limitation in tracking sinusoidal signals was studied in quite a general setting under the assumption that the complete reference information is available to the controller. This paper on the other hand treats a special setup under the assumption that only incomplete information of the reference is available to the controller.

The organization of this paper is as follows. In Section

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2, the problem under consideration in this paper is precisely formulated based on our previous works. In Section 3, we present our main result for a SISO LTI system and the its proof. Then we discuss the relationship between the main result and our previous results. Section 4 extends the main result for the SISO system in Section 3 to a special class of MIMO systems with at most two nonminimum phase zeros. Section 5 is the conclusion.

The notation used throughout this paper is fairly standard. For any complex number, vector and matrix, denote their conjugate, conjugate transpose, real and imaginary part by $\bar{(\cdot)}$, $(\cdot)^*$, $\mathrm{Re}(\cdot)$ and $\mathrm{Im}\,(\cdot)$, respectively. The phase or argument of a nonzero complex number is denoted by $\angle(\cdot)$. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open right and left half plane be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$ containing functions that are analytical in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ constitute orthogonal complements in $\mathcal{L}_2$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices. Finally, the inner product between two complex vectors u, v is defined by $\langle u, v\rangle := u^* v$.

II. Problem Statements

The system under consideration in this paper is shown in Figure 1. Here $P(s)$ is the transfer function of a given

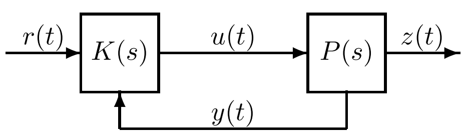

Fig. 1. A two-parameter control structure with partial reference information

plant whose output $z(t)$ and measurement $y(t)$ may not be the same, $K(s)$ is the transfer function of a two degree of freedom (2DOF) controller which is to be designed. We write $P(s) = \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}$ where $G(s)$ is the transfer function from $u(t)$ to $z(t)$ and $H(s)$ is the transfer function from $u(t)$ to $y(t)$. One typical sinusoidal tracking problem is to design a controller $K(s)$ so that the closed loop system is internally stabilized and the plant output $z(t)$ asymptotically tracks a sinusoidal reference signal $r(t)$ of the form

$$r(t) = \bar{v}e^{-j\omega t} + ve^{j\omega t} = 2\,\mathrm{Re}\,v\cos\omega t + 2\,\mathrm{Im}\,v\sin\omega t. \quad (1)$$

In [12], a more general version of the sinusoidal tracking problem is studied in which the reference might be a linear combination of a step and several sinusoidal waves of different frequencies. More importantly, in [12] the controller is assumed to know the magnitude and phase information of all harmonics of the reference $r(t)$ in advance. Such a case will be called the full reference information case. Specialized to the single frequency sinusoid tracking where the reference signal is given in (1), the full reference information case is equivalent to the case when the controller $K(s)$ also takes the derivative of $r(t)$, in addition to the reference $r(t)$ itself, as one of its input, as shown in Figure 2. In this paper, we will assume that the controller does not know the magnitude and phase of $r(t)$, i.e., the vector v, and it can only access the instantaneous values of $r(t)$. If the controller finds that the information on vector v is needed, it has to spend time and effort to estimate it. This latter case will be called the partial reference information case. The intuition tells us that the lack of complete information in the partial reference information case would likely result in performance deterioration, but how much deterioration will be resulted exactly? This is precisely the question that we try to answer in this paper.

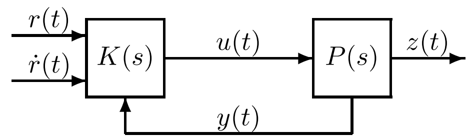

Fig. 2. A two-parameter control structure with full reference information

The transient tracking error is measured by its energy:

$$J(v) = \int_0^\infty \|r(t) - z(t)\|^2 dt = \int_0^\infty \|e(t)\|^2 dt. \quad (2)$$

In order for the tracking problem to be meaningful and solvable, we make the following assumptions throughout the paper.

Assumption 1:

1) $P(s)$, $G(s)$ and $H(s)$ have the same unstable poles.
2) $G(s)$ has no zero at $-j\omega, j\omega$.

The first item in the assumption means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. It is satisfied in the special cases of output feedback, where $y(t) = z(t)$, and state feedback, where $y(t)$ is the state vector of system $G(s)$. A more precise way of stating this is that if $P(s) = \begin{bmatrix} N(s) \\ L(s) \end{bmatrix} M^{-1}(s)$ is a coprime factorization, then we assume that $N(s)M^{-1}(s)$ and $L(s)M^{-1}(s)$ are also coprime factorizations. The second item is of course necessary for the solvability of the tracking problem.

In the full reference information case, $J(v)$ can be minimized for each individual v. The best achievable performance is then given by

$$J_{opt}(v) = \inf_K J(v)$$

which depends on v of course. One possible assessment of the overall performance limitation is given by the average of $J_{opt}(v)$ when v is taken as a random vector with zero mean, unit covariance, and uncorrelated conjugate:

$$J_{opt} = \boldsymbol{E}\{J_{opt}(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\}.$$

The explicit expressions for $J_{opt}(v)$ and J_{opt} were obtained in [12].

In the partial reference information case, since the magnitude and phase of the reference are not available to the feedback controller, it is only meaningful to consider the averaged tracking performance of the system over a reasonable set of magnitudes and phases. Here we again take the average when v is considered as a random vector with zero mean, unit covariance, and uncorrelated conjugate. Hence the averaged performance is given by

$$E = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\}. \quad (3)$$

The limit of E, under any controller design, is given by

$$E_{opt} = \inf_K E. \quad (4)$$

Mathematically, the difference between J_{opt} and E_{opt} lies in the order of the expectation over v and the infimum over the controller K. Immediately we know $E_{opt} \geq J_{opt}$ from their definitions. It is the purpose of this paper to derive an explicit formula for E_{opt}, hence a good understanding of the exact amount of E_{opt} in excess of J_{opt}.

To find an explicit formula for E_{opt} and compare it with J_{opt}, some preliminary results in [12] are reviewed. Suppose that $G(s)$ has nonminimum phase zeros $z_1, \ldots, z_m$. It is shown in [12] that, for a given frequency ω, the transfer function of the system $G(s)$ can be factorized into

$$G(s) = G_{\omega 1}(s) \cdots G_{\omega m}(s) G_0(s)$$

where $G_0(s)$ has only minimum phase zeros,

$$G_{\omega i}(s) = \eta_{\omega i}\eta_{\omega i}^* \left[\frac{z_i^* + j\omega}{z_i - j\omega}\frac{z_i - s}{z_i^* + s} - 1\right] + I$$

and $\eta_{\omega i}$, $i = 1, \ldots, m$, are frequency dependent unit directional vectors associated with z_i, $i = 1, \ldots, m$. And then $N(s)$ can be factorized into

$$N(s) = G_{\omega 1}(s) \cdots G_{\omega m}(s) N_0(s) \quad (5)$$

where $N_0(s)$ is an outer (for example see [16]). Moreover, it is obtained in [12] that, for given unit directional vectors $\eta_{\omega i}$, $i = 1, \ldots, m$, there exist $\eta_{-\omega i}$, $i = 1, \ldots, m$, such that

$$\eta_{-\omega 1} = \eta_{\omega 1}$$

and

$$\eta_{-\omega i} = G_{\omega 1}(-j\omega) \cdots G_{\omega i-1}(-j\omega)\eta_{\omega i}, \ i = 2, \ldots, m. \quad (6)$$

The results in [12], when specialized to the single frequency reference given by (1), give the explicit expressions for $J_{opt}(v)$ and J_{opt}.

Lemma 1: [12] Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then the tracking performance limit is given by

$$J_{opt}(v) = \sum_{i=1}^{m} 2\mathrm{Re}(z_i) \left|\frac{\langle \eta_{-\omega i}, \bar{v}\rangle}{z_i + j\omega} + \frac{\langle \eta_{\omega i}, v\rangle}{z_i - j\omega}\right|^2$$

and

$$J_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right).$$

III. SISO Systems

In this section, we give a rather complete answer for the case when $G(s)$ is a SISO system. In this case, item 1 in Assumption 1 simply means that $G(s)$ and $H(s)$ have the same unstable poles.

Theorem 1: Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, , z_m$. Then

$$E_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right) + \frac{2}{\omega}\sin^2\left[2\sum_{i=1}^{m}\angle(z_i - j\omega)\right].$$

Proof: Let $G(s) = N(s)M^{-1}(s)$ be a coprime factorization. Then by using the parameterization of all stabilizing 2DOF controllers [16], we can see that the achievable transfer function from $r(t)$ to $z(t)$ is $N(s)Q(s)$ where $Q(s)$ is an arbitrary $\mathcal{H}_\infty$ transfer function which can be designed. Hence, for a fixed v, the tracking performance $J(v)$ defined in (2) is written to

$$\begin{aligned} J(v) &= \|[1 - N(s)Q(s)]R(s)\|_2^2 \\ &= \left\|[1 - N(s)Q(s)]\begin{bmatrix}\frac{1}{s + j\omega} & \frac{1}{s - j\omega}\end{bmatrix}\begin{bmatrix}\bar{v} \\ v\end{bmatrix}\right\|_2^2. \end{aligned}$$

The averaged cost function E is then given by

$$\begin{aligned} E &= \left\|[1 - N(s)Q(s)]\begin{bmatrix}\frac{1}{s + j\omega} & \frac{1}{s - j\omega}\end{bmatrix}\right\|_2^2 \\ &= \left\|[1 - N(s)Q(s)]\frac{\sqrt{2}(s + \omega)}{s^2 + \omega^2}\begin{bmatrix}\frac{s - j\omega}{\sqrt{2}(s + \omega)} & \frac{s + j\omega}{\sqrt{2}(s + \omega)}\end{bmatrix}\right\|_2^2 \\ &= \left\|[1 - N(s)Q(s)]\frac{\sqrt{2}(s + \omega)}{s^2 + \omega^2}\right\|_2^2. \end{aligned}$$

The last equality follows from the fact that $\begin{bmatrix}\frac{s - j\omega}{\sqrt{2}(s + \omega)} & \frac{s + j\omega}{\sqrt{2}(s + \omega)}\end{bmatrix}$ is co-inner. Hence the averaged tracking performance E is equal to the performance of the system in tracking the signal

$$\begin{aligned} r(t) &= \frac{\sqrt{2}}{2}(1 + j)e^{-j\omega t} + \frac{\sqrt{2}}{2}(1 - j)e^{j\omega t} \\ &= \sqrt{2}\cos\omega t + \sqrt{2}\sin\omega t, \end{aligned}$$

i.e., $E = J\left(\frac{\sqrt{2}}{2}(1 - j)\right)$. It follows from Lemma 1 that the performance limit is given by

$$\begin{aligned} E_{opt} &= J_{opt}\left(\frac{\sqrt{2}}{2}(1 - j)\right) \\ &= \sum_{i=1}^{m} 2\mathrm{Re}(z_i)\left|\frac{\langle \eta_{-\omega i}, \frac{\sqrt{2}}{2}(1 + j)\rangle}{z_i + j\omega} + \frac{\langle \eta_{\omega i}, \frac{\sqrt{2}}{2}(1 - j)\rangle}{z_i - j\omega}\right|^2. \end{aligned} \quad (7)$$

For the SISO system $G(s)$, we select the unit directional vectors $\eta_{\omega i}$ and the inner functions $G_{\omega i}(s)$, $i = 1, \ldots, m$, associated with z_i, $i = 1, \ldots, m$, as follows

$$\eta_{\omega i} = 1 \quad \text{and} \quad G_{\omega i}(s) = \frac{z_i^* + j\omega}{z_i - j\omega}\frac{z_i - s}{z_i^* + s}, \quad i = 1, \ldots, m.$$

Then following (6), we have

$$\eta_{-\omega i} = \frac{z_1^* + j\omega}{z_1 - j\omega}\frac{z_1 + j\omega}{z_1^* - j\omega}\cdots\frac{z_{i-1}^* + j\omega}{z_{i-1} - j\omega}\frac{z_{i-1} + j\omega}{z_{i-1}^* - j\omega}, \quad i = 2, \ldots, m.$$

Expanding (7) gives $E_{opt} = E_a + E_b$ where

$$\begin{aligned} E_a &= \sum_{i=1}^{m}\left[\frac{2\mathrm{Re}(z_i)}{(z_i^* - j\omega)(z_i + j\omega)} + \frac{2\mathrm{Re}(z_i)}{(z_i^* + j\omega)(z_i - j\omega)}\right] \\ &= 2\sum_{i=1}^{m}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right) \end{aligned}$$

and

$$E_b = \sum_{i=1}^{m}\left[-\frac{j2\mathrm{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^* - j\omega)(z_i - j\omega)} + \frac{j2\mathrm{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^* + j\omega)(z_i + j\omega)}\right].$$

In the remaining part of this proof, induction is used.

First of all, denote $\angle(z_i^* - j\omega)(z_i - j\omega)$ by ϕ_i. Then

$$(z_i^* - j\omega)(z_i - j\omega) = |z_i^* - j\omega||z_i - j\omega|e^{j\phi_i}$$

and

$$-2\mathrm{Re}(z_i)\omega = |z_i^* - j\omega||z_i - j\omega|\sin\phi_i.$$

The first term of E_b can then be written as

$$\begin{aligned} &-\frac{j2\mathrm{Re}(z_1)}{(z_1^* - j\omega)(z_1 - j\omega)} + \frac{j2\mathrm{Re}(z_1)}{(z_1^* + j\omega)(z_1 + j\omega)} \\ &\quad = \frac{j\sin\phi_1}{\omega}\left(e^{-j\phi_1} - e^{j\phi_1}\right) = \frac{2}{\omega}\sin^2\phi_1. \qquad (8) \end{aligned}$$

Assume that

$$\sum_{i=1}^{k-1}\left[-\frac{j2\mathrm{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^* - j\omega)(z_i - j\omega)} + \frac{j2\mathrm{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^* + j\omega)(z_i + j\omega)}\right] = \frac{2}{\omega}\sin^2(\phi_1 + \cdots + \phi_{k-1}). \quad (9)$$

Notice the fact that $\eta_{-\omega k}\eta_{\omega k}^* = e^{-j2(\phi_1 + \cdots + \phi_{k-1})}$ and $\eta_{\omega k}\eta_{-\omega k}^* = e^{j2(\phi_1 + \cdots + \phi_{k-1})}$. Then it holds

$$\begin{aligned} &\left[-\frac{j2\mathrm{Re}(z_k)\eta_{-\omega k}\eta_{\omega k}^*}{(z_k^* - j\omega)(z_k - j\omega)} + \frac{j2\mathrm{Re}(z_k)\eta_{\omega k}\eta_{-\omega k}^*}{(z_k^* + j\omega)(z_k + j\omega)}\right] \\ &= \frac{j\sin\phi_k}{\omega}\left[e^{-j2(\phi_1 + \cdots + \phi_{k-1}) - \phi_k} - e^{j2(\phi_1 + \cdots + \phi_{k-1}) + \phi_k}\right] \\ &= \frac{2}{\omega}\sin\left[2(\phi_1 + \cdots + \phi_{k-1}) + \phi_k\right]\sin\phi_k \qquad (10) \end{aligned}$$

Consequently, it follows from (9) and (10) that

$$\sum_{i=1}^{k}\left[-\frac{j2\mathrm{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^* - j\omega)(z_i - j\omega)} + \frac{j2\mathrm{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^* + j\omega)(z_i + j\omega)}\right] = \frac{2}{\omega}\sin^2(\phi_1 + \cdots + \phi_k). \quad (11)$$

Here, in the last step, we used elementary trigonometrical identities. Therefore

$$E_b = \frac{2}{\omega}\sin^2(\phi_1 + \cdots + \phi_m) = \frac{2}{\omega}\sin^2 2\left[\sum_{i=1}^{m}\angle(z_i - j\omega)\right].$$

This completes the proof. □

Notice that in the full reference information case we have the following performance limit, as stated in Lemma 1,

$$J_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right).$$

Theorem 1 gives an exact picture on how the lack of the reference state information affects the best tracking performance. Compared with the performance limit in the full reference information case, the performance limit in the partial reference information case has an extra nonnegative term which is caused by the controller in estimating the state of the reference or equivalently in predicting the future values of the reference.

Finally we present an extended version of Theorem 1 to the case when $G(s)$ contains a time delay.

Theorem 2: Let $G(s) = e^{-\tau s}G_r(s)$ where $G_r(s)$ is a real rational transfer function with nonminimum phase zeros $z_1, \ldots, z_m$. Then

$$\begin{aligned} E_{opt} = 2\tau &+ 2\sum_{i=1}^{m}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right) \\ &+ \frac{2}{\omega}\sin^2\left[-\omega\tau + 2\sum_{i=1}^{m}\angle(z_i - j\omega)\right]. \end{aligned}$$

The proof is omitted since it is just a minor modification of that of Theorem 1.

IV. MIMO Systems

It appears that extending the SISO result in the last section to the case when $G(s)$ is MIMO is difficult in general. Here we consider a special case of MIMO systems with no more than two nonminimum phase zeros z_1 and z_2. This special case is manageable and the result reveals some interesting insights and also the possible difficulties in the general case. The directional vectors associated with z_1 and z_2 are denoted by $\eta_{\omega 1}$ and $\eta_{\omega 2}$ respectively. Assume that $P(s), G(s), H(s)$ satisfy Assumption 1.

Theorem 3: Let $G(s)$ have two nonminimum phase zeros z_1, z_2 and let θ be the angle between the associated directional vectors $\eta_{\omega 1}$ and $\eta_{\omega 2}$. Then

$$\begin{aligned} E_{opt} = 2&\sum_{i=1}^{2}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right) \\ &+ \frac{2}{\omega}\sin^2\theta\sum_{i=1}^{2}\sin^2[\angle(z_i - j\omega)(z_i^* - j\omega)] \\ &+ \frac{2}{\omega}\cos^2\theta\sin^2\left[\sum_{i=1}^{2}\angle(z_i - j\omega)(z_i^* - j\omega)\right]. \end{aligned}$$

Proof: By the same procedure as that used for a SISO LTI system in the last section, we have

$$E = \left\| \left[I - N(s)Q(s) \right] \frac{\sqrt{2}(s+\omega)}{s^2+\omega^2} \right\|_2^2. \quad (12)$$

Suppose that the output dimension is n. Denote the i-th column of the $n \times n$ identify matrix by e_i, $i = 1, \ldots, n$. It follows from (12) that

$$E = \sum_{l=1}^{n} \left\| \left[I - N(s)Q(s) \right] e_l \frac{\sqrt{2}(s+\omega)}{s^2+\omega^2} \right\|_2^2. \quad (13)$$

From (13), we can see that the averaged tracking performance E is equal to a sum of the performances of the system in tracking n different references

$$r(t) = e_l \left[\frac{\sqrt{2}}{2}(1+j)e^{-j\omega t} + \frac{\sqrt{2}}{2}(1-j)e^{j\omega t} \right], \quad l = 1, \ldots, n.$$

Since the terms in (13) depend on different columns of $Q(s)$, the overall optimum over $Q(s)$ is equal to the sum of the optimal values of the individual terms. Applying Lemma 1, we get

$$\begin{aligned} E_{opt} &= \sum_{l=1}^{n} J_{opt} \left(\frac{\sqrt{2}}{2}(1-j)e_l \right) \\ &= \sum_{l=1}^{n} \sum_{i=1}^{2} 2\mathrm{Re}(z_i) \left| \frac{\langle \eta_{-\omega i}, \frac{\sqrt{2}}{2}(1+j)e_l \rangle}{z_i + j\omega} \right. \\ &\qquad \left. + \frac{\langle \eta_{\omega i}, \frac{\sqrt{2}}{2}(1-j)e_l \rangle}{z_i - j\omega} \right|^2. \quad (14) \end{aligned}$$

Expanding (14) and noticing that $\sum_{l=1}^{n} e_l e_l^T = I$, we have

$$\begin{aligned} E_{opt} = \sum_{i=1}^{2} & 2\mathrm{Re}(z_i) \\ & \times \left[\frac{\langle \eta_{-\omega i}, \eta_{-\omega i} \rangle}{(z_i + j\omega)(z_i^* - j\omega)} + \frac{\langle \eta_{\omega i}, \eta_{\omega i} \rangle}{(z_i - j\omega)(z_i^* + j\omega)} \right. \\ & \left. + \frac{-j\langle \eta_{\omega i}, \eta_{-\omega i} \rangle}{(z_i - j\omega)(z_i^* - j\omega)} + \frac{j\langle \eta_{-\omega i}, \eta_{\omega i} \rangle}{(z_i + j\omega)(z_i^* + j\omega)} \right]. \end{aligned}$$

Denote

$$\begin{aligned} E_a = \sum_{i=1}^{2} & 2\mathrm{Re}(z_i) \\ & \times \left[\frac{\langle \eta_{-\omega i}, \eta_{-\omega i} \rangle}{(z_i + j\omega)(z_i^* - j\omega)} + \frac{\langle \eta_{\omega i}, \eta_{\omega i} \rangle}{(z_i - j\omega)(z_i^* + j\omega)} \right] \end{aligned}$$

and

$$\begin{aligned} E_b = \sum_{i=1}^{2} & 2\mathrm{Re}(z_i) \\ & \times \left[\frac{-j\langle \eta_{\omega i}, \eta_{-\omega i} \rangle}{(z_i - j\omega)(z_i^* - j\omega)} + \frac{j\langle \eta_{-\omega i}, \eta_{\omega i} \rangle}{(z_i + j\omega)(z_i^* + j\omega)} \right]. \end{aligned}$$

It is clear that

$$E_a = 2 \left(\frac{1}{z_1^* + j\omega} + \frac{1}{z_1 - j\omega} + \frac{1}{z_2^* + j\omega} + \frac{1}{z_2 - j\omega} \right).$$

Due to $\eta_{-\omega 1} = \eta_{\omega 1}$, it holds

$$\langle \eta_{\omega 1}, \eta_{-\omega 1} \rangle = \langle \eta_{-\omega 1}, \eta_{\omega 1} \rangle = 1. \quad (15)$$

It follows from (6) that the vector $\eta_{-\omega 2}$ is given by

$$\eta_{-\omega 2} = \left[I + \eta_{\omega 1}\eta_{\omega 1}^* \left(\frac{z_1^* + j\omega}{z_1 - j\omega} \frac{z_1 + j\omega}{z_1^* - j\omega} - 1 \right) \right] \eta_{\omega 2}.$$

Define $\phi_i = \angle(z_i^* - j\omega)(z_i - j\omega)$. Then we have

$$\begin{aligned} \langle \eta_{\omega 2}, \eta_{-\omega 2} \rangle &= 1 + \cos^2\theta \left(\frac{z_1^* + j\omega}{z_1 - j\omega} \frac{z_1 + j\omega}{z_1^* - j\omega} - 1 \right) \\ &= \sin^2\theta + e^{-j2\phi_1} \cos^2\theta. \quad (16) \end{aligned}$$

Consequently, it holds

$$\langle \eta_{-\omega 2}, \eta_{\omega 2} \rangle = \sin^2\theta + e^{j2\phi_1} \cos^2\theta. \quad (17)$$

It follows from (8) and (15) that

$$-\frac{j2\mathrm{Re}(z_1)\langle \eta_{\omega 1}, \eta_{-\omega 1} \rangle}{(z_1^* - j\omega)(z_1 - j\omega)} + \frac{j2\mathrm{Re}(z_1)\langle \eta_{-\omega 1}, \eta_{\omega 1} \rangle}{(z_1^* + j\omega)(z_1 + j\omega)} = \frac{2}{\omega} \sin^2\phi_1.$$

Following (16), (17) and the discussion in the proof of Theorem 1, we have

$$\begin{aligned} &-\frac{j2\mathrm{Re}(z_2)\langle \eta_{\omega 2}, \eta_{-\omega 2} \rangle}{(z_2^* - j\omega)(z_2 - j\omega)} + \frac{j2\mathrm{Re}(z_2)\langle \eta_{-\omega 2}, \eta_{\omega 2} \rangle}{(z_2^* + j\omega)(z_2 + j\omega)} \\ &\quad = \frac{2}{\omega} \sin^2\theta \sin^2\phi_2 + \frac{2}{\omega} \cos^2\theta \sin(2\phi_1 + \phi_2) \sin\phi_2. \end{aligned}$$

Consequently, it holds

$$E_b = \frac{2}{\omega} \sin^2\theta \left(\sin^2\phi_1 + \sin^2\phi_2 \right) + \frac{2}{\omega} \cos^2\theta \sin^2(\phi_1 + \phi_2).$$

Plugging in the definitions of ϕ_1 and ϕ_2 gives the expression to be proved. □

This theorem shows that, in the partial reference information case, the tracking performance limit E_{opt} depends on not only the phases of $z_1 - j\omega$ and $z_2 - j\omega$ but also the angle θ between $\eta_{\omega 1}$ and $\eta_{\omega 2}$. There are two extreme cases. One is that $\eta_{\omega 1}$ and $\eta_{\omega 2}$ are in a common one dimension subspace, i.e., $\theta = 0$ while the other is that $\eta_{\omega 1}$ and $\eta_{\omega 2}$ are orthogonal, i.e., $\theta = \pi/2$. In the first case, the two minimum phase zeros can be considered to appear in the same channel and the performance limit is given by

$$\begin{aligned} E_{opt} = 2\sum_{i=1}^{2} & \left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right) \\ & + \frac{2}{\omega} \sin^2 \left[\sum_{i=1}^{2} \angle(z_i - j\omega)(z_i^* - j\omega) \right]. \end{aligned}$$

In the second case, the two nonminimum phase zeros can be considered to appear separately in two orthogonal channels and the performance limit is given by

$$E_{opt} = 2\sum_{i=1}^{2}\left(\frac{1}{z_i^*+j\omega}+\frac{1}{z_i-j\omega}\right) + \frac{2}{\omega}\sum_{i=1}^{2}\sin^2[\angle(z_i-j\omega)(z_i^*-j\omega)].$$

In general, the performance limit is a convex combination of the two extreme cases depending on θ.

It is worth mentioning that, if the plant has only one nonminimum phase zero z_1, the performance limit E_{opt} is given by

$$E_{opt} = 2\left(\frac{1}{z_1+j\omega}+\frac{1}{z_1-j\omega}\right)+\frac{2}{\omega}\sin^2[2\angle(z_1-j\omega)].$$

Notice that z_1 is a real number in this case. Then we can obtain this result by straightforwardly following the proof of Theorem 3.

Theorem 3 also shows the potential difficulty in extending the result further to MIMO systems with more than two nonminimum phase zeros since the relative angles between each pair of the directional vectors associated with the nonminimum phase zeros will come into the picture. The number of such pairs grow combinatorially as the number of nonminimum phase zeros grow.

V. Conclusions

In this paper, the performance limitation of a feedback system in tracking a sinusoidal signal is studied under the assumption that the controller can only access the instantaneous value of the reference signal. This is in contrast with the previous study where the controller is assumed to have the complete information (past and future values) of the reference signal. A formula for the best achievable average tracking error, depending on the nonminimum phase zeros of the plant and their interactions with the reference frequency, is obtained for general SISO systems, with or without time delay. The worsening of the performance limitation due to the insufficient information is clearly shown. The study is also extended to a class of MIMO systems. It is shown that for MIMO systems, not only the plant nonminimum phase zeros but also the relative directions of the directional vectors associated with these zeros play a key role in the performance limitation. We believe that the results are significant in further understanding linear system structures and its effects on achievable control performances.

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