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On-line Tuning of Controllers for Systems with Constraints

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Abstract— **Model Predictive Control (MPC) is traditionally applied to slow processes. Recently, an explicit solution to MPC was introduced, offering a possibility to extend the area of application to high-bandwidth processes. One of the main drawbacks of the explicit form of MPC is the loss of flexibility to adjust the objective of the controller on-line. We address this issue for constrained time-invariant discrete-time linear systems by defining a parameterized cost function based on piecewise linear norms. An explicit solution to MPC is obtained by solving a more general formulation of a multi-parametric linear program (mpLP), where both the cost and the constraints are parameter dependent. This fully parameterized explicit solution enables a re-adjustment of the parameters of the cost during the controller operation, without need for recomputation of the explicit solution. Furthermore, we show that the properties of the solution for the specific problem setup make stability analysis for a range of control parameters straightforward, providing bounds on the tunable parameters of the cost for which invariance and stability of the closed-loop system are preserved.**

Index Terms— **model predictive control, explicit solution, multi-parametric linear program, parameterized cost, on-line tuning, stability analysis, constraints**

I. INTRODUCTION

Model predictive control is a widely applied, well established and understood advanced control strategy. It is based on the concept of constrained finite time optimal control (CFTOC), where in each time instance a constrained optimal control problem is solved over a finite time horizon, giving a sequence of optimal control moves. In the receding horizon policy of the MPC only the first control input is applied to the plant and the whole procedure is repeated in the next sampling instance. If a controlled plant is modeled as a linear, time-invariant system subject to constraints, the solution to a linear or quadratic program is required in each time step, depending on the type of the cost function. One of the main attractive features of the MPC is the ability to address system and safety constraints in a systematic and straightforward way [1], [2], [3]. However, mostly because a computationally demanding constrained optimization problem is solved in each time interval, MPC has traditionally been used to control only "slow" processes which allow long sampling intervals. A recent breakthrough was made with the introduction of the *explicit solution* to the MPC [4]. In this approach, MPC problem is formulated as a *multi-parametric* convex optimization problem, where states of a system are treated as parameters and control inputs as optimization variables. For piecewise linear (norms 1 and ∞) and quadratic costs, i.e. for multi-parametric linear programs (mpLPs) and multi-parametric quadratic programs (mpQPs), the optimal control input is defined as an affine state feedback law over a polyhedral partition of the state space. By solving the multi-parametric program beforehand, the bulk of the computational burden is moved off-line, while the on-line operation of the controller reduces to the simple procedure of finding the polyhedral region to which the current state vector belongs to and an evaluation of the corresponding affine optimal control law. One significant drawback of the explicit solution is that whenever an adjustment of control parameters is needed, the complete solution has to be recomputed. In this paper we address this issue and provide a solution for a class of problems.

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We consider MPC for discrete-time constrained linear systems using a cost function based on piecewise linear norms. We propose a systematic and efficient procedure based on a formulation of an mpLP with parameterized cost for the computation of the explicit controller with tunable control weight. Properties of this explicit solution make the analysis of the stability and control invariance of the tunable controller straightforward, providing to a plant operator important information about the admissible range of the adjustable control parameter(s).

II. PROBLEM STATEMENT

Consider discrete-time linear time-invariant system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$. The system (1) is subject to constraints:

$$\mathbf{P}_x\mathbf{x}_k + \mathbf{P}_u\mathbf{u}_k \leq \mathbf{p}_c \tag{2}$$

for all time instances $k \geq 0$. Define the following cost function:

$$J(\mathbf{U}_0^{N-1}, \mathbf{x}_0) := \|\mathbf{P}_N\mathbf{x}_N\|_\ell + \sum_{k=0}^{N-1} \|\mathbf{Q}\mathbf{x}_k\|_\ell + \|\mathbf{R}\mathbf{u}_k\|_\ell, \tag{3}$$

where N is a prediction horizon, $\mathbf{P}_N$ is a matrix defining the weight on the terminal state $\mathbf{x}_N$, $\|\cdot\|_\ell$ with $\ell \in \{1, \infty\}$ denotes the vector norm and $\mathbf{U}_0^{N-1} = \left[\mathbf{u}_0^T, \ldots, \mathbf{u}_{N-1}^T\right]^T \in \mathbb{R}^{m\cdot N}$ is the vector of control moves over the time horizon. CFTOC requires the solution to the following problem:

$$J^*(\mathbf{x}_0) := \min_{\mathbf{U}_0^{N-1}} J(\mathbf{U}_0^{N-1}, \mathbf{x}_0), \tag{4}$$

$$\text{subj. to } \begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \\ \mathbf{P}_x\mathbf{x}_k + \mathbf{P}_u\mathbf{u}_k \leq \mathbf{p}_c, \\ \mathbf{x}_N \in \mathcal{T}_{set}, \end{cases} \tag{5}$$

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where $\mathcal{T}_{set}$ is a *terminal set*, i.e. the set of admissible states at the final time instance $k = N$. By substituting:

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B} \mathbf{u}_{k-1-j}, \tag{6}$$

the problem (5) can be written in the compact form as a linear program (cf. [5]):

$$\begin{aligned} &\min_{\mathbf{z}} \mathbf{c}^T \mathbf{z}, \\ &\text{subj. to} \quad \mathbf{G}\mathbf{z} \leq \mathbf{S}\mathbf{x}_0 + \mathbf{w} \end{aligned} \tag{7}$$

In the receding horizon scheme, the optimal control input $\mathbf{u}_0$ is applied to the system and in the next time instance a new state vector is measured and the computation procedure is repeated. The solution to the problem (4) can be obtained by solving linear program (7) in each time instance for a single measured state vector $\mathbf{x}_0$, or for all states by solving a *multi-parametric linear program* (mpLP), in which case a state vector $\mathbf{x}_0$ is treated as a vector of parameters. The following standard result characterizes the solution to MPC:

Theorem 2.1 ([5], Theorem 7.4.1): The solution to the optimal control problem (1)–(5) with $\ell \in \{1, \infty\}$ is a *polyhedral piecewise affine (PPWA)* (affine in every polyhedron) state feedback control law of the form:

$$\mathbf{u}_k^* = \mathbf{F}_{k,i} \mathbf{x}_k + \mathbf{G}_{k,i} \quad \text{if} \ \ \mathbf{x}_k \in \mathcal{R}_{k,i}, \tag{8}$$

where $\mathcal{R}_{k,i}$, $i = 1, \dots, R_k$ are polyhedra defining a polyhedral partition of the set $\mathcal{X}_k$ of feasible states $\mathbf{x}_k$ at time step $k = 0, \dots, N-1$. □

Theorem 2.1 defines an explicit functional dependence of the optimal control law $\mathbf{u}_k^*$ on the state vector $\mathbf{x}_k$. The explicit solution, once computed, makes the evaluation of the optimal control law straightforward and MPC a viable option also for processes with fast dynamics. However, the precomputed explicit solution is optimal for fixed parameters of the controller, i.e. for selected weighting matrices $\mathbf{Q}$, $\mathbf{R}$, terminal weight $\mathbf{P}_N$ and horizon N. A change of any of these parameters, sometimes needed during the operation of the controller, requires the recomputation of the whole explicit controller. As a remedy to this inconvenience we propose an extension of the vector of parameters with the parameters which may need a readjustment during the controller operation. More precisely, we consider the weight matrices $\mathbf{Q}$, $\mathbf{P}_N$ and $\mathbf{R}$. This extension requires a more general formulation of the mpLP with simultaneous parameterization of the the cost coefficients and the constraints, which is commonly referred to as *rim* mpLP [6]. Since this is, to our knowledge, the first reported application of *rim* mpLP for computation of the explicit optimal control law, we discuss in more details some aspects of the practical implementation of the algorithm.

III. SOLVING MPLP WITH PARAMETERIZED COST AND CONSTRAINTS

Consider the following *rim* mpLP:

$$J^*(\boldsymbol{\theta}) = \min_{\mathbf{z}} \quad J(\mathbf{z}, \boldsymbol{\theta}) = (\mathbf{c} + \mathbf{D}\boldsymbol{\theta})^T \mathbf{z} \tag{9}$$

$$\text{subj. to} \quad \mathbf{G}\mathbf{z} \leq \mathbf{S}\boldsymbol{\theta} + \mathbf{w}, \tag{10}$$

where $\mathbf{z} \in \mathbb{R}^s$ is the vector of optimization variables, $\boldsymbol{\theta} \in \mathbb{R}^n$ is the vector of parameters and $\mathbf{G} \in \mathbb{R}^{q \times s}$. Without details, we only state the relevant properties of problem (9) and the corresponding solution. For an in-depth treatment of the subject, the reader is referred to [7].

The following theorem summarizes properties of the solution to the mpLP (9):

Theorem 3.1: Let $\mathcal{P}^*$ be the set of parameters $\boldsymbol{\theta}$ for which the linear program (9) has a finite optimal solution. Then:

i) $\mathcal{P}^*$ is a closed polyhedral set in $\mathbb{R}^n$,

ii) The optimizer $\mathbf{z}^*(\boldsymbol{\theta})$ is a *polyhedral piecewise affine* (PPWA) function over the set $\mathcal{P}^*$, i.e

$$\mathbf{z}^*(\boldsymbol{\theta}) = \boldsymbol{\Phi}_i \boldsymbol{\theta} + \boldsymbol{\gamma}_i, \quad \text{if} \quad \boldsymbol{\theta} \in \mathcal{CR}_i, \tag{11}$$

where $\{\mathcal{CR}_i\}_{i=1}^R$, are non-overlapping polyhedra and $\mathcal{P}^* = \bigcup_{i=1}^R \mathcal{CR}_i$,

iii) The value function $J^*(\boldsymbol{\theta})$ is continuous and *polyhedral piecewise quadratic* (PPWQ) over the set $\mathcal{P}^*$. □

The regions $\mathcal{CR}_i$ are called *critical regions*. In an mpLP, critical regions are defined as polyhedral sets of parameters uniquely determined by the optimal basic solution to the corresponding linear program [6]. A more general definition of critical region is the one using the concept of *active constraints* [5].

Definition 3.1 (Active Constraints): The set of active constraints $\mathcal{A}(\boldsymbol{\theta})$ of the problem (9)-(10) for a given vector of parameters $\boldsymbol{\theta}$ is defined as:

$$\begin{aligned} \mathcal{A}(\boldsymbol{\theta}) := \{ i \in \mathcal{I} \mid \forall \mathbf{z} \ : \ J(\mathbf{z}, \boldsymbol{\theta}) = J^*(\boldsymbol{\theta}) \Rightarrow \\ \Rightarrow \mathbf{G}_{(i)}\mathbf{z} - \mathbf{S}_{(i)}\boldsymbol{\theta} - w_i = 0 \}, \end{aligned} \tag{12}$$

where $\mathbf{G}_{(i)}$, $\mathbf{S}_{(i)}$ and w_i denote the $i-$th row of the matrices $\mathbf{G}$, $\mathbf{S}$ and vector $\mathbf{w}$ respectively, and $\mathcal{I} = \{1, \dots, q\}$. Similarly, the *inactive constraints* are defined as:

$$\begin{aligned} \mathcal{N}(\boldsymbol{\theta}) := \{ i \in \mathcal{I} \mid \exists \mathbf{z} \ : \ J(\mathbf{z}, \boldsymbol{\theta}) = J^*(\boldsymbol{\theta}) \wedge \\ \wedge \mathbf{G}_{(i)}\mathbf{z} - \mathbf{S}_{(i)}\boldsymbol{\theta} - w_i < 0 \}, \end{aligned} \tag{13}$$

Based on the notion of active constraints, critical regions are defined as subsets of the set $\mathcal{P}^*$ related to a unique set of active constraints $\mathcal{A}$:

$$\mathcal{CR}_{\mathcal{A}} = \{ \boldsymbol{\theta} \in \mathcal{P}^* \mid \mathcal{A}(\boldsymbol{\theta}) = \mathcal{A} \} \tag{14}$$

A. Computing the critical regions

The algorithm described here is an extension of the geometric algorithm for solving an mpLP of the type (7), which is based on the strategy of direct exploration of the parameter space [5]. The algorithm computes the polyhedral representation of the critical region in $\mathcal{H}-$form (intersection of half-spaces), the optimizer $\mathbf{z}^*(\boldsymbol{\theta})$ and the value function $J^*(\boldsymbol{\theta})$ using the Karush-Kuhn-Tucker (KKT) optimality conditions, i.e. primal-dual feasibility and complementarity slackness condition.

The corresponding dual problem of (9)-(10) is defined as:

$$J^*(\boldsymbol{\theta}) = \max_{\boldsymbol{\pi}} \quad (\mathbf{S}\boldsymbol{\theta} + \mathbf{w})^T \boldsymbol{\pi}, \tag{15}$$

$$\text{subj. to} \quad \begin{cases} \mathbf{G}^T \boldsymbol{\pi} = \mathbf{D}\boldsymbol{\theta} + \mathbf{c}, \\ \boldsymbol{\pi} \leq \mathbf{0}. \end{cases} \tag{16}$$

Primal and dual feasibility and complementarity slackness conditions are given by:

$$\mathbf{G}\mathbf{z}^* \le \mathbf{S}\boldsymbol{\theta} + \mathbf{w} \tag{17}$$

$$\mathbf{G}^T\boldsymbol{\pi} = \mathbf{D}\boldsymbol{\theta} + \mathbf{c}, \quad \boldsymbol{\pi} \le \mathbf{0}, \tag{18}$$

$$\pi_i(\mathbf{G}_{(i)}\mathbf{z}^* - \mathbf{S}_{(i)}\boldsymbol{\theta} - w_i) = 0. \tag{19}$$

For an arbitrary vector of parameters $\boldsymbol{\theta} = \bar{\boldsymbol{\theta}} \in \mathcal{P}^*$ we solve the primal and dual linear programs (9)-(10) and (15)-(16) and for the chosen parameter vector obtain the sets of active and inactive constraints, i.e. $\mathcal{A}(\bar{\boldsymbol{\theta}})$ and $\mathcal{N}(\bar{\boldsymbol{\theta}})$. Using the indices of active and inactive constraints, the primal feasibility condition (17) can be written as:

$$\mathbf{G}_{\mathcal{A}}\mathbf{z}^*(\boldsymbol{\theta}) \;=\; \mathbf{S}_{\mathcal{A}}\boldsymbol{\theta} + \mathbf{w}_{\mathcal{A}}, \tag{20}$$

$$\mathbf{G}_{\mathcal{N}}\mathbf{z}^*(\boldsymbol{\theta}) \;<\; \mathbf{S}_{\mathcal{N}}\boldsymbol{\theta} + \mathbf{w}_{\mathcal{N}}. \tag{21}$$

Similarly, for dual feasibility (16), using complementarity slackness (19), we get:

$$\mathbf{G}_{\mathcal{A}}^T\boldsymbol{\pi}_{\mathcal{A}} \;=\; \mathbf{D}\boldsymbol{\theta} + \mathbf{c}, \tag{22}$$

$$\boldsymbol{\pi}_{\mathcal{N}} \;=\; \mathbf{0}, \tag{23}$$

$$\boldsymbol{\pi}_{\mathcal{A}} \;\le\; \mathbf{0}. \tag{24}$$

Assuming the uniqueness of the optimal solution to primal and dual problem for $\boldsymbol{\theta} = \bar{\boldsymbol{\theta}}$, primal optimizers $\mathbf{z}^*(\boldsymbol{\theta})$ and $\boldsymbol{\pi}^*(\boldsymbol{\theta})$ can be obtained by solving the equations (20) and (22) directly:

$$\mathbf{z}^*(\boldsymbol{\theta}) \;=\; \mathbf{G}_{\mathcal{A}}^{-1}\mathbf{S}_{\mathcal{A}}\boldsymbol{\theta} + \mathbf{G}_{\mathcal{A}}^{-1}\mathbf{w}_{\mathcal{A}}, \tag{25}$$

$$\boldsymbol{\pi}_{\mathcal{A}}^*(\boldsymbol{\theta}) \;=\; \mathbf{G}_{\mathcal{A}}^{-T}\mathbf{D}\boldsymbol{\theta} + \mathbf{G}_{\mathcal{A}}^{-T}\mathbf{c}. \tag{26}$$

By substituting (25) and (26) into (17) and (24), we obtain the $\mathcal{H}-$representation of the polyhedral critical region:

$$\mathcal{CR}_{\bar{\boldsymbol{\theta}}} = \{\boldsymbol{\theta} \mid \mathbf{H}_{\bar{\boldsymbol{\theta}}}\boldsymbol{\theta} < \mathbf{k}_{\bar{\boldsymbol{\theta}}}\}, \tag{27}$$

where

$$\mathbf{H}_{\bar{\boldsymbol{\theta}}} \;=\; \begin{bmatrix} \mathbf{G}_{\mathcal{N}}\mathbf{G}_{\mathcal{A}}^{-1}\mathbf{S}_{\mathcal{A}} - \mathbf{S}_{\mathcal{N}} \\ \mathbf{G}_{\mathcal{A}}^{-T}\mathbf{D} \end{bmatrix},$$

$$\mathbf{k}_{\bar{\boldsymbol{\theta}}} \;=\; \begin{bmatrix} \mathbf{w}_{\mathcal{N}} - \mathbf{G}_{\mathcal{N}}\mathbf{G}_{\mathcal{A}}^{-1}\mathbf{w}_{\mathcal{A}} \\ -\mathbf{G}_{\mathcal{A}}^{-T}\mathbf{c} \end{bmatrix}$$

Strict inequalities in (27) are due to strict complementarity condition ($\boldsymbol{\pi}_{\mathcal{A}} < 0$) implied by the uniqueness of the primal and dual solution.

Note that critical region (27) is an *open* polyhedron. In general, when strict complementarity is not satisfied, critical regions are neither closed nor open. For practical reasons, critical regions are usually replaced by their closures (changing "$<$" into "$\le$" in (27)) and in the rest of the paper we use the term critical regions in the sense of closures, unless stated otherwise.

The exploration of the parameter space proceeds further until all critical regions $\mathcal{CR}_i \subseteq \mathcal{P}^*$ are found. The important property of the mpLP without parameterized cost is that the primal optimizer, if unique, is continuous. Even if the solution is not unique, like in the case of dual degeneracy, for mpLP (7) it is always possible to find a continuous PPWA primal optimizer function [6]. In general, this is not the case for the *rim* mpLP (9)-(10), as shown by the following example.

Example 3.1: Consider the following mpLP:

$$\min_{z} \quad \theta_2\, z,$$

$$\text{subj. to} \quad \begin{cases} 0 & \le z \le & \theta_1 + 1 \\ -1 & \le \theta_1 \le & 1 \\ -1 & \le \theta_2 \le & 1 \end{cases}$$

The solution consists of four critical regions:

$$\begin{aligned} \mathcal{CR}_1 &= \{\boldsymbol{\theta} \mid \theta_1 \in [-1, 1],\, \theta_2 \in [-1, 0)\}, \\ \mathcal{CR}_2 &= \{\boldsymbol{\theta} \mid \theta_1 \in [-1, 1],\, \theta_2 \in (0, 1]\}, \\ \mathcal{CR}_3 &= \{\boldsymbol{\theta} \mid \theta_1 \in (-1, 1],\, \theta_2 = 0\}, \\ \mathcal{CR}_4 &= \{\boldsymbol{\theta} \mid \theta_1 = -1,\, \theta_2 = 0\}. \end{aligned}$$

The primal optimizer $z^*(\boldsymbol{\theta})$ (see Fig. 1) is given by:

$$z^*(\boldsymbol{\theta}) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{\theta} + 1, & \text{if } \boldsymbol{\theta} \in \mathcal{CR}_1, \\ 0, & \text{if } \boldsymbol{\theta} \in \mathcal{CR}_2 \cup \mathcal{CR}_4. \end{cases}$$

For $\boldsymbol{\theta} \in \mathcal{CR}_3$ the optimizer is not uniquely defined: $\mathbf{z}^*(\boldsymbol{\theta}) \in \{0, 1 + \theta_1\}$. $\square$

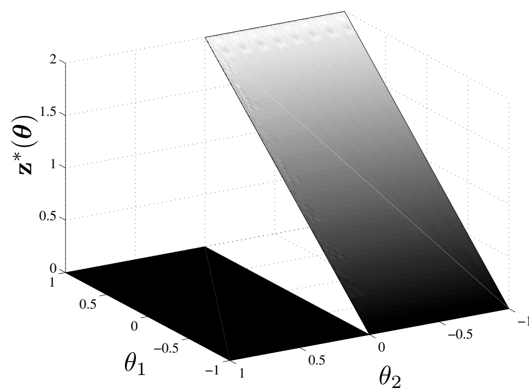

Fig. 1. Primal optimizer $z^*(\boldsymbol{\theta})$ in example 3.1.

Until now we have assumed that the primal and dual optimizers are uniquely defined inside critical regions, i.e. that no primal or dual degeneracy occurs. The solution procedure for *rim* mpLPs is more involved than for mpLPs without parameters in the cost, since both primal and dual solutions are needed for the computation of the critical regions. A detailed procedure for handling cases of primal and dual degeneracy is given in [8].

IV. Obtaining a tunable explicit MPC

Having the necessary tools, we proceed towards a tunable explicit solution to MPC. We will limit our discussion by considering only the tuning of the weights $\mathbf{R}$ on the control moves, probably the most interesting feature in practice. Intuitively it is clear that increasing these weights relatively to the state weights $\mathbf{Q}$ "slows down" the controller, making its action less aggressive, and *vice versa*: if a faster response is required, the weights on the control moves in the cost function of MPC should be decreased. Higher values of the weights $\mathbf{R}$ also make the controller less resilient to disturbances. For unstable or marginally stable systems, setting weights on the control input too high may lead to instability. Hence, we propose a simple and systematic method for the verification of invariance and stability of the resulting "tunable" explicit controller for a specified range of parameters $\mathbf{R}$.

A. Computation of the tunable controller

For the sake of simplicity, we focus on the $\infty-$norm. Also, we consider the tuning of a single control weight and only mention the possible extension to systems with multiple weights on control inputs, when an independent tuning of different control moves is required.

Consider the cost function:

$$J(\mathbf{U}_0^{N-1}, \mathbf{x}_0, r) := \|\mathbf{P}\mathbf{x}_N\|_\infty + \\ + \sum_{k=0}^{N-1} \|\mathbf{Q}\mathbf{x}_k\|_\infty + r\|\mathbf{u}_k\|_\infty, \tag{28}$$

where r is a positive scalar parameter. Redefine the objective of the MPC:

$$J^*(\mathbf{x}_0, r) := \min_{\mathbf{U}_0^{N-1}} J(\mathbf{U}_0^{N-1}, \mathbf{x}_0, r), \tag{29}$$

$$\text{subj. to} \begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \\ \mathbf{P}_x\mathbf{x}_k + \mathbf{P}_u\mathbf{u}_k \leq \mathbf{p}_c, \\ r_{min} \leq r \leq r_{max}, \\ \mathbf{x}_N \in \mathcal{T}_{set}, \end{cases} \tag{30}$$

and introduce the vector of optimizers (cf. [5]):

$$\mathbf{z} := \left[\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u, \mathbf{u}_0^T, \ldots, \mathbf{u}_{N-1}^T\right]^T \tag{31}$$

where ε_k^x, ε_k^u are variables representing upper bounds on the components of vectors $\mathbf{x}_k$ and $\mathbf{u}_k$ respectively. Using the state update equation (6), the following formulation of the problem (29)-(30) is obtained:

$$\min_{\mathbf{z}} J(\mathbf{z}, r) = \sum_{i=0}^{N} \varepsilon_i^x + r \sum_{j=0}^{N-1} \varepsilon_j^u, \tag{32}$$

$$\text{subj. to} \quad -\mathbf{1}_m\varepsilon_k^u \leq \pm\mathbf{u}_k, \tag{33}$$

$$-\mathbf{1}_n\varepsilon_k^x \leq \pm\mathbf{Q}\mathbf{x}_k, \tag{34}$$

$$-\mathbf{1}_n\varepsilon_N^x \leq \pm\mathbf{P}\mathbf{x}_N, \tag{35}$$

$$\mathbf{P}_x\mathbf{x}_k + \mathbf{P}_u\mathbf{u}_k \leq \mathbf{p}_c, \tag{36}$$

$$-r \leq -r_{min}, \tag{37}$$

$$r \leq r_{max}, \tag{38}$$

$$\mathbf{x}_N \in \mathcal{T}_{set}, \tag{39}$$

where $k = 1, \ldots, N-1$ and $\mathbf{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^n$. The problem (32)-(39) can be rewritten in a compact form as the *rim* mpLP discussed in section III:

$$\min_{\mathbf{z}} \quad (\mathbf{c} + \mathbf{d}_r r)^T\mathbf{z}, \tag{40}$$

$$\text{subj. to} \quad \mathbf{G}_x\mathbf{z} \leq \mathbf{S}_x\mathbf{x}_0 + \mathbf{w}_x, \tag{41}$$

$$\mathbf{0} \leq \mathbf{s}_r r + \mathbf{w}_r, \tag{42}$$

where the vector of parameters $\boldsymbol{\theta} = \begin{bmatrix} \mathbf{x}_0^T & r \end{bmatrix}^T$. Solving the mpLP (40)-(42) provides the solution to the MPC not only for all feasible state vectors $\mathbf{x}_0$, but also for all control input weights for the specified range $[r_{min}, r_{max}]$. For an illustration, consider the following numerical example:

Example 4.1: Consider the system described by the state equations:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \tag{43}$$

and the following cost function:

$$J(\mathbf{U}_0^{N-1}, \mathbf{x}_0, r) = \sum_{k=0}^{N-1} (\|\mathbf{Q}\mathbf{x}_k\| + r|u_k|), \tag{44}$$

where $N = 3$, $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $r \in [0.01, 10]$. The system is subject to state and input constraints:

$$-2 \leq u_k \leq 2, \quad -5 \leq \mathbf{x}_k \leq 5, \quad -2 \leq \mathbf{x}_N \leq 2$$

The explicit solution to the problem, with $\mathbf{x}_0$ and r as parameters, is shown on Fig. 2.

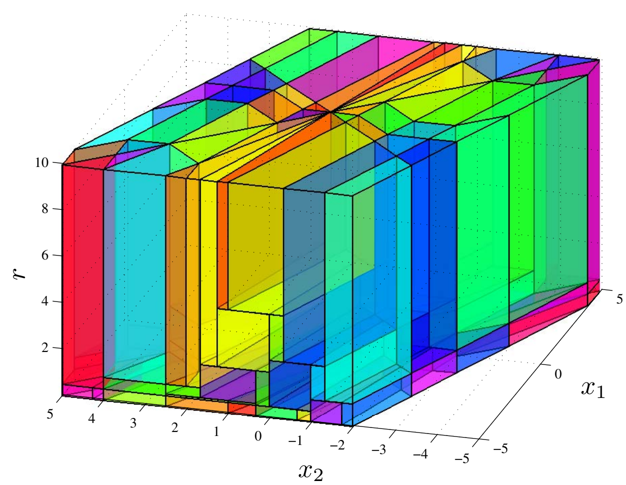

Fig. 2. Explicit solution to the MPC problem in example 4.1 with control weight as a parameter (154 critical regions).

The extension of the problem formulation for systems with multiple control inputs and more than one tunable control weight is straightforward. Consider a vector of control parameters $\mathbf{r} = [r_1, \ldots, r_m]^T$, one weight for each component of the control vector $\mathbf{u} \in \mathbb{R}^m$. For each element of the control vector and for each of N time instances an additional variable $\varepsilon_{k,i}^u$ has to be introduced. The MPC problem is now formulated as:

$$\min_{\mathbf{z}} J(\mathbf{z}, \mathbf{r}) = \sum_{i=0}^{N} \varepsilon_i^x + \sum_{j=0}^{N-1} \sum_{p=1}^{m} r_p \varepsilon_{j,p}^u, \tag{45}$$

$$\text{subj. to} \quad -\varepsilon_{j,l}^u \leq \pm u_{j,l}, \tag{46}$$

$$+ \text{ constraints} \quad (34) - (39)$$

for $j = 0, \dots, N-1$ and $l = 1, \dots, m$. Note that this setup significantly enlarges the optimization problem in all aspects since the number of optimization variables, number of parameters and the number of constraints are increased. Hence, the explicit solution, though possibly tractable, may become too complex for any practical application.

B. Determining the admissible range of tunable parameters

The mpLP (40)-(42) is written in the form which points out the special structure of the problem, i.e. the fact that the constraints on the parameter r of the cost are separate from the constraints on the parameters (states) $\mathbf{x}_0$ and the optimizer $\mathbf{z}$. Due to this special structure of the problem, the explicit solution has two important properties:

i) The set $\mathcal{P}^*_{x0}$ of parameters $\mathbf{x}_0$ for which there exists a solution to the (40)-(42) does not depend on the parameter r.
ii) The optimizer function $\mathbf{z}^*$ (hence, the optimal control law) for a specific critical region $\mathcal{CR}_i$ is not a function of parameter r.

Critical regions $\mathcal{CR}_i$ obtained as a solution to (40)-(42) have a "decoupled" form:

$$\mathcal{CR}_i = \left\{ \boldsymbol{\theta} \;\middle|\; \begin{bmatrix} \mathbf{H}^i_x & \mathbf{0} \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \boldsymbol{\theta} \le \begin{bmatrix} \mathbf{k}^i_x \\ -r^i_{min} \\ r^i_{max} \end{bmatrix} \right\}, \tag{47}$$

where $\boldsymbol{\theta} = \begin{bmatrix} \mathbf{x}_0^T & r \end{bmatrix}^T$. The partitioning of the feasible parameter space occurs independently in $\mathbf{x}_0$ and r, as shown on Fig. 3. An important fact follows directly from the properties of the solution: since the number of critical regions in the solution is finite, for a specified bounded range of parameters $r \in [r_{min}, r_{max}]$ the number of possible optimal control moves for a single state $\mathbf{x}_0^*$ is also finite. Therefore, in order to analyze the stability of different controller realizations for $r \in [r_{min}, r_{max}]$, it is necessary to analyze stabilizing properties of *finitely many* control partitions of the set $\mathcal{P}^*_{\mathbf{x}_0}$.

The first step in this analysis is to determine the intervals $\mathcal{I}^i_r$ of the parameter "r" such that the partitioning of the set $\mathcal{P}^*_{\mathbf{x}_0}$ does not change for $r \in \mathcal{I}^i_r$ (see Fig. 3). Note that the intervals $\mathcal{I}^i_r$ are open sets. If we were solving the explicit MPC problem for a fixed value of "r" corresponding to the boundary of $\mathcal{I}^i_r$, a dual degeneracy would occur in some parts of the solution. The procedure for obtaining the intervals $\mathcal{I}_r$ is easily extended to the case of the multi-dimensional vector $\mathbf{r} \in \mathbb{R}^m$, since all components of $\mathbf{r}$ are independent variables. In a multi-dimensional case intervals $\mathcal{I}^i_{\mathbf{r}}$ are simply "hypercubes": $\mathcal{I}^i_{\mathbf{r}} = \mathcal{I}^i_{r_1} \times \cdots \times \mathcal{I}_{r_m}$.

For such a general case, let $\mathcal{P}^*_{\mathbf{r}}$ denote the feasible range of the vector of parameters $\mathbf{r}$. It is necessary to identify *all* intervals $\mathcal{I}^i_{\mathbf{r}}$, i.e. to cover the set $\mathcal{P}^*_{\mathbf{r}}$:

$$\bigcup_{i=1}^{N_{\mathcal{I}_r}} cl\left\{\mathcal{I}^i_{\mathbf{r}}\right\} = \mathcal{P}^*_{\mathbf{r}}, \tag{48}$$

where $N_{\mathcal{I}_r}$ is the total number of intervals $\mathcal{I}^i_r$ and "$cl\{\cdot\}$" stands for a closure of a set. For a fixed $\mathbf{r}^* \in \mathcal{I}^i_{\mathbf{r}}$ for some

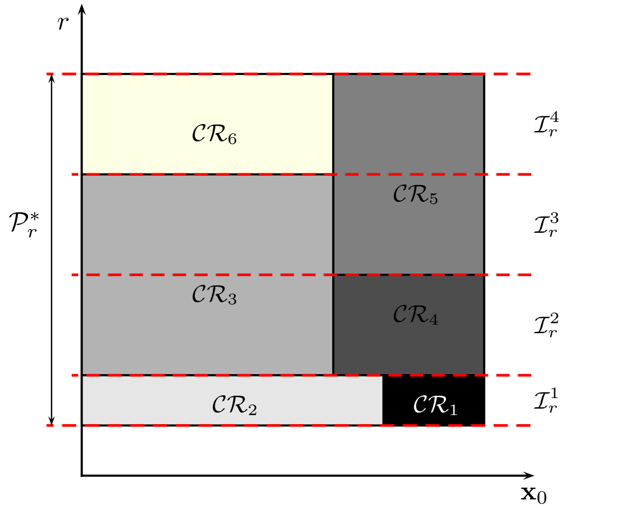

Fig. 3. Determining intervals $\mathcal{I}^i_r$ inside which the partitioning of $\mathbf{x}_0$-space does not change.

$i \in \{1, \dots, N_{\mathcal{I}_r}\}$ define the following set of indices with a fixed order:

$$\mathcal{X}_{\mathbf{r}^*} = \left\{ l \;\middle|\; \exists\, \mathbf{x}_0 \in \mathcal{P}^*_{\mathbf{x}_0} \;:\; \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{r}^* \end{bmatrix} \in \mathcal{CR}_l \right\}$$

and let $l(j)$ denote the j-th element of the set $\mathcal{X}_{\mathbf{r}^*}$. For the chosen $\mathbf{r}^*$ we obtain the controller partition consisting of non-overlapping polyhedral regions $\mathcal{R}^{\{j\}}_{\mathbf{r}^*} \subseteq \mathcal{P}^*_{\mathbf{x}_0}$:

$$\mathcal{R}^j_{\mathbf{r}^*} = \left\{ \mathbf{x}_0 \;\middle|\; \mathbf{H}^{l(j)}_x \mathbf{x}_0 \le \mathbf{k}^{l(j)}_x \right\}, \quad j = 1, \dots, \mathcal{C}(\mathcal{X}_{\mathbf{r}^*}),$$

where $\mathcal{C}(\mathcal{X}_{\mathbf{r}^*})$ is the cardinal number of the set $\mathcal{X}_{\mathbf{r}^*}$. From the expression for the optimizer $\mathbf{z}^*(\boldsymbol{\theta})$ for each of the critical regions $\mathcal{CR}_{l(j)}$ it is easy to obtain the affine control feedback corresponding to each of the controller regions $\mathcal{R}^{\{j\}}_{\mathbf{r}^*}$:

$$\mathbf{u}^*_{\mathbf{r}^*}(\mathbf{x}_0) = \mathbf{F}^j_{\mathbf{r}^*}\mathbf{x}_0 + \mathbf{g}^j_{\mathbf{r}^*}$$

The same procedure is performed for all intervals $\mathcal{I}^i_{\mathbf{r}}$. As a result, we obtain $N_{\mathcal{I}_r}$ control partitions. For each one, stability and invariance can be verified using standard tools. It is possible to achieve similar result by simply picking up a number of values for parameter "r" and computing the standard explicit MPCs for these values. Note, however, that the procedure presented here yields *all* possible realizations of the explicit MPC for a given range of parameters.

Example 4.2: We apply the procedure described in this section to the explicit solution in example 4.1. The analysis of the solution gives 10 intervals of the parameter "r" for which a unique control partition exists. For each of these intervals the corresponding control partition is computed and for each one the stability of the origin and invariance are verified. In the stability analysis we are looking for a common quadratic, piecewise quadratic or piecewise affine Lyapunov function as the stability certificate for the particular control partition. The tools for performing the stability analysis are available as parts of the *Multi-parametric Toolbox* for Matlab [9]. The results of the analysis are given in Table I. For the control partition corresponding to the range of parameter

TABLE I

DECOMPOSITION AND ANALYSIS OF THE SOLUTION FOR EXAMPLE 4.1

$\mathcal{I}_r$	regions	invariant	Lyapunov function found
$[0.01, 0.5)$	42		
$(0.5, 0.5714)$	42		
$(0.5714, 1)$	42		
$(1, 1.5)$	42		
$(1.5, 2)$	44	YES	YES
$(2, 2.5)$	44		
$(2.5, 3)$	46		
$(3, 4)$	46		
$(4, 4.5)$	48		
$(4.5, 10]$	50	YES	NO

$\mathcal{I}_r^{10} = (4.5, 10]$ no Lyapunov function is found. Indeed, the simulations of the response of the controlled system for different initial conditions show that the origin is not a stable critical point (see Fig. 4). The tunable explicit controller

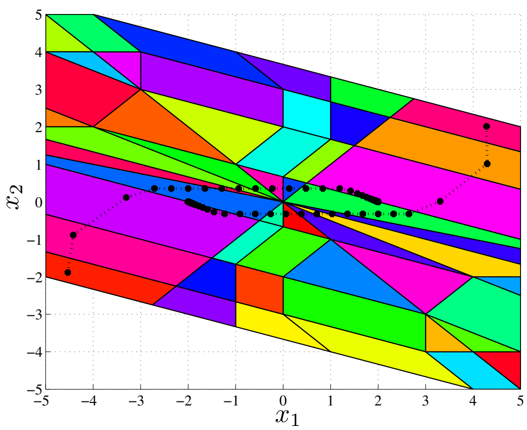

Fig. 4. Response of the closed-loop system from example 4.1 with the control weight $r = 7.25$.

comprises 9 control partitions which provide the stability and invariance of the closed-loop system within the parameter "r" ranging from 0.01 to 4.5. Each partition represents the explicit solution to the MPC problem for a certain interval of the parameter "r" and the on-line "tuning" of the controller behavior amounts to switching among these partitions. Note that we do not have to store all partitions separately, since many of the control regions are geometrically the same and share the same optimal feedback law. By turning the "dial" on the panel, the operator can adjust the behavior of the controller. The responses for different positions of the "dial" are shown on Fig. 5.

The question that naturally arises is whether it is possible to extend the concept of tunable controller to formulations based on quadratic costs. An exact solution to a parametric quadratic program with parameterized quadratic terms in the cost generally contains non-convex, non-polyhedral critical regions and as such is not practical [7]. An approximate solution based on combined quadratic-linear cost is reported in [8].

V. CONCLUSION

A systematic procedure for tuning the explicit solution to MPC for discrete-time constrained linear systems is proposed. Solving the problem for a cost function based on 1 and ∞ norms amounts to formulating and solving a generalized mpLP with simultaneous parameterization of cost and constraints. From the properties of the solution to the mpLP it follows that the number of different realizations of the explicit controllers for a continuous range of the cost parameters is finite. This is not a surprising result, it simply follows from the character of the solution of an LP problem and the "robustness" of the optimal solution of an LP to the variations in the cost. The "tunable" explicit controller is easy to analyze and implement. The main drawback is that a single optimal control law is valid for a range of parameters and changes abruptly at the borders of these ranges. Therefore, it is not possible to achieve a continuous tuning of the behavior of the controller.

For MPC based on quadratic cost, an approximate formulation of the tunable explicit solution using quadratic-linear cost is reported in the literature.

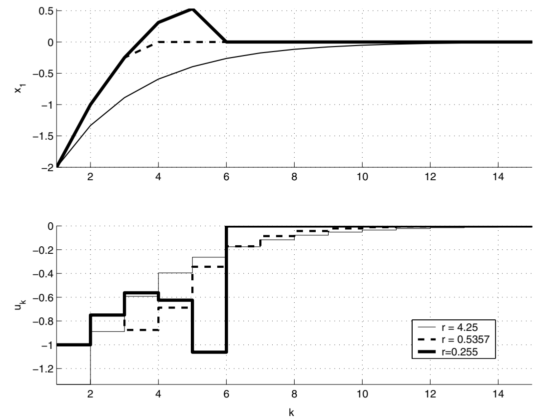

Fig. 5. Response of the system in example 4.1 and the corresponding control action as the tuning "dial" changes parameter r from 0.01 to 4.5.

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