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Semi-Global Output Feedback Stabilization of Non-Uniformly Observable and Nonsmoothly Stabilizable Systems

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Abstract— **The problem of semi-global output feedback stabilization is investigated for nonlinear systems. The main contribution of this paper is to prove that without imposing any growth condition, it is possible to achieve semi-global stabilization by nonsmooth output feedback for a chain of odd power integrators perturbed by a triangular vector field, which is in general not smoothly stabilizable nor uniformly observable.**

I. INTRODUCTION

The purpose of this paper is to address the problem of semi-global stabilization by output feedback, for a class of highly nonlinear systems that are neither uniformly observable nor smoothly stabilizable. Specifically, we are interested in the question of when semi-global stabilization by nonsmooth output feedback can be achieved for the triangular system

$$\begin{aligned} \dot{x}_1 &= x_2^{p_1} + f_1(x_1) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + f_{n-1}(x_1, \cdots, x_{n-1}) \\ \dot{x}_n &= u + f_n(x_1, \cdots, x_n) \\ y &= x_1, \end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. The mappings $f_i : \mathbb{R}^i \to \mathbb{R}$, $i = 1, \cdots, n$, are C^1 functions with $f_i(0, \cdots, 0) = 0$ and $p_1, \cdots, p_{n-1}$ are odd positive integers.

The problem of semi-global stabilization by nonsmooth output feedback can be formulated as follows. Given a bound $r > 0$, find, if possible, a nonsmooth dynamic output compensator, which may depend on r, of the form

$$\begin{aligned} \dot{\hat{z}} &= \eta(\hat{z}, y), \qquad \hat{z} \in \mathbb{R}^{n-1} \\ u &= u(\hat{z}, y) \end{aligned} \tag{1.2}$$

such that the following two properties hold:

- Local Stability: The closed-loop system (1.1)-(1.2) is locally asymptotically stable at the origin $(x, \hat{z}) = (0, 0)$;
- Semi-Global Attraction: All the trajectories of the closed-loop system starting from the compact set $\Gamma_x \times \Gamma_{\hat{z}} \triangleq [-r, r]^n \times [-r, r]^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^{n-1}$ converge to the origin.

It has been known that for nonlinear control systems, global stabilizability by state feedback plus global observability is usually not sufficient for achieving global stabilizability by output feedback. As a matter of fact, counter-examples were given in [13] illustrating that even for a simple feedback linearizable or minimum-phase system in the plane, which is uniformly observable and stabilizable by smooth state feedback, global output feedback stabilization is still not possible. The impossibility of this kind indicates that in the nonlinear case, semi-global, in stead of global, stabilization by output feedback is perhaps the more realistic control objective to be pursued.

While there are numerous papers in the literature devoted to the topic of semi-global output feedback stabilization, a major progress was reported in the work [17], where it was shown that for nonlinear systems, uniform observability [5] and global stabilizability by smooth state feedback implies semi-global stabilizability by smooth output feedback. As a consequence, semi-global stabilization by smooth output feedback was shown to be possible for minimum-phase nonlinear systems including system (1.1) with $p_1 = \cdots = p_{n-1} = 1$ [18], as well as for a class of non-minimum-phase nonlinear systems [8].

Note that when $p_i \equiv 1,\ i = 1, \cdots, n-1$, the triangular system (1.1) is feedback linearizable and hence globally stabilizable by smooth state feedback [7]. In addition, according to the characterization given in [5], the system is also uniformly observable. These two conditions are, however, violated when some of p_i are larger than one. Indeed, the triangular system (1.1) is no longer uniformly observable [5], [17], simply because the system state of (1.1) can only be represented as a *nonsmooth* rather than smooth function of the system input, output, and their derivatives (see, for instance, Example 4.4). Furthermore, system (1.1) is not smoothly stabilizable, even locally, for the reason that its linearized system may have uncontrollable modes associated with eigenvalues on the right-half plane, as illustrated by the simple planar system $\dot{x}_1 = x_2^3 + x_1, \dot{x}_2 = u, y = x_1$.

Although the nonlinear system (1.1) is neither uniformly observable nor smoothly stabilizable, it is *globally* stabilizable by *nonsmooth state* feedback, as shown in [14]. More recently, it has been further proved in [15], among the other things, that the *local* stabilization of the triangular system (1.1) is achievable by *nonsmooth output* feedback. This was accomplished by means of the theory of homogeneous systems [1], particularly, using the homogeneous approximation [3], [4], [9], [10] and the robust stability of homogeneous systems [6], [16]. In view of the results obtained in [14], [15], one might naturally make the conjecture that the triangular system (1.1) is semi-globally stabilizable by nonsmooth output feedback, without requiring any growth condition.

It turns out that this conjecture is true and semi-global stabilization by nonsmooth output feedback is indeed possible, for a significant class of non-uniformly observable and nonsmoothly stabilizable systems such as (1.1). The main contribution of this paper is the following theorem.

Theorem 1.1: For the triangular system (1.1), there exists a nonsmooth dynamic output compensator of the form (1.2), such that the closed-loop system (1.1)-(1.2) is semi-globally asymptotically stable.

The significance of Theorem 1.1 over the existing results can be summarized as follows. On the one hand, it generalizes the semi-global output feedback stabilization results in [17], [18] for nonlinear systems that are require to be uniformly observable and smoothly stabilizable, to a wider class of non-uniformly

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observable and nonsmoothly stabilizable systems such as (1.1). On the other hand, it shows that the local output feedback stabilization result in [15] (see Theorem 3.5) can be extended, without requiring any growth condition on (1.1), to the semi-global case, which is certainly a substantial progress from either a theoretical or practical viewpoint. Finally, compared with the previous global output feedback stabilization results [19], [20], [15], all the restrictive conditions such as $p_1 = \cdots = p_{n-1}$ and a high-order global Lipschitz-like condition in [19], or those growth requirements imposed on the nonlinear system (1.1) in [15] have been removed. The price we paid is that only semi-global rather than global stabilizability is achieved.

In the remainder of this paper, we shall prove Theorem 1.1 by explicitly constructing a nonsmooth, dynamic output feedback compensator of the form (1.2) for the triangular system (1.1). While the design of Hölder continuous state feedback control laws is adopted from [14], the nonsmooth observer construction is new and carried out in a subtle manner. It integrates the idea of the recursive observer design [15] with the technique of the saturated state estimates [12], [17]. The proof of semi-global stabilizability is motivated by the work [17] yet much simpler. In fact, it is strong reminiscent of the one given in [21], where a simple and intuitive argument was developed without involving intricate Lyapunov functions [17]. The use of simple Lyapunov functions, together with a delicate choice of the level sets, makes it possible to simplify the analysis and synthesis of our semi-global, nonsmooth output feedback control scheme. In the feedback linearizable case, the result presented in this paper provides an alternative yet simpler solution to the semi-global output feedback stabilization problem considered, e.g., in [18].

II. Preliminaries

In this section, we list several useful lemmas that will be used in the sequel.

Definition 2.1: A saturation function with the threshold $M \geq 0$ is defined as

$$\mathrm{sat}_M(x) : \mathbb{R} \to \mathbb{R} = \begin{cases} -M & \text{if} \quad x < -M \\ x & \text{if} \quad |x| \leq M \\ M & \text{if} \quad x > M. \end{cases} \tag{2.1}$$

Clearly, the saturation function thus defined is continuous, bounded by M and has the following property.

Lemma 2.2: Assume that $p \geq 1$ is an odd integer and $a \in [-M, M]$. Then,

$$|a - \mathrm{sat}_M(b)| \leq 2\min\{|a^p - b^p|^{1/p}, M\}, \quad \forall b \in \mathbb{R}.$$

The following lemma characterizes a useful property of smooth functions on a compact set.

Lemma 2.3: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping and $\Gamma = [-N, N]^n$ a compact set in $\mathbb{R}^n$, with $N > 0$ being a real number. Then, for $\sigma_i \in (0, 1]$, $i = 1, \cdots, n$, there exists a constant $K \geq 1$ depending on N, such that $\forall (x_1, \cdots, x_n) \in \Gamma$ and $\forall (y_1, \cdots, y_n) \in \Gamma$,

$$|f(x_1, \cdots, x_n) - f(y_1, \cdots, y_n)| \leq K(|x_1 - y_1|^{\sigma_1} + \cdots + |x_n - y_n|^{\sigma_n}).$$

The proofs of Lemmas 2.2 and 2.3 are straightforward and left to reader as an exercise.

III. A Simpler Paradigm

In order to tackle the semi-global stabilization problem for the highly nonlinear system (1.1) by nonsmooth output feedback, it is essential to understand how the problem was solved in the feedback linearizable case, i.e., $p_i = 1$, $1 \leq i \leq n$ in (1.1). In that case, system (1.1) becomes smoothly stabilizable and uniformly observable. As a consequence of the work [17], semi-global stabilization of (1.1) can be achieved by smooth output feedback. This was done based on the high-gain observer [11], [5], the idea of saturating the estimated state [12], and the technique of dynamic extension. Due to the use of uniform observability and smoothness of state feedback, the semi-global design method of [17], [18] is, however, hard to be extended to the nonlinear system (1.1) that is non-uniformly observable and nonsmoothly stabilizable. Furthermore, the proof of semi-global stability of the closed-loop system in [17] involved a subtle construction of control Lyapunov functions only defined on bounded sets, making the stability analysis less intuitive and difficult to be adopted for system (1.1) with $p_i \geq 1$.

In this section, we revisit the feedback linearizable case, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1) \\ &\vdots \\ \dot{x}_n &= u + f_n(x_1, \cdots, x_n), \\ y &= x_1. \end{aligned} \tag{3.1}$$

The objective is to develop an alternative semi-global output feedback control strategy involving ideas that do not seem to have been fully exploited yet. Specifically, we shall generalize the simple and intuitive argument proposed in [21] to the highly nonlinear system (1.1) with $p_i \geq 1$. As we shall see in a moment, the output feedback control scheme developed here is reminiscent of [21] and simpler than those in [17], [18]. More importantly, it can be carried over, with a subtle twist, to a class of inherently nonlinear systems such as (1.1), as shown in section 4.

To begin with, we observe that by adding an integrator, it is easy to get recursively a smooth state feedback controller

$$u^*(x_1, \cdots, x_n) = -\xi_n \beta_n(x_1, \cdots, x_n) \tag{3.2}$$

such that the closed-loop system (3.1)-(3.2) satisfies

$$\dot{V}_c(x) \leq -3(\xi_1^2 + \cdots + \xi_n^2) + \xi_n(u - u^*(x_1, \cdots, x_n)), \tag{3.3}$$

where $V_c(x) = \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2)$, $\xi_i = x_i - x_i^*$, $i = 1, \cdots, n$,

$$x_1^* = 0, \ x_2^* = -\xi_1\beta_1(x_1), \ \cdots, \ x_n^* = -\xi_{n-1}\beta_{n-1}(x_1, \cdots, x_{n-1})$$

with $\beta_i(\cdot) > 0$ being known smooth functions.

Since $V_c(\cdot)$ is positive definite and proper, one can define the level set $\Omega_x = \{x \in \mathbb{R}^n | V_c(x) \leq 2r_0\}$, where $r_0 > 0$ is a constant such that $\Gamma_x = [-r, r]^n \subset \{x \in \mathbb{R}^n | V_c(x) \leq r_0\}$. Then, denote

$$M = \max_{x \in \Omega_x} ||x||_\infty > 0$$

as a saturation threshold, where $|| \cdot ||_\infty$ stands for l_∞ norm of vectors. The relations among the compact sets thus introduced are illustrated in Fig. 1.

Because the states $(x_2, \cdots, x_n)$ of (3.1) are not measurable, the controller (3.2) is not realizable. To get an implementable controller, we shall design an $(n-1)-th$ order observer to estimate, instead of the states $(x_2, \cdots, x_n)$, the unmeasurable variables $(z_2, \cdots, z_n)$ defined by

$$z_2 = x_2 - L_2 x_1, \quad \cdots, \quad z_n = x_n - L_n x_{n-1}, \tag{3.4}$$

where $L_i \geq 1$, $i = 2, \cdots, n$ are gains to be assigned later.

From (3.4) it follows that

$$\begin{aligned} \dot{z}_2 &= \big[x_3 + f_2(\cdot)\big] - L_2\big[x_2 + f_1(\cdot)\big] \\ &\vdots \\ \dot{z}_n &= \big[u + f_n(\cdot)\big] - L_n\big[x_n + f_{n-1}(\cdot)\big]. \end{aligned} \tag{3.5}$$

In view of (3.5), we design the implementable dynamic compensator

$$\begin{aligned}\dot{\hat{z}}_2 &= \big[\hat{x}_3 + \hat{f}_2(\cdot)\big] - L_2\big[\hat{x}_2 + f_1(\cdot)\big]\\ \dot{\hat{z}}_3 &= \big[\hat{x}_4 + \hat{f}_3(\cdot)\big] - L_3\big[\hat{x}_3 + \hat{f}_2(\cdot)\big]\\ &\vdots\\ \dot{\hat{z}}_n &= \big[u + \hat{f}_n(\cdot)\big] - L_n\big[\hat{x}_n + \hat{f}_{n-1}(\cdot)\big], \qquad (3.6)\end{aligned}$$

where

$$\hat{x}_2 = \hat{z}_2 + L_2 x_1, \quad \cdots, \quad \hat{x}_n = \hat{z}_n + L_n \hat{x}_{n-1} \qquad (3.7)$$

$$\hat{f}_i(\cdot) \triangleq f_i(x_1, \text{sat}_M(\hat{x}_2), \cdots, \text{sat}_M(\hat{x}_i)), \quad i = 2, \cdots, n. \quad (3.8)$$

Using the certainty equivalence principle, we obtain the realizable controller

$$u = \hat{u}^*(\cdot) \triangleq u^*(x_1, \text{sat}_M(\hat{x}_2), \cdots, \text{sat}_M(\hat{x}_n)). \qquad (3.9)$$

Let $e_i = z_i - \hat{z}_i,\ i = 2, \cdots, n$ be the estimate errors. Then,

$$x_i - \hat{x}_i = e_i + L_i e_{i-1} + \cdots + L_i \cdots L_3 e_2. \qquad (3.10)$$

Consequently, the error dynamics are given by

$$\begin{aligned}\dot{e}_2 &= \big[e_3 + L_3 e_2 + \big(f_2(\cdot) - \hat{f}_2(\cdot)\big)\big] - L_2 e_2.\\ &\vdots \qquad (3.11)\\ \dot{e}_n &= \big[f_n(\cdot) - \hat{f}_n(\cdot)\big] - \big[L_n e_n + L_n^2 e_{n-1} + \cdots\\ &\quad + L_n^2 L_{n-1} \cdots L_3 e_2 + L_n\big(f_{n-1}(\cdot) - \hat{f}_{n-1}(\cdot)\big)\big].\end{aligned}$$

In view of the fact that $|\text{sat}_M(\hat{x}_i)| \le M$ and Lemmas 2.2-2.3, there exists a constant $K \ge 1$, which depends on M and is independent of L_i's, so that on the set $B_M \times \mathbb{R}^{n-1} \triangleq \{(x, \hat{z}) \in \mathbb{R}^{2n-1} | (x_1, \cdots, x_n) \in [-M, M]^n\}$, the following estimations hold ($i = 2, \cdots, n,$).

$$\begin{aligned}|f_i(\cdot) - \hat{f}_i(\cdot)|\Big|_{B_M \times \mathbb{R}^{n-1}} &\le \frac{K}{n}\sum_{j=2}^{i} |x_i - \text{sat}_M(\hat{x}_i)|\\ &\le K|e_i| + KL_i|e_{i-1}| + \cdots + KL_i \cdots L_3|e_2| \quad (3.12)\\ |\hat{u}^*(\cdot) - u^*(\cdot)|\Big|_{B_M \times \mathbb{R}^{n-1}} &\le K \min\{|e_n| + L_n|e_{n-1}| + \cdots\\ &\quad + L_n \cdots L_3|e_2|, 1\}, \qquad (3.13)\end{aligned}$$

where the notation $f(\cdot)\big|_\Gamma$ denotes the restriction of a function $f(\cdot)$ on the set Γ.

Using (3.3) and (3.13) yields

$$\begin{aligned}\dot{V}_c\Big|_{B_M \times \mathbb{R}^{n-1}} &\le -3(\xi_1^2 + \cdots + \xi_n^2)\\ &\quad + K|\xi_n| \min\{|e_n| + L_n|e_{n-1}| + \cdots + L_n \cdots L_3|e_2|, 1\}\\ &\le -2\Big(\sum_{i=1}^{n} \xi_i^2\Big) + 2\min\{C_n e_n^2 + C_{n-1}(L_n)e_{n-1}^2\\ &\quad + \cdots + C_2(L_n, \cdots, L_3)e_2^2, C_n\}, \qquad (3.14)\end{aligned}$$

where $C_n \ge 1$ is a constant independent of L_i's, $C_{n-1}(L_n) \ge C_n, \cdots, C_2(L_n, \cdots, L_3) \ge C_n$ are polynomial functions of their arguments. They can be obtained by completing the square, as done in [19].

Now, choose $V_e(e) = \frac{1}{2}(e_n^2 + \cdots + e_2^2)$ for the error dynamics (3.11). On the set $B_M \times \mathbb{R}^{n-1}$, we have

$$\begin{aligned}\dot{V}_e\Big|_{B_M \times \mathbb{R}^{n-1}} &\le \big|e_n[f_n(\cdot) - \hat{f}_n(\cdot)]\big| + \big|e_n L_n[f_{n-1}(\cdot) - \hat{f}_{n-1}(\cdot)]\big|\\ &\quad - L_n e_n^2 + \big|e_n(L_n^2 e_{n-1} + \cdots + L_n^2 L_{n-1} \cdots L_3 e_2)\big|\\ &\quad + \cdots + \big|e_2[f_2(\cdot) - \hat{f}_2(\cdot)]\big| + \big|e_2(e_3 + L_3 e_2)\big| - L_2 e_2^2\\ &\le -\big(L_n - \bar{C}_n\big)e_n^2 - \big(L_{n-1} - \bar{C}_{n-1}(L_n)\big)e_{n-1}^2\\ &\quad - \cdots - \big(L_2 - \bar{C}_2(L_n, \cdots, L_3)\big)e_2^2, \qquad (3.15)\end{aligned}$$

where $\bar{C}_n \ge 1$ is a constant independent of L_i's, $\bar{C}_{n-1}(L_n) \ge 1, \cdots, \bar{C}_2(L_n, \cdots, L_3) \ge 1$ are fixed polynomial functions of their own arguments. They can be calculated by the completion of the square, as done in [19].

From (3.15), it is easy to see that the gain assignments

$$\begin{aligned}L_n &= L_n(L) \triangleq \bar{C}_n + LC_n \ge 1\\ L_{n-1} &= L_{n-1}(L) \triangleq \bar{C}_{n-1}(L_n) + LC_{n-1}(L_n) \ge 1\\ &\vdots \qquad (3.16)\\ L_2 &= L_2(L) \triangleq \bar{C}_2(L_n, \cdots, L_3) + LC_2(L_n, \cdots, L_3) \ge 1\end{aligned}$$

with $L > 0$ being a parameter to be determined later, render

$$\dot{V}_e\Big|_{B_M \times \mathbb{R}^{n-1}} \le -LW_e, \qquad (3.17)$$

where $W_e \triangleq C_n e_n^2 + C_{n-1}(L)e_{n-1}^2 + \cdots + C_2(L)e_2^2$ and $C_{n-1}(L) \ge C_n, \cdots, C_2(L) \ge C_n$ are fixed positive polynomial functions of L.

Motivated by [17], we define

$$\mu(L) = \frac{1}{2}\sum_{i=2}^{n}(2r + L_i(L)r)^2 \ge \max_{(x,\hat{z}) \in \Gamma_x \times \Gamma_{\hat{z}}} V_e(x, \hat{z}) > 0$$

and choose the following Lyapunov function, which is different from the complicated one used in [17], for the closed-loop system (3.1)-(3.9)-(3.6)-(3.7)-(3.8):

$$V(x, \hat{z}) = \frac{V_c(x)}{2} + \frac{r_0}{2}\frac{\ln(1 + V_e(e))}{\ln(1 + \mu(L))}. \qquad (3.18)$$

Moreover, define the corresponding level set

$$\Omega = \{(x, \hat{z}) \in \mathbb{R}^{2n-1} | V(x, \hat{z}) \le r_0\}. \qquad (3.19)$$

Then, it is easy to verify the following facts (see Fig. 2):

1) For every $L > 0$, $V(\cdot)$ is a positive definite and proper function and Ω is a compact set in $\mathbb{R}^n \times \mathbb{R}^{n-1}$ (i.e., $(x, \hat{z})$-space). Once $L > 0$ is fixed, V and Ω are fixed too;

2) $\forall L > 0, \quad \Omega \supset \Gamma_x \times \Gamma_{\hat{z}}$;

3) $\forall L > 0, \quad B_M \times \mathbb{R}^{n-1} \supset \Omega$.

To prove the semi-global asymptotic stability, it remains to show that one can take advantage of the uniform boundedness of Ω with respect to L and choose $L > 0$, such that $\dot{V}\big|_\Omega \le 0$.

In view of the relationship $B_M \times \mathbb{R}^{n-1} \supset \Omega$, we deduce from (3.17) and (3.14) that $\forall L > 0$,

$$\begin{aligned}\dot{V}\Big|_\Omega &= \frac{1}{2}\dot{V}_c\Big|_\Omega + \frac{r_0}{2\ln(1 + \mu(L))}\frac{\dot{V}_e}{1 + V_e}\Big|_\Omega \qquad (3.20)\\ &\le -\Big(\sum_{i=1}^{n}\xi_i^2\Big) + \min\{W_e, C_n\} - \frac{r_0 L}{\ln(1 + \mu(L))}\frac{W_e}{2 + 2V_e}\end{aligned}$$

Observe that $C_i(L) \ge C_n \ge 1,\ i = 2, \cdots, n$ implies

$$\frac{W_e}{2 + 2V_e} \ge \frac{1}{3}\min\{W_e, C_n\}.$$

We have,

$$\dot{V}\Big|_{\Omega} \le -(\sum_{i=1}^{n}\xi_i^2) - \Big[\frac{r_0 L}{3\ln(1+\mu(L))} - 1\Big]\min\{W_e, C_n\}.$$

By construction, $\mu(L) > 0$ is a fixed *polynomial* function of L. Thus, there is a constant $L^* > 0$ such that

$$\frac{r_0 L}{3\ln(1+\mu(L))} \ge 2, \qquad \forall L \ge L^*.$$

Choosing $L = L^*$, we have immediately,

$$\dot{V}\Big|_{\Omega} \le -(\sum_{i=1}^{n}\xi_i^2) - \min\{W_e, C_n\}.$$

That is, the uniformly observable, feedback linearizable system (3.1) is semi-globally stabilizable via smooth output feedback.

IV. Proof of Theorem 1.1

Now we are ready to prove the main result of this paper — Theorem 1.1. The proof is constructive and carried out by generalizing the semi-global output feedback design method illustrated in the previous section, with a subtle twist, to the non-uniformly observable and nonsmoothly stabilizable system (1.1). In particular, we construct explicitly a nonsmooth dynamic output compensator by integrating the tool of adding a power integrator [14], the recursive nonsmooth observer design algorithm [15], and the idea of saturating the estimated states [12].

Proof of Theorem 1.1: First of all, using the nonsmooth state feedback control scheme [14], we can design a globally stabilizing state feedback controller as follows.

Let $\xi_1 = x_1$ and choose $V_1(x_1) = \frac{1}{2}\xi_1^2$. Since $f_1(x_1)$ is a C^1 function with $f_1(0) = 0$, there exists a smooth function $\rho_1(\cdot) \ge 0$ such that $|f_1(x_1)| \le |x_1|\rho_1(x_1)$. Hence,

$$\dot{V}_1 \le \xi_1 x_2^{*p_1} + \xi_1(x_2^{p_1} - x_2^{*p_1}) + \rho_1(x_1)\xi_1^2.$$

Setting $x_2^{*p_1} = -\xi_1\beta_1(x_1) \triangleq -\xi_1[(n+2) + \rho_1(x_1)]$ yields

$$\dot{V}_1 \le -(n+2)\xi_1^2 + \xi_1(x_2^{p_1} - x_2^{*p_1}).$$

Next, let $\xi_2 = x_2^{p_1} - x_2^{*p_1}$ and choose

$$V_2(x_1, x_2) = V_1(x_1) + \int_{x_2^*}^{x_2} (s^{p_1} - x_2^{*p_1})^{2-\frac{1}{p_1}} \mathrm{d}s$$

which is C^1, positive definite and proper [14]. Note that $|f_2(x_1, x_2)| \le (|x_1| + |x_2|)\bar{\rho}_2(x_1, x_2)$ with $\bar{\rho}_2(\cdot) \ge 0$ being a smooth function. With this in mind, it follows that

$$\dot{V}_2 \le -(n+1)\xi_1^2 + \xi_2 x_3^{*p_2} + \xi_2^{2-\frac{1}{p_1}}(x_3^{p_2} - x_3^{*p_2}) + \rho_2(x_1, x_2)\xi_2^2,$$

where $\rho_2(\cdot) \ge 0$ is a smooth function.

Then, it is not difficult to show the existence of a smooth function $\beta_2(\cdot,\cdot) \ge 0$, such that

$$\beta_2(x_1, x_2^{p_1}) \ge (n+1) + \rho_2(x_1, x_2), \qquad \forall (x_1, x_2).$$

Then, setting $x_3^{*p_2p_1} = -\xi_2\beta_2(x_1, x_2^{p_1})$ yields

$$\dot{V}_2 \le -(n+1)(\xi_1^2 + \xi_2^2) + \xi_2^{2-\frac{1}{p_1}}(x_3^{p_2} - x_3^{*p_2}).$$

Following the inductive argument in [14], we can obtain a set of virtual controllers

$$\xi_i = x_i^{p_{i-1}\cdots p_1} - x_i^{*p_{i-1}\cdots p_1}, \quad i = 1, \cdots, n, \qquad (4.1)$$

where $x_1^* = 0$ and

$$\begin{aligned} x_2^{*p_1} &= -\xi_1\beta_1(x_1) \\ &\vdots \qquad\qquad (4.2) \\ x_n^{*p_{n-1}\cdots p_1} &= -\xi_{n-1}\beta_{n-1}(x_1, x_2^{p_1}, \cdots, x_{n-1}^{p_{n-2}\cdots p_1}) \\ x_{n+1}^{*p_{n-1}\cdots p_1} &= -\xi_n\beta_n(x_1, x_2^{p_1}, \cdots, x_n^{p_{n-1}\cdots p_1}) \\ &\triangleq u^*(x_1, x_2^{p_1}, \cdots, x_n^{p_{n-1}\cdots p_1}) \qquad (4.3) \end{aligned}$$

with $\beta_i : \mathbb{R}^i \to \mathbb{R}^+, i = 1, \cdots, n,$ and $u^* : \mathbb{R}^n \to \mathbb{R}$ being smooth functions, and a C^1 Lyapunov function $V_c(x)$, which is positive definite and proper (whose form can be found in [14]), such that

$$\dot{V}_c \le -3(\sum_{i=1}^{n}\xi_i^2) + \xi_n^{2-\frac{1}{p_{n-1}\cdots p_1}}(u - u^*) \qquad (4.4)$$

Similar to the feedback linearizable case, we define the level set $\Omega_x = \{x \in \mathbb{R}^n | V_c \le 2r_0\}$, where $r_0 > 0$ is a constant such that $\Gamma_x \subset \{x \in \mathbb{R}^n | V_c \le r_0\}$. Moreover, denote $M = \max_{x\in\Omega_x} ||x||_\infty$ as a saturation threshold.

As done in the last section, to get an implementable controller a reduced-order observer must be designed for the estimation of the unmeasurable states $(x_2, \cdots, x_n)$ of system (1.1). Motivated by the nonsmooth observer design in [15], next we construct a reduced-order observer to estimate, instead of $(x_2, \cdots, x_n)$, the unmeasurable variables $(z_2, \cdots, z_n)$ defined by

$$\begin{aligned} z_2 &= x_2^{p_1} - L_2 x_1 \quad \Leftrightarrow \quad x_2^{p_1} = z_2 + L_2 x_1 \\ &\vdots \qquad\qquad (4.5) \\ z_n &= x_n^{p_{n-1}} - L_n x_{n-1} \quad \Leftrightarrow \quad x_n^{p_{n-1}} = z_n + L_n x_{n-1}, \end{aligned}$$

where $L_i \ge 1, 2 \le i \le n$ are gain constants to be assigned later.

By (4.5), the z-dynamics can be described by

$$\begin{aligned} \dot{z}_2 &= p_1 x_2^{p_1-1}\big[x_3^{p_2} + f_2(\cdot)\big] - L_2\big[x_2^{p_1} + f_1(\cdot)\big] \\ &\vdots \qquad\qquad (4.6) \\ \dot{z}_n &= p_{n-1}x_n^{p_{n-1}-1}\big[u + f_n(\cdot)\big] - L_n\big[x_n^{p_{n-1}} + f_{n-1}(\cdot)\big]. \end{aligned}$$

In view of (4.6), we design, similar to what we did in the last section (see (3.5)-(3.6)), the realizable observer

$$\begin{aligned} \dot{\hat{z}}_2 &= -L_2\big[\hat{x}_2^{p_1} + f_1(\cdot)\big] \\ &\vdots \\ \dot{\hat{z}}_n &= -L_n\big[\hat{x}_n^{p_{n-1}} + \hat{f}_{n-1}(\cdot)\big] \qquad (4.7) \end{aligned}$$

where for $i = 2, \cdots, n,$

$$\begin{aligned} \hat{x}_i^{p_{n-1}} &= \hat{z}_i + L_i\hat{x}_{i-1} \Leftrightarrow \hat{z}_i = \hat{x}_i^{p_{i-1}} - L_i\hat{x}_{i-1} \quad (4.8) \\ \hat{f}_i(\cdot) &\triangleq f_i(x_1, \mathrm{sat}_M(\hat{x}_2), \cdots, \mathrm{sat}_M(\hat{x}_i)). \qquad (4.9) \end{aligned}$$

By the certainty equivalence principle, we replace the unmeasurable state $(x_2, \cdots, x_n)$ in the virtual controller x_{n+1}^* by the saturated state estimate $(\hat{x}_2, \cdots, \hat{x}_n)$, which is generated by the observer (4.7)-(4.9). In doing so, we obtain the realizable controller

$$u^{p_n\cdots p_1} = \hat{u}^*(\cdot) \triangleq u^*(x_1, [\mathrm{sat}_M(\hat{x}_2)]^{p_1}, \cdots, [\mathrm{sat}_M(\hat{x}_n)]^{p_{n-1}\cdots p_1}). \qquad (4.10)$$

For $i = 2, \cdots, n$, define the estimate errors

$$e_i = z_i - \hat{z}_i = x_i^{p_{i-1}} - L_i x_{i-1} - \hat{z}_i. \qquad (4.11)$$

Note that $x_i^{p_{i-1}} - \hat{x}_i^{p_{i-1}} = e_i + L_i(x_{i-1} - \hat{x}_{i-1}),\ i = 2, \cdots, n$. Thus, the error dynamics can be expressed as

$$\begin{aligned} \dot{e}_2 &= p_1 x_2^{p_1-1}\big[x_3^{p_2} + f_2(\cdot)\big] - L_2 e_2 \\ &\vdots \\ \dot{e}_n &= p_{n-1} x_n^{p_{n-1}-1}\big[u + f_n(\cdot)\big] - L_n e_n \\ &\quad -(x_{n-1} - \hat{x}_{n-1}) - L_n\big[f_{n-1}(\cdot) - \hat{f}_{n-1}(\cdot)\big]. \end{aligned} \tag{4.12}$$

To analyze the error dynamics, we now introduce several useful propositions that can be proved by direct but tedious calculations. The detailed proofs are omitted here due to the limit of space.

The following notation is used in the remainder of this section.

$$B_M \times \mathbb{R}^{n-1} \triangleq \{(x, \hat{z}) \in \mathbb{R}^{2n-1} | (x_1, \cdots, x_n) \in [-M, M]^n\}.$$

Proposition 4.1: There exists a generic constant $K \geq 1$, which depends on M and is independent of all the L_i's, such that on the set $B_M \times \mathbb{R}^{n-1}$ the following estimations hold:

$$\begin{aligned} &|f_2(\cdot) - \hat{f}_2(\cdot)|\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K|e_2|^{\frac{1}{p_1}} \\ &\qquad\qquad \vdots \\ &|f_n(\cdot) - \hat{f}_n(\cdot)|\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K\Big(|e_n|^{\frac{1}{p_{n-1}}} + L_n^{\frac{1}{p_{n-1}}}|e_{n-1}|^{\frac{1}{p_{n-1}p_{n-2}}} \\ &\qquad + \cdots + L_n^{\frac{1}{p_{n-1}}} \cdots L_3^{\frac{1}{p_{n-1}\cdots p_2}} |e_2|^{\frac{1}{p_{n-1}\cdots p_1}}\Big). \end{aligned} \tag{4.13}$$

Proposition 4.2: There is a generic constant $K \geq 1$, which depends on M and is independent of L_i's, so that on the set $B_M \times \mathbb{R}^{n-1}$,

$$\begin{aligned} &|u - u^*|\Big|_{B_M \times \mathbb{R}^{n-1}} \\ &\leq K \min\Big\{(\sum_{i=1}^{n} |\xi_i|^{\frac{1}{p_{n-1}\cdots p_1}}) + |e_n|^{\frac{1}{p_{n-1}}} + L_n^{\frac{1}{p_{n-1}}}|e_{n-1}|^{\frac{1}{p_{n-1}p_{n-2}}} \\ &\quad + \cdots + L_n^{\frac{1}{p_{n-1}}} \cdots L_3^{\frac{1}{p_{n-1}\cdots p_2}} |e_2|^{\frac{1}{p_{n-1}\cdots p_1}}, 1\Big\}. \end{aligned} \tag{4.14}$$

Proposition 4.3: There is a generic constant $K \geq 1$, which depends on M and is independent of all the L_i's, such that on the set $B_M \times \mathbb{R}^{n-1}$ the following inequalities hold ($i = 2, \cdots, n$):

$$|x_i|\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K(|\xi_1|^{\frac{1}{p_{i-1}\cdots p_1}} + \cdots + |\xi_i|^{\frac{1}{p_{i-1}\cdots p_1}}) \tag{4.15}$$

$$|f_i(\cdot)|\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K(|\xi_1|^{\frac{1}{p_{i-1}\cdots p_1}} + \cdots + |\xi_i|^{\frac{1}{p_{i-1}\cdots p_1}}) \tag{4.16}$$

$$\begin{aligned} &|u|\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K\Big[(\sum_{i=1}^{n} |\xi_i|^{\frac{1}{p_{n-1}\cdots p_1}}) + |e_n|^{\frac{1}{p_{n-1}}} \\ &+ L_n^{\frac{1}{p_{n-1}}}|e_{n-1}|^{\frac{1}{p_{n-1}p_{n-2}}} + \cdots + L_n^{\frac{1}{p_{n-1}}} \cdots L_3^{\frac{1}{p_{n-1}\cdots p_2}} |e_2|^{\frac{1}{p_{n-1}\cdots p_1}}\Big]. \end{aligned} \tag{4.17}$$

Using the Young's inequality, it is not difficult to deduce from (4.4) and (4.14) that

$$\begin{aligned} \dot{V}_c\Big|_{\Omega} \leq& -2(\sum_{i=1}^{n} \xi_i^2) + 2\min\Big\{C_n e_n^{2p_{n-2}\cdots p_1} \\ &+ C_{n-1}(L_n) e_{n-1}^{2p_{n-3}\cdots p_1} + \cdots + C_2(L_n, \cdots, L_3) e_2^2,\ C_n\Big\}, \end{aligned} \tag{4.18}$$

where $C_n \geq 1$ is a constant independent of L_i's, while $C_{n-1}(L_n) \geq C_n, \cdots, C_2(L_n, \cdots, L_3) \geq C_n$ are fixed polynomial functions of their own arguments. They can be obtained in a manner similar to the one in [15].

For the error dynamics, consider the Lyapunov function

$$V_e = \frac{1}{2}(e_n^{2p_{n-2}\cdots p_1} + \cdots + e_2^2) \tag{4.19}$$

whose derivative along the trajectories of (4.12) on the set $B_M \times \mathbb{R}^{n-1}$ satisfies

$$\begin{aligned} \dot{V}_e\Big|_{B_M \times \mathbb{R}^{n-1}} &\leq K(\sum_{i=1}^{n} \xi_i^2) - [L_n - \bar{C}_n] e_n^{2p_{n-2}\cdots p_1} \\ &- [L_{n-1} - \bar{C}_{n-1}(L_n)] e_{n-1}^{2p_{n-3}\cdots p_1} - \cdots - [L_2 - \bar{C}_2(L_n, \cdots, L_3)] e_2^2, \end{aligned} \tag{4.20}$$

where K and $\bar{C}_n \geq 1$ are positive constants independent of L_i's, while $\bar{C}_{n-1}(L_n) \geq 1, \cdots, \bar{C}_2(L_n, \cdots, L_3) \geq 1$ are fixed polynomial functions of their own arguments, which can be computed using a similar argument as done in [15].

Now, it is clear that by choosing the gain constants

$$\begin{aligned} L_n &= L_n(L) \triangleq \bar{C}_n + LC_n \geq 1 \\ L_{n-1} &= L_{n-1}(L) \triangleq \bar{C}_{n-1}(L_n) + LC_{n-1}(L_n) \geq 1 \\ &\vdots \\ L_2 &= L_2(L) \triangleq \bar{C}_2(L_n, \cdots, L_3) + LC_2(L_n, \cdots, L_3) \geq 1 \end{aligned} \tag{4.21}$$

with $L > 0$ being a parameter to be determined later, one has

$$\dot{V}_e\Big|_{B_M \times \mathbb{R}^{n-1}} \leq K(\xi_1^2 + \cdots + \xi_n^2) - LW_e, \tag{4.22}$$

where $W_e \triangleq C_n e_n^{2p_{n-2}\cdots p_1} + C_{n-1}(L) e_{n-1}^{2p_{n-3}\cdots p_1} + \cdots + C_2(L) e_2^2$ and $C_{n-1}(L) \geq 1, \cdots, C_2(L) \geq 1$ are fixed positive polynomial functions of L.

For the closed-loop system (1.1)-(4.10)-(4.7)-(4.8)-(4.9), we choose the Lyapunov function

$$V(x, \hat{z}) = \frac{V_c(x)}{2} + \frac{r_0}{2}\frac{\ln(1 + V_e(e))}{\ln(1 + \mu(L))}, \tag{4.23}$$

where

$$\mu(L) \triangleq \frac{1}{2}\sum_{i=2}^{n} (r^{p_{i-1}} + L_i(L) r + r)^{2p_{i-2}\cdots p_1} \geq \max_{(x,\hat{z}) \in \Gamma_x \times \Gamma_{\hat{z}}} V_e > 0.$$

Associated with $V(x, \hat{z})$, define the level set

$$\Omega = \{(x, \hat{z}) \in \mathbb{R}^{2n-1} | V(x, \hat{z}) \leq r_0\}.$$

As shown in Section 3, similar properties of V and Ω still hold according to Fig. 2.

1. For every $L > 0$, $V(\cdot)$ is a positive definite and proper function and Ω is a compact set in $\mathbb{R}^n \times \mathbb{R}^{n-1}$. Once $L > 0$ is fixed, both V and Ω are also fixed;

2. $\forall L > 0, \Omega \supset \Gamma_x \times \Gamma_{\hat{z}}$;

3. $\forall L > 0, B_M \times \mathbb{R}^{n-1} \supset \Omega$.

By construction, $B_M \times \mathbb{R}^{n-1} \supset \Omega$. With this in mind, it follows from (4.22) and (4.18) that $\forall L > 0$,

$$\begin{aligned} \dot{V}\Big|_{\Omega} &= \frac{1}{2}\dot{V}_c\Big|_{\Omega} + \frac{r_0}{2\ln(1 + \mu(L))}\frac{\dot{V}_e}{1 + V_e}\Big|_{\Omega} \\ &\leq -\Big[1 - \frac{Kr_0}{2\ln(1 + \mu(L))(1 + V_e)}\Big](\xi_1^2 + \cdots + \xi_n^2) \\ &\quad - \Big[\frac{r_0 L}{3\ln(1 + \mu(L))} - 1\Big]\min\{W_e, C_n\}. \end{aligned} \tag{4.24}$$

The remaining part of the proof is to find a suitable constant L such that $\dot{V}\Big|_{\Omega} \leq 0$. Recall that $\mu(L)$ is a fixed polynomial function of L and the constant K is independent of L. Hence, there exists a constant $L^* > 0$ such that $\forall L \geq L^*$,

$$\begin{aligned} \frac{Kr_0}{2\ln(1+\mu(L))} &\leq \frac{1}{2} \\ \text{and} \quad \frac{r_0 L}{3\ln(1+\mu(L))} &\geq 2. \end{aligned}$$

Choosing $L = L^*$, we can estimate $\dot{V}\Big|_{\Omega}$ as follows:

$$\dot{V}\Big|_{\Omega} \leq -\frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2) - \min\{W_e, C_n\}.$$

In summary, system (3.1) is semi-globally stabilizable by the nonsmooth dynamic output compensator (4.10)-(4.7)-(4.8)-(4.9), although it is non-uniformly observable and nonsmoothly stabilizable. ∎

Example 4.4: Consider the nonsmoothly stabilizable system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_1 e^{x_1} \\ \dot{x}_2 &= x_3 + x_1 x_2^2 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{4.25}$$

which is not uniformly observable, because $x_2 = (\dot{y} - ye^y)^{1/3}$ and

$$x_3 = \frac{d}{dt}\left[(\dot{y} - ye^y)^{\frac{1}{3}}\right] - yx_2^2 = \frac{\ddot{y} - \dot{y}e^y - \dot{y}ye^y}{3(\dot{y} - ye^y)^{2/3}} - y(\dot{y} - ye^y)^{\frac{2}{3}}.$$

Thus, the method in [17], [18] is invalid. On the other hand, the work [15] gives only a local output feedback stabilization result due to the non-homogeneous terms $x_1 e^{x_1}$ and $x_1 x_2^2$. By Theorem 1.1, we now know that system (4.25) is semi-globally stabilizable via output feedback.

V. Conclusion

In this paper, we have proved that *without requiring uniform observability and smooth stabilizability* by state feedback, it is still possible to achieve semi-global stabilization via nonsmooth output feedback, for a significant class of nonlinear systems such as (1.1). This was made possible by developing a nonsmooth semi-global output feedback control scheme, which extended the output feedback design approach [17], [21] and integrated the recursive nonsmooth observer design algorithm [15] with the idea of saturating the estimated state [12], [17]. In the case when the nonlinear system is uniformly observable and smoothly stabilizable, the result of this paper has provided an alternative yet simpler semi-global output feedback design method.

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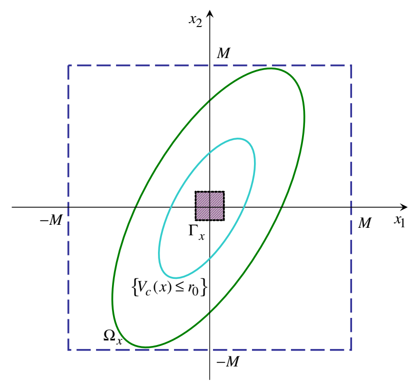

Fig. 1 The level set on x-space and the saturation threshold.

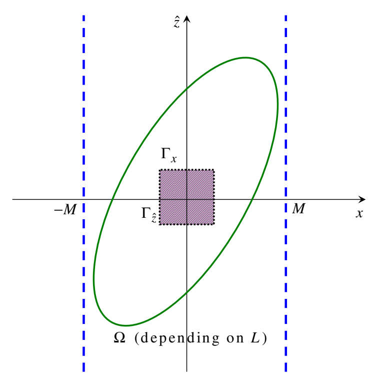

Fig. 2 The level set on $(x, \hat{z})$-space.