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# Robust Stabilization of Nonlinear Time Delay Systems Using Convex Optimization

Antonis Papachristodoulou

*Abstract*— **We address the problem of robust, global, delay-dependent and delay-independent stabilization of nonlinear time-delay systems with memory state feedback. The methodology we use is based on a linear-like representation of the time-delay system for which we construct appropriate Lyapunov-Krasovskii functionals. The resulting conditions take the form of infinite-dimensional state-dependent Linear Matrix Inequalities which can be treated as sum of squares matrices. The sum of squares program that emerges can then be solved using semidefinite programming and SOSTOOLS, which results in an algorithmic construction of the control law and the Lyapunov-Krasovksii functional. An example is presented that shows the effectiveness of the methodology.**

## I. INTRODUCTION

Functional Differential Equations (FDEs) [1] are the simplest adequate framework for modeling systems that involve transportation of data or which have an aftereffect. Examples come from population dynamics [2] and network congestion control for the Internet [3]. Time-delay systems are important for providing robust system descriptions; the presence of delays should be taken into account analysis and design as they can cause instabilities and loss of performance. Inevitably the stability and stabilization of time-delay systems has attracted the attention of many researchers in the area [4].

The analysis of systems described by FDEs is complicated by their infinite-dimensional nature which many times gives rise to conditions that are difficult to test algorithmically. In particular, in the case of analyzing delay-dependent stability of linear time-delay systems, the complete Lyapunov-Krasovskii (L-K) functional yields Linear Matrix Inequality (LMI) conditions that are infinite dimensional [5] which are difficult to test in general. Several ways have been proposed recently to address this problem, such as a discretization method by Gu [6] and other techniques [7]. In particular, a technique based on the sum of squares (SOS) decomposition that was proposed in [8] does not only allow the efficient solution of such LMIs in the linear case, but it can also be used to treat the more interesting case of stability analysis for nonlinear time-delay systems algorithmically.

This paper is about controller synthesis for uncertain systems described by FDEs. There are various classifications of state feedback control synthesis approaches for FDEs of retarded type, based on whether the feedback is instantaneous (memoryless) or contains delayed information (memory); whether the stabilization is for specific delay sizes (delay-dependent) or not (delay-independent); whether a cost function is being minimized (optimal control) or the pure stabilization problem is being considered; and whether the systems considered are linear or nonlinear, uncertain or not. Memory controllers are a more natural choice for feedback control, as time-delay systems are infinite dimensional. Such controllers can achieve better performances than memoryless controllers; in some cases memoryless controllers are incapable of stabilizing the system.

There is a series of papers concerned with the design of state feedback controllers for robust feedback stabilization of linear time delay systems, see for example [9], [10]. Similar results were obtained for output feedback compensators [11]. As far as optimal control is concerned, controllers for robust optimal control of linear time delay systems have been developed, such as $H_\infty$ [12], [13], [14] and with guaranteed cost [15], [16], [17]. Some of the above methods take the size of the delay into account during the controller synthesis (delay-dependent stabilization), and some don't (delay-independent stabilization). In [18] the authors consider the construction of L-K functionals using the discretization approach proposed in [6], by solving the resulting infinite dimensional LMIs.

As far as nonlinear time delay systems are concerned, solutions of the global asymptotic stabilization problem of feedforward systems and systems consisting of chains of integrators with a delay in the input have been produced [19], [20]. A synthesis procedure for nonlinear time-delay systems was developed based on the backstepping method for controller design [21], as well as other control L-K functional constructions in [22].

Here we address the problem of memory controller synthesis for delay-independent and delay-dependent stabilization of uncertain nonlinear time delay systems, by constructing L-K functionals algorithmically. Even in the case of nonlinear systems described by ODEs, this problem is difficult to solve algorithmically. In [23], using a linear-like representation of the system dynamics and a new methodology for solving state-dependent LMIs, a special class of Lyapunov functions was constructed algorithmically to address the state feedback control synthesis problem for nonlinear systems even with guaranteed cost or $H_\infty$ performance objectives. Here we take a similar approach. The time-delay system is represented in a linear-like fashion, and the resulting stabilization conditions are in the form of state-dependent LMIs [24]. These are then treated as *SOS matrices* and solved using the SOS decomposition [25] and SOSTOOLS [26]. This way we

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A. Papachristodoulou is with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA. Email: antonis@cds.caltech.edu.

construct memory controllers for robust delay-independent and delay-independent stabilization.

The paper is organized as follows. In Section II we revisit the basic tools that will be needed to formulate and solve the problem at hand. In Section III we present the problem we wish to solve, and derive the relevant state-dependent infinite dimensional LMIs. In Section IV we illustrate our technique with some examples. We conclude the paper in Section V.

The notation we use is standard and can be found in [1]. $\mathbb{R}^n$ denotes the $n$-dimensional real Euclidean space with norm $|\cdot|$. $C_n = C([a,b], \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $[a,b] = [-\tau, 0]$ we designate the norm of an element $\phi \in C([-\tau,0],\mathbb{R}^n)$ by $\|\phi\| = \sup_{-\tau \le \theta \le 0} |\phi(\theta)|$. For $\sigma \in \mathbb{R}$ and $A \ge 0$ and $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$ then for any $t \in [\sigma, \sigma + A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$.

## II. Preliminaries

In this section we revisit some tools that will be needed to formulate and solve the problem of interest. These are state-dependent Riccati equation techniques and an algorithmic approach for solving the infinite dimensional LMIs, using the SOS decomposition and SOSTOOLS [26].

### A. State-Dependent Riccati Equation Techniques

Consider the nonlinear differential equation

$$\dot{x} = f(x) + g(x)u$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are polynomial functions with $f(0) = 0$. We seek a

$$u = k(x)$$

where $k(x)$ is a polynomial function of $x$ so that the zero equilibrium of the feedback system

$$\dot{x} = f(x) + g(x)k(x)$$

is globally asymptotically stable. The problem of designing $k(x)$ and finding an appropriate Lyapunov function $V(x)$ is not jointly convex, so the problem is algorithmically hard.

To tackle the synthesis problem, the following methodology has been proposed. It is based on representing the nonlinear dynamics in the following linear-like form with state-dependent coefficients:

$$\dot{x} = A(x)Z(x) + B(x)u \tag{1}$$

where $A(x)$ and $B(x)$ are polynomial matrices in $x$ and $Z(x)$ is an $N \times 1$ vector of monomials in $x$ satisfying $Z(x) = 0$ if and only if $x = 0$. In addition, we define $M(x)$ an $N \times n$ polynomial matrix such that $M_{ij} = \frac{\partial Z_i}{\partial x_j}$. It was shown in [27] there are many ways of 'recasting' the nonlinear system to the state-dependent coefficient form. We let $u = K(x)Z(x)$ and we have [23]:

*Theorem 1:* For system (1), suppose there exist an $N \times N$ symmetric polynomial matrix $P$, an $m \times N$ polynomial matrix $K(x)$, a constant $\epsilon_1 > 0$ and a sum of squares $\epsilon_2(x)$ such that the following two parameter dependent Linear Matrix Inequalities are satisfied:

$$P - \epsilon_1 I \ge 0 \tag{2}$$

$$\begin{aligned} &- PA^T(x)M^T(x) - M(x)A(x)P - K^T(x)B^T(x)M^T(x) \\ &\quad - M(x)B(x)K(x) - \epsilon_2(x)I \ge 0 \end{aligned} \tag{3}$$

Then the state feedback stabilization problem is solvable, and a controller that globally stabilizes the system is given by:

$$u(x) = K(x)P^{-1}Z(x). \tag{4}$$

Furthermore, if (3) holds with $\epsilon_2(x) > 0$ for $x \neq 0$, then the zero equilibrium is globally asymptotically stable.
The proof of the above theorem can be found in [23], and is based on the fact that

$$V(x) = Z(x)^T P^{-1} Z(x) \tag{5}$$

is a control Lyapunov function for the above system. The problem that remains is how to test the above conditions algorithmically and construct the resulting nonlinear control law.

### B. Solving state Dependent LMIs

Here we present a methodology for solving state-dependent LMIs or in general, infinite dimensional LMIs that appear in time delay systems when investigating delay-dependent stability. What we mean by state-dependent LMIs is an infinite dimensional convex optimization problem of the form:

$$\text{minimize} \quad \sum_{i=1}^{m} a_i c_i \tag{6}$$

$$\text{subject to} \quad F_0(x) + \sum_{i=1}^{m} c_i F_i(x) \ge 0 \tag{7}$$

where the $a_i$'s are the cost coefficients on the decision variables $c_i$, and the $F_i(x)$ are some symmetric matrix functions of $x \in \mathbb{R}^n$. We therefore seek $c_i$ that minimize the cost function (6) and for which the LMI (7) is satisfied for all $x \in \mathbb{R}^n$. If we restrict our attention to the case in which the $F_i(x)$ are symmetric polynomial matrices in $x$, the sum of squares (SOS) decomposition [25] can provide an appropriate computational relaxation. This is stated in the following Proposition:

*Proposition 2:* Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$, where by degree we mean the maximum degree of all the polynomial entries. Also, let $Z(x)$ be a column vector whose entries are monomials in $x$ with degree no greater than $d$. Then $v^T F(x) v$ is a SOS, where $v \in \mathbb{R}^N$ if and only if there exists a positive semidefinite matrix $Q$ such that

$$v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x)) \tag{8}$$

where $\otimes$ denotes the Kronecker product. Furthermore, if $v^T F(x) v$ is a SOS, then $F(x) \ge 0$ for all $x \in \mathbb{R}^n$.
The proof can be found in [23]. A matrix for which $v^T F(x) v$ is a SOS, is termed a *SOS matrix*. Therefore instead of

solving the optimization problem (6–7), we can solve the problem:

$$\text{minimize } \sum_{i=1}^{m} a_i c_i \tag{9}$$

$$\text{subject to } \quad v^T(F_0(x) + \sum_{i=1}^{m} c_i F_i(x))v \text{ is SOS} \tag{10}$$

(9–10) can be solved using semidefinite programming and SOSTOOLS [28]. In the next section we will be using the SOS decomposition to test the SOS matrix property.

## III. Stabilization of Time-Delay Systems

We are now ready to state the problem we wish to solve, and provide a solution. Consider a nonlinear time-delay system of the form:

$$\begin{aligned} \dot{x}(t) &= f(x_t, p) + g(x_t, p)u \\ x(\theta) &= \phi(\theta), \quad \theta \in [-\tau, 0] \end{aligned}$$

where for convenience $f(0,p) = 0$. Here $x_t \in C_n = C([-\tau, 0], \mathbb{R}^n)$ is the state, $u \in C_m$ is the input and $p \in \Delta$ is a parameter set, which we assume takes the form:

$$\Delta = \{p \in \mathbb{R}^k | g_i(p) \leq 0, i = 1, \ldots, N\}$$

We assume that the $g_i(p)$ are polynomial functions of $p$. The initial condition is $\phi(\theta) \in C_n$ and $\tau > 0$ is the time delay of this system. For the purposes of this paper, we will assume that both $f(x_t, p)$ and $g(x_t, p)$ are polynomial functions in $x(t), x(t-\tau)$ and $p$ — although this is not restrictive as the case in which the system state is non-polynomial can be treated in a unified manner [29]. The aim of this work is to construct (design) $u$ of the form:

$$u = k(x(t), x(t-\tau)) \tag{11}$$

i.e. a polynomial function of $x(t)$ and $x(t-\tau)$, so that the zero equilibrium of the resulting closed loop system enjoys certain stability properties. In particular, the controller $u$ has to be such, so that the zero equilibrium of the closed loop system

$$\begin{aligned} \dot{x}(t) = &f(x(t), x(t-\tau), p) \\ &+ g(x(t), x(t-\tau))k(x(t), x(t-\tau)) \end{aligned}$$

is robustly (asymptotically) stable in a delay-independent or delay-dependent fashion.

For notational simplicity, we denote $z_1 = x(t)$ and $z_2 = x(t-\tau)$ and let $Z(y)$ be a vector of monomials in $y = (y_1, \ldots, y_n)$ such that $Z = 0$ if and only if $y$ is zero. In order to proceed we write the above system in state-dependent linear like representation:

$$\begin{aligned} \dot{x}(t) = &A_0(z_1, z_2, p)Z(z_1) + A_1(z_1, z_2, p)Z(z_2) \\ &+ B(z_1, z_2, p)u \end{aligned} \tag{12}$$

where $A_0$, $A_1$ and $B$ are polynomial matrices in $(z_1, z_2, p)$. The control law is assumed to have the form

$$u = k(z_1, z_2) = K_0(z_1, z_2)Z(z_1) + K_1(z_1, z_2)Z(z_2).$$

Similarly, we define $M_{ij} = \frac{\partial Z_i}{\partial x_j(t)}(x(t))$. The closed loop system is therefore

$$\dot{x}(t) = (A_0 + BK_0)Z(z_1) + (A_1 + BK_1)Z(z_2)$$

where we have suppressed the fact that $A_i$ are polynomial matrices in $(z_1, z_2, p)$ etc.

In the next two subsections we will construct controllers for delay-independent stabilization.

### *A. Delay-independent stabilization*

Delay-independent stabilization aims in constructing a controller and a Lyapunov functional so that the resulting system is delay-independent stable. Recall that for a system of the form:

$$\dot{x} = f(x_t)$$

with $f(0) = 0$ the following L-K functional was used in [8] to prove delay-independent stability:

$$V(x_t) = a_0(x(t)) + \int_{-\tau}^{0} a_1(x(t+\theta))d\theta$$

Here by $a_i$ we mean polynomials of degree at least 2 in their arguments, that need to satisfy certain SOS conditions.

For the state-feedback stabilization of the system described by (12) in a robust, global, delay-independent fashion, we have the following Proposition:

*Proposition 3:* For system (12), suppose there exist $N \times N$ dimensional symmetric matrices $P$ and $Q$, $m \times N$ dimensional polynomial matrices $S_0$ and $S_1$, a constant $\epsilon_1 > 0$ and a SOS $\epsilon_2(z_1)$ such that the following are satisfied:

$$P - \epsilon_1 I \geq 0, \tag{13}$$

$$Q \geq 0, \text{ and} \tag{14}$$

$$-\begin{bmatrix} \begin{pmatrix} M(A_0P + BS_0) + \epsilon_2(z_1) \\ +(S_0^T B^T + PA_0^T)M^T + Q \end{pmatrix} & M(A_1P + BS_1) \\ (S_1^T B^T + PA_1^T)M^T & -Q \end{bmatrix} \tag{15}$$

is a SOS matrix for $p \in \Delta$. Then the state feedback stabilization problem is solvable, and the controller is given by:

$$u(x) = S_0(x)P^{-1}Z(x(t)) + S_1(x)P^{-1}Z(x(t-\tau)). \tag{16}$$

This control law stabilizes the zero equilibrium of (12) globally in a robust delay-independent way. Moreover, if $\epsilon_2(z_1) > 0$ for $z_1 \neq 0$, then the equilibrium is robustly globally asymptotically stable independent of the size of the delay.

*Proof:* Consider an L-K functional of the form:

$$\begin{aligned} V(x_t) = &Z^T(z_1)P^{-1}Z(z_1) \\ &+ \int_{-\tau}^{0} Z^T(x(t+\theta))P^{-1}QP^{-1}Z(x(t+\theta))d\theta \end{aligned}$$

$V$ is positive definite since the conditions $P > 0$ and $Q \geq 0$ are imposed by (13–14). The time derivative of $V$ along the system trajectories is:

$$\begin{aligned}\dot V &= Z^T(z_1)(P^{-1}MA_0 + A_0^T M^T P^{-1} + P^{-1}QP^{-1})Z(z_1)\\ &+ Z^T(z_1)(P^{-1}MBK_0 + K_0^T B^T M^T P^{-1})Z(z_1)\\ &+ Z^T(z_2)(A_1^T + K_1^T B^T)M^T P^{-1} Z(z_1)\\ &+ Z^T(z_1)P^{-1}M(A_1+BK_1)Z(z_2)\\ &- Z^T(z_2)P^{-1}QP^{-1}Z(z_2).\end{aligned}$$

Now condition (15) implies that

$$\begin{bmatrix} \begin{pmatrix} PA_0^T M^T + MA_0 P \\ +S_0^T B^T M^T + MBS_0 + Q \end{pmatrix} & MA_1P + MBS_1 \\ S_1^T B^T M^T + PA_1^T M^T & -Q \end{bmatrix}$$

is negative semidefinite for all $z_1, z_2$ and $p \in \Delta$ (recall that the matrices $S_i$ are dependent on $z_1, z_2$ and $A_i$ are dependent on $z_1, z_2$ and $p$ even though this is not written explicitly). Pre and post-multiplying this last expression by a block-diagonal matrix diag$\{P^{-1}, P^{-1}\}$ and renaming $S_0 = K_0 P$ and $S_1 = K_1 P$, we conclude that $\dot V$ is non-positive for all $p \in \Delta$. This proves robust global stability of the zero equilibrium of the closed loop system independent of the size of the delay. If $\epsilon_2(z_1) > 0$ for $z_1 \neq 0$, then $\dot V < 0$, and therefore the zero equilibrium is robustly, globally, delay-independent asymptotically stable. ∎

The conditions in the above Proposition can be tested algorithmically using the SOS decomposition, as explained in Section II. Recall that the set $\Delta$ is captured by certain inequalities, $g_i(p) \le 0$. These inequalities can be adjoined to condition (15) using SOS multipliers, in a way reminiscent to the S-procedure, as it was done in [8]. The resulting SOS programme can be solved algorithmically with the aid of SOSTOOLS.

We now turn to the more interesting delay-dependent robust stabilization case.

### B. Delay-dependent stabilization

If the size of the delay for which stabilization is required is known a-priori, a more attractive stabilization condition would be a delay-dependent one. A better performance may be achieved by resorting to this type of stabilization rather than a delay-independent one.

Recall that for a system of the form:

$$\dot x = f(x_t)$$

with $f(0) = 0$ the following L-K functional was considered in [8] for delay-dependent stability, which resembles closely the complete L-K functional in the linear case,

$$\begin{aligned}V(x_t) &= a_0(x(t))\\ &+ \int_{-\tau}^0 \int_{-\tau}^0 a_1(\theta, \xi, x(t+\theta), x(t+\xi), x(t)) d\theta d\xi\\ &+ \int_{-\tau}^0 \int_{t+\theta}^t a_2(x(\zeta)) d\zeta d\theta + \int_{-\tau}^0 \int_{t+\xi}^t a_3(x(\zeta)) d\zeta d\xi\end{aligned}$$

where again the $a_i$'s are polynomials of degree at least 2 in the state arguments. In particular $a_1$ can be taken to be bipartite in $(\theta, \xi)$ and $(x(t+\theta), x(t+\xi), x(t))$.

Here we use a more structured L-K functional, similar to the one shown above, for delay-dependent stabilization. In particular we have the following Proposition:

*Proposition 4:* For the system (12), suppose there exist $N \times N$ dimensional symmetric matrices $P$ and $Q$, $m \times N$ dimensional polynomial matrices $S_0$ and $S_1$, a matrix polynomial $R(\xi, \theta)$ of size $N \times N$, a constant $\epsilon_1 > 0$, positive semi-definite matrices $T_1$ and $T_2$, and a SOS polynomial $\epsilon_2(z_1)$ such that (18–19) hold. Then the state feedback stabilization problem is solvable, and the controller is given by:

$$u(x) = S_0(x)P^{-1}Z(x(t)) + S_1(x)P^{-1}Z(x(t-\tau)). \quad (17)$$

This stabilizes the zero equilibrium of system (12) globally in a robust way, for a delay size equal to $\tau$. Moreover, if $\epsilon_2(z_1) > 0$ for $z_1 \neq 0$, then the equilibrium is robustly globally asymptotically stable for a delay size equal to $\tau$.

*Proof:* Consider the following functional:

$$\begin{aligned}V(x_t) = \int_{-\tau}^0 \int_{-\tau}^0 &\begin{bmatrix} \frac{Z(x(t))}{\tau} \\ Z(x(t+\theta)) \\ Z(x(t+\xi)) \end{bmatrix}^T \\ &\begin{bmatrix} P^{-1} & \frac12 P^{-1} & \frac12 P^{-1} \\ \frac12 P^{-1} & \frac12 P^{-1}QP^{-1} & \frac12 P^{-1}R(\xi,\theta)P^{-1} \\ \frac12 P^{-1} & \frac12 P^{-1}R^T(\xi,\theta)P^{-1} & \frac12 P^{-1}QP^{-1} \end{bmatrix} \\ &\begin{bmatrix} \frac{Z(x(t))}{\tau} \\ Z(x(t+\theta)) \\ Z(x(t+\xi)) \end{bmatrix} d\theta d\xi \\ &+ \int_{-\tau}^0 \int_{t+\theta}^t Z^T(x(\zeta))P^{-1}T_1P^{-1}Z(x(\zeta)) d\zeta d\theta \\ &+ \int_{-\tau}^0 \int_{t+\xi}^t Z^T(x(\zeta))P^{-1}T_2P^{-1}Z(x(\zeta)) d\zeta d\xi.\end{aligned}$$

Condition (18) imposes positive definiteness of $V$ since we can pre- and post-multiply it by a block diagonal matrix diag$\{P^{-1}, P^{-1}, P^{-1}\}$, as $P > 0$, and $T_1 \ge 0$, $T_2 \ge 0$. The derivative of $V$ along the trajectories of the system (12), with the control law (17) imposed, denoting $z_1 = x(t)$, $z_2 = x(t-\tau)$, $z_3 = x(t+\theta)$ and $z_4 = x(t+\xi)$, is given by (20). Condition (19) guarantees that $\dot V$ is non-positive for all $\theta \in [-\tau, 0]$, $\xi \in [-\tau, 0]$ and $p \in \Delta$. To see this, pre- and post-multiply (19) by diag$\{P^{-1}, P^{-1}, P^{-1}, P^{-1}\}$ and rename of variables $S_0 = K_0 P$ and $S_1 = K_1 P$. Note that the resulting SOS matrices depend on the delay, and so the resulting controller

$$u(x) = S_0(x)P^{-1}Z(x(t)) + S_1(x)P^{-1}Z(x(t-\tau)) \quad (21)$$

robustly, globally stabilizes the zero equilibrium of the system for a delay of size $\tau$. Moreover, if $\epsilon_2(z_1) > 0$ for $z_1 \neq 0$, then $\dot V < 0$ and the equilibrium is robustly globally asymptotically stable for a delay size equal to $\tau$. ∎

*Remark 5:* Different Lyapunov functionals can be used for the construction of appropriate delay-dependent stabilization conditions, such as the one used in [13] or the methodology proposed in [18].

$$\begin{bmatrix} P & \frac{1}{2}P & \frac{1}{2}P \\ \frac{1}{2}P & \frac{1}{2}Q & \frac{1}{2}R(\xi,\theta) \\ \frac{1}{2}P & \frac{1}{2}R^T(\xi,\theta) & \frac{1}{2}Q \end{bmatrix} - \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is SOS matrix } \forall\theta \in [-\tau, 0], \xi \in [-\tau, 0], \tag{18}$$

$$-\left[\begin{array}{c|c|c|c} \begin{matrix}(PA_0^T + S_0^T B^T)M^T + Q + 2P \\ +M(A_0 P + BS_0) + \tau(T_1 + T_2)\end{matrix} & M(A_1 P + BS_1) - P & \frac{1}{2}(PA_0^T + S_0^T B^T)M^T + \frac{1}{2}R(0,\theta) & \frac{1}{2}(PA_0^T + S_0^T B^T)M^T + \frac{1}{2}R^T(\xi,0) \\ \hline (PA_1^T + S_1^T B^T)M^T - P & -Q & \frac{1}{2}(PA_1^T + S_1^T B^T)M^T - \frac{1}{2}R(-\tau,\theta) & \frac{1}{2}(PA_1^T + S_1^T B^T)M^T - \frac{1}{2}R^T(\xi,-\tau) \\ \hline \frac{1}{2}M(A_0 P + BS_0) + \frac{1}{2}R^T(0,\theta) & \frac{1}{2}M(A_1 P + BS_1) - \frac{1}{2}R^T(-\tau,\theta) & -\frac{1}{\tau}T_1 & -\frac{1}{2}\frac{\partial R(\xi,\theta)}{\partial\theta} - \frac{1}{2}\frac{\partial R(\xi,\theta)}{\partial\xi} \\ \hline \frac{1}{2}M(A_0 P + BS_0) + \frac{1}{2}R(\xi,0) & \frac{1}{2}M(A_1 P + BS_1) - \frac{1}{2}R(\xi,-\tau) & -\frac{1}{2}\frac{\partial R^T(\xi,\theta)}{\partial\theta} - \frac{1}{2}\frac{\partial R^T(\xi,\theta)}{\partial\xi} & -\frac{1}{\tau}T_2 \end{array}\right]$$

$$- \begin{bmatrix} \epsilon_2(z_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{is SOS matrix } \forall\theta \in [-\tau, 0], \xi \in [-\tau, 0] \text{ and } p \in \Delta. \tag{19}$$

$$\begin{aligned} \dot{V}(x_t) &= Z^T(z_1)((A_0^T + K_0^T B^T)M^T P^{-1} + P^{-1}M(A_0 + BK_0))Z(z_1) + Z^T(z_2)(A_1^T + K_1^T B^T)M^T P^{-1}Z(z_1) \\ &+ Z^T(z_1)P^{-1}M(A_1 + BK_1)Z(z_2) + 2Z^T(z_1)P^{-1}Z(z_1) - Z^T(z_1)P^{-1}Z(z_2) - Z^T(z_2)P^{-1}Z(z_1) \\ &+ \int_{-\tau}^0 (Z^T(z_1)(A_0^T + K_0^T B^T) + Z^T(z_2)(A_1^T + K_1^T B^T))M^T P^{-1}Z(z_3)d\theta + Z^T(z_1)P^{-1}QP^{-1}Z(z_1) \\ &+ \int_{-\tau}^0 Z^T(z_4)P^{-1}M((A_0 + BK_0)Z(z_1) + (A_1 + BK_1)Z(z_2))d\xi - Z^T(z_2)P^{-1}QP^{-1}Z(z_2) \\ &+ \int_{-\tau}^0 (Z^T(z_1)P^{-1}R(0,\theta) - Z^T(z_2)P^{-1}R(-\tau,\theta))P^{-1}Z(z_3)d\theta \\ &+ \int_{-\tau}^0 Z^T(z_4)P^{-1}(R(\xi,0)P^{-1}Z(z_1) - R(\xi,-\tau)P^{-1}Z(z_2))d\xi \\ &+ \int_{-\tau}^0\int_{-\tau}^0 Z^T(z_4)P^{-1}\left(\frac{\partial R(\xi,\theta)}{\partial\xi} + \frac{\partial R(\xi,\theta)}{\partial\theta}\right)P^{-1}Z(z_3)d\theta d\xi + \tau Z^T(z_1)P^{-1}T_1 P^{-1}Z(z_1) \\ &+ \tau Z^T(z_1)P^{-1}T_2 P^{-1}Z(z_1) - \int_{-\tau}^0 Z^T(z_3)P^{-1}T_1 P^{-1}Z(z_3)d\theta - \int_{-\tau}^0 Z^T(z_4)P^{-1}T_2 P^{-1}Z(z_4)d\xi \end{aligned} \tag{20}$$

The conditions $\theta \in [-\tau, 0]$ and $\xi \in [-\tau, 0]$ can be adjoined to the SOS matrix condition (19) using SOS matrix multipliers, in a way similar to the S-procedure [8]. The resulting SOS conditions can be tested algorithmically using SOSTOOLS; so the resulting infinite dimensional LMI conditions can be solved directly, without resorting to discretization [18] or other approaches [7]. To get robustness with respect to the delay size, one can impose an extra condition that the above LMIs are valid for all delays $\tau \in [0, \overline{\tau}]$.

## IV. Example

Consider the system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)x_2(t) + 2x_1(t-\tau)x_2(t) - x_1(t-\tau) + u \\ \dot{x}_2(t) &= -px_1(t) + x_2(t-\tau) \end{aligned}$$

where $p \in [0.5, 1.5]$. We represent the system as follows:

$$\begin{aligned} \dot{x} = &\begin{bmatrix} x_2(t) & x_1(t-\tau) \\ -p & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} -1 + x_2(t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \end{aligned}$$

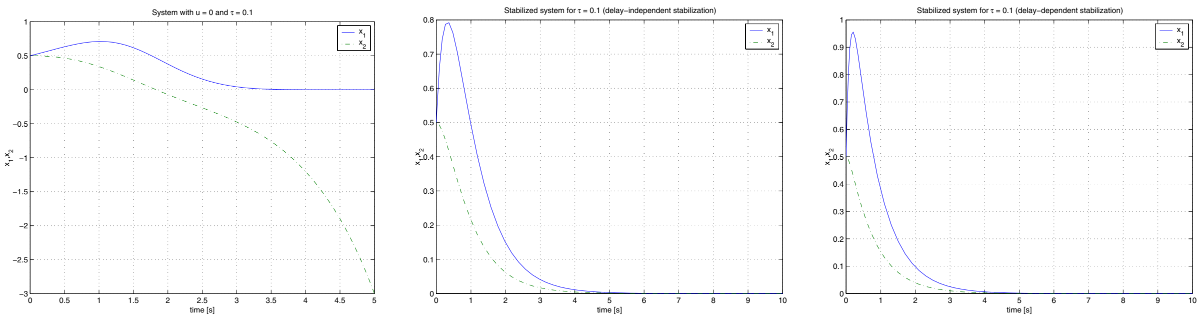
When $u = 0$, the above system has two equilibria, one which is at the origin and the other one at $\left(\frac{1}{3p}, \frac{1}{3}\right)$. A sample simulation for the system with nominal value $p = 1$, initialized with the constant initial function $(x_1, x_2) = (0.5, 0.5)$ with delay $\tau = 0.1$ is shown in Figure 1A. We aim to construct controllers for robust global delay-independent and delay-dependent stabilization of the zero equilibrium, based on the conditions for stabilizability that we proposed in this paper. Note that the equilibrium of the above system is unstable in the undelayed, unforced, linearized case.

### A. Delay-Independent stabilization

In this case we setup a SOS program, which needs to satisfy all the conditions in Proposition 3. For the above system an appropriate control law was constructed which guarantees that the zero equilibrium of the above system is robustly globally delay-independent stable. The control law that was constructed contains terms up to 2nd order in $x_1(t), x_2(t), x_1(t-\tau)$ and $x_2(t-\tau)$, and is omitted due to space restrictions. A simulation of the closed loop system for $p = 1$ can be seen in Figure 1B.

### B. Delay-dependent stabilization

Similarly, Proposition 4 was used to construct a feedback control law that ensures robust global delay-dependent stability of the equilibrium of the loop system for $\tau = 0.1$.




A B C

Fig. 1. Simulations from initial conditions $(x_1, x_2) = (0.5, 0.5)$. Figure A: open loop system. Figure B: closed loop system as it was stabilized in a delay-independent fashion. Figure C: closed loop system as it was stabilized in a delay-dependent fashion.

Simulations of the closed loop system can be seen in Figure 1C for $p = 1$. The controller structure is also a polynomial up to 2nd order in $x_1(t), x_2(t), x_1(t-\tau)$ and $x_2(t-\tau)$.

## V. Conclusions

In this paper we presented a new technique for controller synthesis for time-delay systems to achieve robust global delay-dependent and delay-independent stability. The infinite dimensional, state-dependent LMIs were solved using the SOS technique. The above methodology can be extended to guaranteed cost control and $H_\infty$ control synthesis procedures, which will be the subject of future research. These constructions were already implemented in [23]. Local (but nonlinear) stabilization is also possible, by imposing that the Lyapunov conditions are only satisfied in a neighborhood of the equilibrium of interest.

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