Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Complex Interval Constraint Propagation for Non Linear Bounded-Error Parameter Identification

N. Ramdani, T. Raïssi, Y. Candau and L. Ibos

Abstract— **This paper is dedicated to bounded error identification with complex valued non-linear models. Complex intervals are characterized by using polar forms and a new inclusion function is given for the addition of sectors. The latter is expressed as an optimization problem solved analytically. The new complex interval arithmetic is used for solving constraint satisfaction problems with complex variables and is used within algorithms for bracketing the interval hull of posterior feasible parameter sets. A case study from dielectric relaxation spectra analysis, involving a fairly large number of parameters is investigated from simulation data.**

I. Introduction

PARAMETER estimation problems are usually solved by probabilistic methods when an explicit characterization of the errors is assumed available. In practice, this is not always possible for many reasons (for instance, there is a modeling error that cannot be characterized by random variables) and it is more natural to assume that the perturbations belong to a known set. In such a case, bounded-error techniques allow the characterization of the set, known as the *posterior feasible set*, of all parameter vectors that are compatible with the measured data, a model structure and the prior error bounds. Many researchers have established several techniques for characterizing the posterior feasible set (see *e.g.* [1]; and the references therein). For linear models for instance, simple-shaped forms such as ellipsoids, parallelotopes, zonotopes or boxes are used to give an enclosure of this set [2]-[3] whereas for non-linear models, techniques based on interval analysis and constraint satisfaction problems are used ([4]-[5]; and references therein) for computing inner and outer enclosures of this set.

In addition, for many real-life engineering problems, it is more convenient to base the experimental modeling on frequency response data : the system is then described by a complex-valued model. In a bounded error context, all the uncertainties are thus described by complex sets. As a result, the derivation of an optimal inclusion function for a complex-valued non-linear model is a major issue for ensuring success for the identification procedure.

In this paper, we will investigate bounded-error model identification for complex-valued non-linear models.

The paper is structured as follows: Section 2 is dedicated to complex interval analysis and contains the first contribution of this paper: polar forms are used for characterizing complex intervals and the smallest polar complex interval, called *sector*, containing the sum of two sectors is given. Section 3 is dedicated to set membership identification and constraint satisfaction problems. Section 4 contains the application of polar complex intervals to bounded error identification of the dielectric properties of a sample material from simulated data, with a model involving nine parameters. The second contribution of this paper is the use of an algorithm for bracketing the interval hull of the posterior feasible set in order to reduce computation time when the model contains a fairly large number of parameters.

II. Complex Intervals Analyis

A. Real intervals

Interval analysis was initially developed to take into account the quantification errors introduced by the rational representation of real numbers with computers [6] and was later extended to validated numerics. An interval $[a]=[a^-,a^+]$ is a connected and closed subset of $\mathbb{R}$. The set of all intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Real arithmetic operations are extended to intervals. Let $\mathbf{f}:\mathbb{R}^n\rightarrow\mathbb{R}^m$; an inclusion function of **f**, denoted by $[\mathbf{f}]$, is defined by:

$$\forall[\mathbf{a}]\in\mathbb{IR}^n,\ \mathbf{f}([\mathbf{a}])\subseteq[\mathbf{f}]([\mathbf{a}]) \quad (1)$$

An inclusion function of **f** can be obtained by replacing each occurrence of a point variable by its corresponding interval variable and by replacing each standard function by an interval evaluation. Such a function is called the natural form. In practice the inclusion function is not unique, it depends on the formal expression of **f**.

N. Ramdani was with the CERTES EA3481, Université Paris XII - Val de Marne. He is now with the LIRMM, UMR CNRS 5506, Université Montpellier 2, 161 rue Ada, 34392 Montpellier Cédex 5, France. (corresponding author phone: +33467418678; fax: +33467418500; e-mail: Nacim.Ramdani@lirmm.fr)

T. Raïssi was with the CERTES EA3481, Université Paris XII-Val de Marne. He is now with the LAPS, UMR CNRS 5131, Université Bordeaux I - ENSEIRB, 351 cours de la Libération, 33405 Talence Cédex, France.

Y. Candau and L. Ibos are with the CERTES EA3481, Université Paris XII-Val de Marne, IUT de Créteil, avenue G. de Gaulle, 94000 Créteil, France.

0-7803-9568-9/05/$20.00 ©2005 IEEE

B. Complex intervals

The simplest complex interval approximation is the rectangular representation where a complicated shaped set is approximated by a rectangle; but the circular form, where a set is approximated by a disc, is more often used. Unfortunately, both of complex interval representations cited above are not closed with respect to the arithmetic operations $\{+, -, *, /\}$. This is due to the fact that a multiplication of a set by a complex number is a rotation, thus, for the rectangular representation, the result of such an operation must be wrapped in a rectangle, which introduces large pessimism. The arithmetic operation $\{*\}$ is then non minimal and a pessimism is introduced when a multiplication of two complex intervals, represented as rectangles or discs, is performed [7]-[11].

In the sequel, we introduce an extension of the polar representation of complex numbers to the case of intervals. Indeed, the polar representation can be preferred for non-linear complex valued models. We prove that both the multiplication and the division are exact operations, i.e. the result of the multiplication of two polar complex intervals is a polar interval. Nevertheless, this property is not satisfied for addition and subtraction. Consequently, a new algorithm which allows to compute the minimal polar complex interval containing the sum of two polar intervals, is derived in [12]. In the sequel, we give main results.

C. Definition of a sector

Consider the intervals $[\rho]=[\rho^-,\rho^+]\subseteq\mathbb{R}^+$ and $[\theta]=[\theta^-,\theta^+]\subseteq\mathbb{R}$; the set defined by

$$Z=\left\{z\in\mathbb{C}\;\middle|\;z=\rho e^{i\theta},\,\rho\in[\rho],\,\theta\in[\theta]\right\}\tag{2}$$

is called a polar complex interval (or a *sector*) denoted by $\{[\rho];[\theta]\}$. A polar interval can be uniquely characterized by two real intervals: its magnitude $[\rho]=[\rho^-,\rho^+]$, and its angle $[\theta]=[\theta^-,\theta^+]$; as illustrated in fig.1. To ensure uniqueness of the representation, we can always choose the bounds of the latter interval such that

$$0\le\theta^+-\theta^-\le 2\pi,\quad 0\le\theta^-<2\pi,\quad 0\le\theta^+<4\pi\tag{3}$$

The set of all polar complex intervals is denoted by $\mathbb{S}(\mathbb{C})$.

D. Arithmetic operations with sectors

Let $Z_1=\{[\rho_1];[\theta_1]\}$ and $Z_2=\{[\rho_2];[\theta_2]\}$ be two sectors, the multiplication operation between Z_1 and Z_2 is defined as follows:

$$Z_1\cdot Z_2\triangleq\left\{z_1\cdot z_2\,\middle|\,z_1\in Z_1,z_2\in Z_2\right\}=\left\{[\rho_1]\cdot[\rho_2];[\theta_1]+[\theta_2]\right\}\tag{4}$$

Since the set of the real intervals is closed with respect to addition and multiplication, the product of two sectors is also a sector; this operation is then minimal. Similar results are derived for the division operation between Z_1 and Z_2 and the power of a complex interval by a real interval. We should note that the argument bounds of the result (4) may not verify (3). In such a case, we add $2k\pi,\,k\in\mathbb{Z}$, to the argument of the computed sector until (3) is met.

By contrast, the set

$$Z_1\oplus Z_2=\left\{z_1+z_2\;\mid\;z_1\in Z_1,\,z_2\in Z_2\right\}\tag{5}$$

known as the *Minkowski sum* [13], is not a sector but has a complex shape; to define addition as an operation in $\mathbb{S}(\mathbb{C})$, one has to determine some element of $\mathbb{S}(\mathbb{C})$ which contains this set. Some degree of pessimism will thus be introduced. To minimize pessimism, we define Z_1+Z_2 as the smallest sector, in the sense of inclusion, containing $Z_1\oplus Z_2$:

$$Z_1+Z_2=\cap Z,\quad Z\in\mathbb{S}(\mathbb{C}),\quad Z_1\oplus Z_2=\cap Z\tag{6}$$

Z_1+Z_2 defined in this way exists as an element of $\mathbb{S}(\mathbb{C})$, because the intersection of any number of closed boxes is a closed box in $\mathbb{R}^2$. Subtraction is defined in the same way.

E. Characterization of the addition of two sectors

Let $Z_1=\{[\rho_1];[\theta_1]\}$ and $Z_2=\{[\rho_2];[\theta_2]\}$ be two sectors and Z their sum; then Z can be uniquely written as $Z=\{[\rho];[\phi]\}$. Then the bounds ρ^- and ρ^+ of $[\rho]$ must verify

$$\left\{\rho^-=\min_{z\in Z_1\oplus Z_2}|z|,\quad\rho^+=\max_{z\in Z_1\oplus Z_2}|z|\right\}\tag{7}$$

$$\text{with }\left|\rho_1e^{i\theta_1}+\rho_2e^{i\theta_2}\right|=\sqrt{\rho_1^2+\rho_2^2+2\rho_1\rho_2\cos(\theta_1-\theta_2)}\tag{8}$$

and the function square root is monotonously increasing. Solving the first of problems (7) is equivalent to solving

$$\min_{\Omega}f(\rho_1,\rho_2,\theta)\tag{9}$$

with the following definitions:

$$f:(\rho_1,\rho_2,\theta)\mapsto\rho_1^2+\rho_2^2+2\rho_1\rho_2\cos(\theta)\tag{10}$$
$$\Omega=[\rho_1]\times[\rho_2]\times[\theta]\subset\mathbb{R}^+\times\mathbb{R}^+\times[0,2\pi]$$

where $\theta=\theta_1-\theta_2$. Thus the problem of finding ρ^- is identical to minimizing a function on a box of $\mathbb{R}^3$. The same applies to ρ^+ by replacing min by max. In the same way, the bounds of

[ϕ] are solution of

$$\left\{\phi^- = \min_{z\in Z_1\oplus Z_2} A(z),\quad \phi^+ = \max_{z\in Z_1\oplus Z_2} A(z)\right\} \tag{11}$$

where $A(z)$, the angle of a complex z, is defined on Z_1+Z_2 such that $A(z)\in\left[\phi^-,\phi^+\right]$, this is always possible because $Z_1\oplus Z_2\subset Z$ and $Z\in\mathbb{S}(\mathbb{C})$. Denote $z_1=\rho_1 e^{i\theta_1}$, $z_2=\rho_2 e^{i\theta_2}$, $z=\rho e^{i\theta}=\rho_1 e^{i\theta_1}+\rho_2 e^{i\theta_2}$ and $x=\rho_1/\rho_2$, then

$$\tan(\phi)=g(x,\theta_1,\theta_2) \tag{12}$$

where function g is defined by

$$g(\rho_1,\rho_2,\theta_1,\theta_2)=\frac{x\sin\theta_1+\sin\theta_2}{x\cos\theta_1+\cos\theta_2} \tag{13}$$

Since the derivative of function tan is strictly positive, the extrema of A are also extrema of g. In conclusion, computing the lower and upper bounds of [ρ] and [ϕ] are optimization problems that will be solved analytically, since they are not very difficult and the number of variables is only 3.

F. Optimality conditions

Let $\Omega=[u_1]\times[u_2]\times[u_3]\subset\mathbb{R}^3$ and f real function on Ω, and consider the problem

$$\max_{\Omega} f \tag{14}$$

Ω is a compact convex set and problem (14) has a solution $\mathbf{u}^*=(u_1^*,u_2^*,u_3^*)$. For any index i, the i^{th} component u_i^* of u^* must verify one of the following conditions :

$$\frac{\partial h}{\partial u_i}(\mathbf{u}^*)=0 \quad\text{and}\quad \frac{\partial^2 h}{\partial u_i^2}(\mathbf{u}^*)\le 0 \tag{15}$$

$$u_i^*=u_i^- \quad\text{and}\quad \frac{\partial h}{\partial u_i}(\mathbf{u}^*)<0 \tag{16}$$

$$u_i^*=u_i^+ \quad\text{and}\quad \frac{\partial h}{\partial u_i}(\mathbf{u}^*)>0 \tag{17}$$

In the case of a minimization problem, the same conditions apply with all inequalities reversed. Observe that each of these conditions is composed of a first part which is an equation (first-order condition) and a second part which is an inequality (second-order condition). A point of $\mathbb{R}^3$ which verifies, for each of its component, one of the first-order conditions will be termed a candidate. If the corresponding second-order condition is also met, it will be termed an acceptable candidate (in fact, a local optimum). The strategy used by the authors to solve (15) is to determine analytically all candidates, by examining all possible combinations of first-order conditions, and to eliminate the candidates that can never be acceptable by investigating second-order conditions,. The authors set up a reasonably efficient algorithm to check the acceptability of remaining candidates, and to select the optimum, by simple comparison among acceptable candidates [12].

In the next section, this new inclusion function is used for performing set inversion via interval analysis.

III. Constraints satisfaction problems

A. Reduction with one constraint

Consider the constraint satisfaction problem

$$H_1:\left(f(\mathbf{x})=[z],\ \mathbf{x}\in[\mathbf{x}]_0\right) \tag{18}$$

where function $f:\mathbb{C}^n\to\mathbb{C}$. The solution set for (18) is given by

$$\mathbb{S}_1=\left\{\mathbf{x}\in[\mathbf{x}]_0 \mid f_1(\mathbf{x})\in[z]\right\}=f_1^{-1}([z])\cap[\mathbf{x}]_0 \tag{19}$$

The reduction problem for (18) is to find, without any bisection, a subbox $[\mathbf{x}]_1\subset[\mathbf{x}]_0$ as small as possible which contains the solution set $\mathbb{S}_1$. A possible approach is based on the extension of theorem 1 of [14] to complex variables: Assume it is possible to proceed with an explicit inversion of (18), which means:

$$\forall i\ \exists g_1^i \ \Big|\ f_1(\mathbf{x})=z \Leftrightarrow x_i=g_1^i\left({}^i\mathbf{x},z\right) \tag{20}$$

where $\mathbf{x}=(x_1,\ldots,x_n)^{\mathrm{T}}$ and ${}^i\mathbf{x}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)^{\mathrm{T}}$. Denote by $\left[g_j^i\right]$, an inclusion function for the solution function g_j^i and by $[x_i]$, a domain for the variable x_i. The projection $\pi_i(\mathbb{S}_1)$ of the solution set $\mathbb{S}_1$ onto the i^{th} axis, satisfies:

$$\pi_i(\mathbb{S}_1)\subset\left[g_1^i\right]\left(\left[{}^i\mathbf{x}\right],[z]\right)\cap[x_i] \tag{21}$$

The interval hull $[\mathbb{S}_1]$ of $\mathbb{S}_1$, i.e. the smallest axis-aligned box which contains $\mathbb{S}_1$ is then easily obtained from (21). This projection algorithm is used for solving binary or ternary primitive constraints involving addition or multiplication of complex variables, or the power of a complex by a real number.

B. Reduction with m constraints

Consider the constraint satisfaction problem

$$H : \left(\mathbf{f}(\mathbf{x}) = [\mathbf{z}],\ \mathbf{x} \in [\mathbf{x}]_0\right) = \bigcap_j H_j \quad (22)$$

$$H_j : \left(f_j(\mathbf{x}) = [z_j],\ \mathbf{x} \in [\mathbf{x}]_0\right) \quad (23)$$

where function $f_j : \mathbb{C}^n \to \mathbb{C}$ and $j \in \{1, 2, \ldots, m\}$. The solution set for (23) is given by

$$\mathbb{S}_j = \left\{\mathbf{x} \in [\mathbf{x}]_0 \middle| f_j(\mathbf{x}) \in [z_j]\right\} = f_j^{-1}\left([z_j]\right) \cap [\mathbf{x}]_0 \quad (24)$$

and the solution set for (22) is given by

$$\mathbb{S} = \left\{\mathbf{x} \in [\mathbf{x}]_0 \middle| \forall j, f_j(\mathbf{x}) \in [z_j]\right\} = \bigcap_{j=1\ldots m} \mathbb{S}_j \quad (25)$$

Since $\pi_i(\bigcap_{j=1\ldots m} \mathbb{S}_j) \subset \bigcap_{j=1\ldots m} \pi_i(\mathbb{S}_j)$, the algorithm used for solving the reduction problem with several constraints is to deal with the constraints in a sequential way in order to build a nested sequence of subboxes of $[\mathbf{x}]$ which contains $\mathbb{S}$ [14]. This is a simplified version of the local Waltz filtering algorithm initially presented in [15] and extended to intervals in [16] and [17]. The algorithm thus derived is as follows:

Algorithm $C_{\mathbb{S}}$(in: $[\mathbf{z}]$, inout: $[\mathbf{x}]$)

Repeat

1. $[x]_b = [x]$;
2. for j=1 to m, for i=1 to n do
3. $[x_i] = [g_j^i]\left([{}^i\mathbf{x}], [z_j]\right) \cap [x_i]$;
4. until ($r\left([\mathbf{x}]_b, [\mathbf{x}]\right) < \nu$)

In algorithm $C_{\mathbb{S}}$, all variables are taken complex and all domains taken as sectors. Line 4 uses a stopping criterion based on a measure of the relative remoteness of a complex sectors $[\mathbf{u}]$ to a complex sector $[\mathbf{v}]$, which is taken as the maximum of the two relative remoteness of magnitude and angle of sectors. When the reduction achieved at a given iteration is smaller than the real number ν, the algorithm stops.

IV. Set Membership Identification with Complex Interval Analysis

A. Bounded-error context

Denote by $\mathbf{y}_m(\mathbf{p}) : \mathbb{R}^n \mapsto \mathbb{C}^m$, the non-linear model output vector, $\tilde{\mathbf{y}} \in \mathbb{C}^m$ the experimental data vector and $\mathbb{E} \subset \mathbb{C}^m$ a feasible domain for output error, known prior to the identification. The feasible domain for model output is then given by

$$\mathbb{Y} = \tilde{\mathbf{y}} + \mathbb{E} \quad (26)$$

Estimating the parameter vector $\mathbf{p} \in \mathbb{R}^n$ in a bounded error context consists in determining the posterior feasible set $\mathbb{S}$:

$$\mathbb{S} = \left\{\mathbf{p} \in \mathbb{P} \mid \mathbf{y}_m(\mathbf{p}) \in \mathbb{Y}\right\} \quad (27)$$

where $\mathbb{P}$ is some prior search space. The characterization of the posterior feasible set $\mathbb{S}$ is a *set inversion problem*; a guaranteed approximation of such a set can be provided by using interval analysis. The algorithm *set inversion via interval analysis* (SIVIA) [4] uses branch-and-bound techniques with interval analysis and constraint propagation in order to bracket $\mathbb{S}$ between two union of boxes with arbitrary precision. However, as the bisections (subdivision of a box) have to be performed in all directions of the parameter space, SIVIA is practicable for problems involving only few parameters.

Nevertheless, in many estimation problems, one is not interested in the exact characterization of the posterior feasible set, but merely in the minimum volume (axis-aligned) box containing it, called minimum outer box (MOB) or *hull* of set $\mathbb{S}$. MOB has quite interesting properties : 1) The length of each of the axis of MOB along the corresponding i^{th} coordinate axis gives the maximum range of possible variation of the solution set, which is indeed an outer enclosure of the parameter uncertainty interval ; and 2) The center of MOB enjoys several optimality conditions [18]. One can also be interested in the maximum volume box contained in the solution set, mainly to prove the existence of the solution if the inner box is not empty.

Several techniques have been established for computing outer and inner enclosures of the posterior feasible set. Milanese and Vicino [18] introduce an algorithm that derive MOB for polynomial functions. Jaulin [14], introduces a modified version of SIVIA, the algorithm HULL, capable of bracketing the posterior feasible set in between inner and outer hulls, for any non-linear model.

B. Computing the interval hull

The algorithm HULL given below is taken from [14]. It computes two boxes $[\mathbb{S}_{in}]$ and $[\mathbb{S}_{out}]$ that bracket the interval hull $[\mathbb{S}]$ of $\mathbb{S}$, as follows: $[\mathbb{S}_{in}] \subset [\mathbb{S}] \subset [\mathbb{S}_{out}]$. At line 1 of HULL, the algorithm CROSS searches for the largest enclosure of inner points. $\mathbf{u}$ is a punctual interval defined as the Center($[\mathbf{x}]$), but when called for the first time, $\mathbf{u}$ may be taken as a known feasible point. Such an initialization reduces significantly the algorithm computation time.

Starting from a feasible point, algorithm CROSS (see below) attempts to find the largest cross contained in the solution set. Since, each vertex of the cross is a feasible point, it can be used to increase $[\mathbb{S}_{in}]$. Note also that at the end of CROSS, the inner enclosure bracketed by $[\mathbb{S}_{in}]$ is pathconnected. In step 1 of CROSS, $\left[\mathbf{v}\right]=\left[\lambda_i\right]\left(\mathbf{u},\left[\mathbf{z}\right],\left[\mathbf{x}\right],\mathbb{S}\right)$ is the intersection of $\mathbb{S}$ and a line parallel to the i^{th} axis which contains $\mathbf{u}$. The degenerated box $\left[\mathbf{v}\right]$ should be contained in $\mathbb{S}$. In general, the projection theorem introduced above for solving the reduction problem with one or several constraints is sufficient. However, since the solution thus obtained is only locally consistent or if the inclusion function used is not minimal, a pessimism may be introduced at line 2. Therefore, we introduced a simple algorithm $[\lambda_i]$ for solving such an issue.

Algorithm HULL(in: $[\mathbf{z}]$, inout: $[\mathbf{x}]$, $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$)

1. CROSS($\mathbf{u}$, $[\mathbf{x}]$, $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$);
2. $[\mathbb{S}_{out}]=\left[[\mathbb{S}_{out}]\cup[\mathbb{S}_{in}]\right]$;
3. $[\mathbf{x}]=C_{\mathbb{S}}\left([\mathbf{z}],[\mathbf{x}]\right)$;
4. if ($[\mathbf{x}]=\varnothing$), return;
5. if ($w\left([\mathbf{x}]\right)<\eta$ or $r\left([\mathbf{x}],[\mathbb{S}_{in}]\right)<\nu$)
   $[\mathbb{S}_{out}]=\left[[\mathbb{S}_{out}]\cup[\mathbf{x}]\right]$, return;
6. bisect $[\mathbf{x}]$ and get the two boxes $[\mathbf{x}]_1$ and $[\mathbf{x}]_2$;
7. HULL($[\mathbf{z}]$, $[\mathbf{x}]_1$, $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$),
   HULL($[\mathbf{z}]$, $[\mathbf{x}]_2$, $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$)

Algorithm CROSS(in: $\mathbf{u}$, $[\mathbf{x}]$, inout: $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$)

For i = 1 to n

1. $[\mathbf{v}]=[\lambda_i]\left(\mathbf{u},[\mathbf{z}],[\mathbf{x}],\mathbb{S}\right)$;
2. if $[\mathbf{v}]=\varnothing$, next i;
3. if $r\left(\mathbf{v}^-,[\mathbb{S}_{in}]\right)>\kappa$, $[\mathbb{S}_{in}]=\left[[\mathbb{S}_{in}]\cup\mathbf{v}^-\right]$,
   CROSS($\mathbf{v}^-$, $[\mathbf{x}]$, inout: $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$);
4. if $r\left(\mathbf{v}^+,[\mathbb{S}_{in}]\right)>\kappa$, $[\mathbb{S}_{in}]=\left[[\mathbb{S}_{in}]\cup\mathbf{v}^+\right]$,
   CROSS($\mathbf{v}^+$, $[\mathbf{x}]$, inout: $[\mathbb{S}_{in}]$, $[\mathbb{S}_{out}]$);

Algorithm $[\lambda_i]\left(in:\mathbf{u},[\mathbf{z}],[\mathbf{x}],\mathbb{S},\text{ out:}[\mathbf{v}]\right)$

1. $\mathbf{v}=\mathbf{u}$, $[v]_i=[x]_i$;
2. $[\mathbf{v}]=C_{\mathbb{S}}\left([\mathbf{z}],[\mathbf{v}]\right)$;
3. while ($\left[\mathbf{f}\left([\mathbf{v}]\right)\right]\not\subset[\mathbf{z}]$) do
4. $[v]_i=u_i+(1-\xi)\cdot\left[v_i^- -u_i, v_i^+ -u_i\right]$);
5. return ;

In line 4 of algorithm $[\lambda_i]$, the degenerated interval $[\mathbf{v}]$ is reduced, while keeping the feasible point $\mathbf{u}$ interior, until it satisfies the inclusion test. The coefficient ξ is taken equal to 0.05.

C. *Selection of the subdivision direction*

The basic idea of branch-and-bound algorithms such as SIVIA or HULL is to subdivide the original parameter vector, evaluate its acceptability, reject unacceptable boxes, subdivide again ambiguous ones until the desired accuracy is achieved. Therefore, any improvement of the algorithm performances that could be achieved by an optimal choice of the subdivision direction deserve attention. In the context of global optimization by interval analysis, several rules for selecting the subdivision direction have been studied in the literature. A first class of rules relies on the width or the relative width of the parameter vector [6]. Whereas a second class uses the width of the inclusion function [19]. In the sequel, the second type of strategy is used.

V. APPLICATION

The problem under investigation in this paper is the estimation of the dielectric relaxation spectra of polymeric materials in a bounded error context. In the frequency range under analysis, *i.e.* [10^{-3} Hz; 10^{7} Hz], dielectric relaxation spectra can be split into a sum of independent contributions, the so-called relaxation modes, corresponding to dipoles motions of the macromolecular chains [20]. They are approximated using semi-empirical models derived from the Debye equation, such as Cole-Cole, Cole-Davidson or Havriliak-Negami laws [20]. In this work, we consider only the Havriliak-Negami model, given by the following expression [21]:

$$\varepsilon_m(\omega)=\varepsilon_\infty+\sum_{i=1}^{n}\frac{\Delta\varepsilon_i}{\left(1+\left(j\omega\tau_i\right)^{\alpha_i}\right)^{\beta_i}} \tag{28}$$

where $\varepsilon_m(\omega)$ is the relative dielectric complex permittivity measured at a constant temperature and pulsation ω, ε_∞ is the high frequency permittivity, τ_i and $\Delta\varepsilon_i$ are respectively the relaxation time and the dielectric strength associated with relaxation mode i, α_i and β_i are shape parameters describing respectively the symmetric and the asymmetric broadening of the distribution function of relaxation times and n is the number of relaxation modes in the dielectric spectrum. In most practical cases, the number of relaxation processes n is usually not greater than 3. In addition, the dielectric parameters usually satisfy the following physical constraints

$$\varepsilon_\infty>1,\ \Delta\varepsilon_i>0,\ \tau_i>0,\ \alpha_i\in\left]0,1\right],\ \beta_i\in\left]0,1\right] \tag{29}$$

A. *Pseudo-Actual data*

In this paper, pseudo-actual are derived by running equation (28) with the actual values given in Table I or II. The feasible domain for output error is taken constant on both real and imaginary parts of the complex dielectric permittivity, as given by the complex domain

$$\mathbb{E} = [-e, e] + j[-e, e] \qquad (30)$$

In this paper, we will also evaluate the computing times induced by the algorithms for two level of uncertainty bounds, namely e = 0.01 and e = 0.1.

B. The hull of the posterior feasible set

In this study, the algorithm CROSS is initialized with one feasible punctual interval, assumed known, found for instance by global optimization with a genetic algorithm.

For κ = 0.001, ν = 0.1 and η = 0.5, the algorithm HULL computes $[\mathbb{S}_{in}]$ and $[\mathbb{S}_{out}]$ given in Table I for prior error bound e = 0.01 in 6 *hours* on a *CeleronD* 2.6Ghz. Note that $[\mathbb{S}_{in}]$ is found in less that 4*mn*. Table II contains the results derived for prior error bound e = 0.1, in 18 *hours*, whereas $[\mathbb{S}_{in}]$ is found in less that 15 *mn*. Computation times are quite long and further work is needed.

For both error bounds, the accuracy of the bracketing of the parameter uncertainty intervals is acceptable for most parameters related to relaxation mode number 2, whereas for the ones related to relaxation mode number 1, of smaller magnitude, the accuracy of the bracketing is quite poor and would benefit from smaller stopping criterion but this would lead to very long computation time.

TABLE I
INNER AND OUTER ENCLOSURES OF THE HULL OF THE POSTERIOR FEASIBLE SET, e = 0.01

Parameters	Actual values	$[\mathbb{S}_{in}]$	$[\mathbb{S}_{out}]$
ε_∞	3	[2.992,3.009]	[2.87,3.04]
$\tau_1\,(\times 10^{-5}s)$	1.5915	[1.445,1.780]	[0.697,3.904]
α_1	0.6	[0.587,0.611]	[0.395,0.788]
β_1	1	[0.963,1]	[0.96,1]
$\Delta\varepsilon_1$	1	[0.981,1.025]	[0.671,1.447]
$\tau_2\,(s)$	0.15915	[0.157,0.162]	[0.086,0.223]
α_2	0.8	[0.799,0.801]	[0.716,0.863]
β_2	0.7	[0.697,0.707]	[0.545,1]
$\Delta\varepsilon_2$	6	[5.98,6.02]	[5.58,6.33]

TABLE II
INNER AND OUTER ENCLOSURES OF THE HULL OF THE POSTERIOR FEASIBLE SET, e = 0.1

Parameters	Actual values	$[\mathbb{S}_{in}]$	$[\mathbb{S}_{out}]$
ε_∞	3	[2.86,3.11]	[2.78,3.15]
$\tau_1\,(\times 10^{-5}s)$	1.5915	[0.749,2.89]	[0.1,10]
α_1	0.6	[0.571,0.794]	[0.57,0.87]
β_1	1	[0.941,1]	[0.9,1]
$\Delta\varepsilon_1$	1	[0.79,1.25]	[0.79,1.52]
$\tau_2\,(s)$	0.15915	[0.150,0.210]	[0.075,0.21]
α_2	0.8	[0.782,0.818]	[0.77,0.91]
β_2	0.7	[0.643,0.732]	[0.64,1]
$\Delta\varepsilon_2$	6	[5.83,6.20]	[5.22,6.20]

VI. CONCLUSION

This paper addresses bounded error parameter identification for complex-valued non-linear models with large number of parameters. Complex intervals are characterized with polar forms and a new inclusion function is defined for computing the smallest sector containing the sum of two polar complex intervals. Allied with projection algorithms, the new complex interval arithmetic makes it possible to solve efficiently constraint satisfaction problems with complex variables. Used within a bracketing algorithm, parameter identification has been achieved in a reasonable computation time for a non-linear complex-valued model containing 9 real-valued parameters. The evaluation of the new inclusion function will be continued and algorithms studied in order to achieve global consistency for complex interval constraint satisfaction problems.

VII. REFERENCESS

[1] M. Milanese, J.P. Norton, H. Piet-Lahanier and E. Walter, *Bounding approaches to system identification*. New York: Plenum, 1996.

[2] C. Durieu, E. Walter and B. Polyak, "Multi-input multi-output ellipsoidal state bounding", *Journal of Optimization Theory and Applications*, vol. 111, no. 2, pp. 273–303, 2001.

[3] D.G. Maksarov and J.P. Norton, "Computationally efficient algorithms for state estimation with ellipsoidal approximation", *Int. J. Adapt. Control Signal Process*, vol. 16, no. 6, pp. 411-434, 2002.

[4] L. Jaulin and E. Walter, "Set inversion via interval analysis for non-linear bounded-error estimation", *Automatica*, vol. 29(4), pp. 1053-1064, 1993.

[5] L. Jaulin, M. Kieffer, O. Didrit, E. Walter, *Applied interval analysis*, London: Springer, 2001.

[6] R. E. Moore, *Interval analysis*. Englewood Cliffs, NJ, Prentice-Hall, 1966.

[7] P. Henrici, "Circular arithmetic and the determination of polynomial zeros", *Springer Lecture Notes*, vol. 228, pp.86–92, 1971.

[8] R. Klatte and C. Ullrich, "Complex sector arithmetic", *Computing*, vol. 24, no. 2-3, pp.139–148, 1980.

[9] K. Nickel, "Arithmetic of complex sets", *Computing*, vol. 24, pp. 97–105, 1980.

[10] M. S. Petkovic and L. D. Petkovic, *Complex interval arithmetic and its applications*, Wiley-VCH, Berlin, 1998.

[11] J. Rokne and P. Lancaster, "Complex interval arithmetic", *ACM*, vol. 14, pp. 111–112, 1971.

[12] Y. Candau, T. Raïssi, N. Ramdani and L. Ibos, "Complex interval arithmetic using polar form", *Reliable Computing*, *submitted.*

[13] R. T. Farouki and H. Pottmann, "Exact Minkowski products of N complex disks", *Reliable Computing*, vol. 8, pp. 43–66, 2002.

[14] L. Jaulin, "Interval constraint propagation with application to bounded-error estimation", *Automatica*, vol. 36, pp. 1547-1552, 2000.

[15] D. L. Waltz, "Generating semantic descriptions from drawings of scenes with shadows". In P. H. Winston, *The psychology of computer vision*, pp. 19–91, New York: McGraw-Hill, 1975.

[16] E. Davis, "Constraint propagation with interval labels", *Artificial Intelligence*, vol. 32, pp. 281–331, 1987.

[17] J. C. Cleary, "Logical arithmetic", *Future Computing Systems*, vol. 2, no. 2, pp. 125–149, 1987.

[18] M. Milanese and A. Vicino, "Estimation theory for nonlinear models and set membership uncertainty", *Automatica*, vol. 27, pp.403–408, 1991.

[19] T. Cendes and D. Ratz, "Subdivision direction selection in interval methods for global optimization", *SIAM Journal of numerical Analysis*, vol. 34, pp. 922–938, 1997.

[20] A.R. Blythe, *Electrical properties of polymers*, Cambridge solid state science series: Cambridge university press, 1979.

[21] S. Havriliak Jr. and S.J. Havriliak, *Dielectric and mechanical relaxation in materials*, Munich: Hanser, 1997.