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Approximate Modeling of a Class of Nonlinear Oscillators using Takagi-Sugeno Fuzzy Systems and Its Application to Control Design

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Abstract— **An effective modeling of nonlinearities and the analysis of the influence on the closed-loop dynamics in mechatronic systems such as servo systems is often crucial for high performance applications. For this we propose an analytical method of approximate modeling of a class of nonlinear mechanical oscillators using fuzzy systems. The emphasis in this work will be on a systematic description of the construction of fuzzy systems from known nonlinear models and an error analysis as a function of model complexity. Finally, its application as a model framework for an effective gain-scheduling control design method will be discussed.**

Index Terms— **Fuzzy modeling, Mechanical Systems, Control oriented models, Approximate analysis**

I. INTRODUCTION

A. Motivation

An effective modeling of nonlinearities and the analysis of the influence on the closed-loop dynamics in mechatronic systems such as servo systems is often crucial for high performance applications. For this reason this work presents an analytical method of approximate description of a class of nonlinear mechanical oscillators. Instead of studying the exact nonlinear models to enlarge the description capability compared to linear models we will use a Takagi–Sugeno (TS) fuzzy system [7], that consists of a time-variable weighted combination of N_r linear state-space models. TS fuzzy systems also called a Polytopic Linear Model (PLM) [1] since the set of linear models define a polytope in the model-parameter space.

In this paper the investigated class of one-degree-of-freedom (1-DOF) mechanical oscillators is composed of force elements such as springs, dampers or shock absorbers, that may be presented by rather complex relations, including nonlinear characteristics and even additional differential equations. The propose model class is sufficiently rich to describe a wide variety of nonlinear effects in mechatronic systems such as the load and position dependency of the input-output behavior of servo-hydraulic and servo-pneumatic systems and the nonlinear behavior of suspension.

The *goal* of this paper is, starting from the above mentioned nonlinear representations of 1-DOF mechanical oscillators, to describe a systematic approach of approximate modeling using TS fuzzy system and its application to model-based control design [2]. Because the performance of a model-based control strategy depends strongly on the quality of the model it is especially desirable to design an approximate model with sufficient accuracy bounds. In this paper an upper bound on the number of models N_r will be applied that is sufficient to construct a TS fuzzy system with predefined accuracy. The TS fuzzy system may be seen here as a compromise between general nonlinear models that can be very accurate but due to their complexity difficult to apply in model-based control schemes and linear time invariant (LTI) systems that can be very simple and easy to use for control design purposes but the expected behavior of a nonlinear system can only be guaranteed for operating conditions that are close to the point of linearization.

B. Overview

This paper is organized as follows: First of all, in *Section 2* the Takagi–Sugeno fuzzy system structure is introduced and some interpretations are given. After that the approximate construction of TS fuzzy systems from known nonlinear models are explained and an error analysis and some approximation properties are investigated. *Section 3* presents the application of the above construction method. In succession a mechanical oscillator with a nonlinear spring and a mechanical oscillator with a nonlinear spring and damper will be discussed. For this the total number of local models and a grid of equilibrium points are determined on a compact space by an analytically derived equation that is a function of a given upper bound of the model error. Further on some simulation results will be presented in comparison with simulations using the known nonlinear models. Finally, in *Section 4* the application of the above derived model for nonlinear state-feedback controller design is briefly described.

II. APPROXIMATION OF NONLINEAR FUNCTIONS USING TAKAGI–SUGENO SYSTEMS

A. Takagi–Sugeno fuzzy systems

The Takagi-Sugeno fuzzy system [7] considered from a system-theoretic perspective as time-variable smoothed weighted combinations of linear and affine state-space systems provides a flexible framework for analysis and synthesis of nonlinear systems. In this paper the so called affine type of a TS fuzzy system in the state-space form is used. At this,

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the state and output equations are defined as follows

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\, \boldsymbol{A}_i\, \boldsymbol{x}(t) + \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\, \boldsymbol{B}_i\, \boldsymbol{u}(t) + \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\, \boldsymbol{a}_i \;, \tag{1a}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\, \boldsymbol{C}_i\, \boldsymbol{x}(t) \;, \tag{1b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the input, and $\boldsymbol{y} \in \mathbb{R}^p$ is the output. The matrices $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}^{n\times m}$ and $\boldsymbol{C}_i \in \mathbb{R}^{p\times n}$ are used to express the local models $i = 1, ..., N_r$ in state-space representation $G_i := \{\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i\}$ and to take into account the offset using $\boldsymbol{a}_i \in \mathbb{R}^{n\times 1}$. The linear model G_i is valid in a region defined by

$$\alpha_i : \qquad \mathbb{R}^l \to \mathbb{R}\,, \qquad \boldsymbol{z} \longmapsto \alpha_i(\boldsymbol{z}) \tag{1c}$$

$$\alpha_i(\boldsymbol{z}) = \frac{w_i(\boldsymbol{z})}{\sum\limits_{k=1}^{r} w_k(\boldsymbol{z})} \;, \quad w_i(\boldsymbol{z}) = \prod_{j=1}^{l} M_{ij}(z_j) \;, \tag{1d}$$

where $M_{ij}(z_j)$ is the membership function of the model i via z_j. We set the vector $\boldsymbol{z} = [z_1, z_2, ..., z_l]^T$, which may be chosen from a set of measurements such as external physical values, components of the state vector $\boldsymbol{x}$ and the input $\boldsymbol{u}$. Remark: Within the context of controller design, $\boldsymbol{z}$ may be interpreted as a scheduling-vector [6]. Furthermore we assume for $i = 1, \ldots, N_r$

$$w_i(\boldsymbol{z}) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N_r} w_i(\boldsymbol{z}) > 0 \quad \forall\, \boldsymbol{z} \quad . \tag{2}$$

Hence $\alpha_i(\boldsymbol{z})$ satisfies for $i = 1, \ldots, N_r$

$$\alpha_i(\boldsymbol{z}) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z}) = 1 \quad \forall\, \boldsymbol{z} \quad . \tag{3}$$

The affine TS fuzzy system (1) can be seen as a smoothed piecewise approximation of a nonlinear surface of the right-hand-side of the differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$. Approximation properties of this were first investigated in [3] for single-input single-output (SISO) systems and were extended in [1], [6] to multi-input multi-output (MIMO) systems.

B. Construction of TS fuzzy systems from known nonlinear models

The characteristics of this modeling method is described as follows based on dynamic linearization. Suppose we want to approximate a known nonlinear model described by the system of differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\,\boldsymbol{x}(t), \boldsymbol{u}(t)\,)\,, \tag{4a}$$

$$\boldsymbol{y}(t) = \boldsymbol{g}(\,\boldsymbol{x}(t), \boldsymbol{u}(t)\,)\,, \tag{4b}$$

with

$$\boldsymbol{f} \;:\; \mathbb{R}^{m+n} \to \mathbb{R}^n \;, \quad \boldsymbol{g} \;:\; \mathbb{R}^{m+n} \to \mathbb{R}^p \;,$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{u} \in \mathbb{R}^m$ is the input (in the sense that it is free) and $\boldsymbol{y} \in \mathbb{R}^p$ the output of the system (in the sense that $\boldsymbol{y}$ is uniquely specified by $\boldsymbol{u}$ and $\boldsymbol{x}(0)$). Let $(\,\boldsymbol{x}_s(t), \boldsymbol{y}_s(t)\,)$ a solution of (4) for $t \in [t_0, \infty)$. Then the right-hand-side of (4) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}\,(\boldsymbol{x}_s + \Delta\boldsymbol{x})}{\mathrm{d}t} &= \boldsymbol{f}(\boldsymbol{x}_s + \Delta\boldsymbol{x}, \boldsymbol{u}_s + \Delta\boldsymbol{u}) \\ &= \boldsymbol{f}(\boldsymbol{x}_s\,,\, \boldsymbol{u}_s) + \boldsymbol{A}(t)\,\Delta\boldsymbol{x} + \boldsymbol{B}(t)\,\Delta\boldsymbol{u} + \boldsymbol{r}_f \;\;,\end{aligned} \tag{5a}$$

$$\begin{aligned}\boldsymbol{y}_s + \Delta\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}_s + \Delta\boldsymbol{x}, \boldsymbol{u}_s + \Delta\boldsymbol{u}) \\ &= \boldsymbol{g}(\boldsymbol{x}_s\,,\, \boldsymbol{u}_s) + \boldsymbol{C}(t)\,\Delta\boldsymbol{x} + \boldsymbol{D}(t)\,\Delta\boldsymbol{u} + \boldsymbol{r}_g \;,\end{aligned} \tag{5b}$$

if $\boldsymbol{f}$ and $\boldsymbol{g}$ are at least one time continuously differentiable with respect to $\boldsymbol{x}$ and $\boldsymbol{u}$. The matrices in (5) are the well-known Jacobians and defined as

$$\boldsymbol{A}(t) := \boldsymbol{A}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) = \left[\frac{\partial \boldsymbol{f}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t))}{\partial \boldsymbol{x}(t)}\right] , \tag{6a}$$

$$\boldsymbol{B}(t) := \boldsymbol{B}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) = \left[\frac{\partial \boldsymbol{f}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t))}{\partial \boldsymbol{u}(t)}\right] , \tag{6b}$$

$$\boldsymbol{C}(t) := \boldsymbol{C}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) = \left[\frac{\partial \boldsymbol{g}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t))}{\partial \boldsymbol{x}(t)}\right] , \tag{6c}$$

$$\boldsymbol{D}(t) := \boldsymbol{D}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) = \left[\frac{\partial \boldsymbol{g}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t))}{\partial \boldsymbol{u}(t)}\right] . \tag{6d}$$

The remaining terms $\boldsymbol{r}_f$ and $\boldsymbol{r}_g$ as $\boldsymbol{r}_f, \boldsymbol{r}_g = f(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t), \Delta\boldsymbol{x}, \Delta\boldsymbol{u})$ in (5) can be estimated for smoothed functions by the well-known bounds of the error of the Taylor series expansion, see the detailed study in [6]. If $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{u}$ are sufficient small (depending on the choice of the number of local linear models, we will discuss this fact later by means of some case studies) the terms $\boldsymbol{r}_f$ and $\boldsymbol{r}_g$ are negligible in the right-hand side of (5). Using this assumption we get the new system from the right-hand sides of (4a) and (4b):

$$\begin{aligned}\dot{\boldsymbol{\xi}}(t) &= \boldsymbol{f}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) + \boldsymbol{A}(t)\;(\boldsymbol{\xi}(t) - \boldsymbol{x}_s(t)) \\ &\qquad + \boldsymbol{B}(t)\;(\boldsymbol{u}(t) - \boldsymbol{u}_s(t)) \;, \\ \boldsymbol{\xi}(t_0) &= \boldsymbol{x}_s(t_0)\end{aligned} \tag{7a}$$

and

$$\begin{aligned}\tilde{\boldsymbol{y}}(t) &= \boldsymbol{g}(\boldsymbol{x}_s(t), \boldsymbol{u}_s(t)) + \boldsymbol{C}(t)\;(\boldsymbol{\xi}(t) - \boldsymbol{x}_s(t)) \\ &\qquad + \boldsymbol{D}(t)\;(\boldsymbol{u}(t) - \boldsymbol{u}_s(t))\end{aligned} \tag{7b}$$

with the state-space vector $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ (in general $\boldsymbol{\xi}(t) \neq \boldsymbol{x}_s(t)$ for $t > t_0$) and the output vector $\tilde{\boldsymbol{y}} \in \mathbb{R}^p$. Let's consider a given finite set

$$\mathbb{G}_s = \{(\boldsymbol{x}_i, \boldsymbol{u}_i) \in \{(\boldsymbol{x}_s(t), \boldsymbol{y}_s(t))\,,\; t \in [0, \infty)\}\;, \quad i = 1, \ldots, N_r\;\}$$

for $i = 1, \ldots, N_r$ where $\mathbb{G}_s \subset \mathbb{G} \subseteq \mathbb{R}^n \times \mathbb{R}^m$. For the approximation of the right-hand side of (7a) we use now a weighted combination of state-space models

$$\begin{aligned}\dot{\tilde{\boldsymbol{\xi}}}(t) = \sum_{i=1}^{N_r} \alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t))\,[&\boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{u}_i)\,(\tilde{\boldsymbol{\xi}}(t) - \boldsymbol{x}_i) \\ &+ \boldsymbol{B}(\boldsymbol{x}_i, \boldsymbol{u}_i)\,(\boldsymbol{u}(t) - \boldsymbol{u}_i) + \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{u}_i)]\,,\end{aligned} \tag{8}$$

with the new state-space vector $\tilde{\boldsymbol{\xi}}(t) \in \mathbb{R}^n$ (in general $\tilde{\boldsymbol{\xi}}(t) \neq \boldsymbol{\xi}(t)$ for $t > t_0$) and the weighting function $\alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t))$ as a function of the so-called scheduling vector

$$\boldsymbol{z} := \left[\begin{array}{cc} \tilde{\boldsymbol{\xi}}^T & \boldsymbol{u}^T \end{array} \right]^T \quad . \tag{9}$$

Now, putting the static terms in (8) together we get

$$\begin{aligned} \dot{\tilde{\boldsymbol{\xi}}}(t) = & \sum_{i=1}^{N_r} \alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t)) \boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \tilde{\boldsymbol{\xi}}(t) \\ & + \sum_{i=1}^{N_r} \alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t)) \boldsymbol{B}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \boldsymbol{u}(t) \\ & + \sum_{i=1}^{N_r} \alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t)) [\boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{u}_i) \\ & - \boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \boldsymbol{x}_i - \boldsymbol{B}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \boldsymbol{u}_i] \,. \end{aligned} \tag{10}$$

Using the abbreviations

$$\boldsymbol{A}_i := \boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{u}_i)\,, \quad \boldsymbol{B}_i := \boldsymbol{B}(\boldsymbol{x}_i, \boldsymbol{u}_i)\,, \tag{11a}$$

$$\boldsymbol{a}_i := \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{u}_i) - \boldsymbol{A}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \boldsymbol{x}_i - \boldsymbol{B}(\boldsymbol{x}_i, \boldsymbol{u}_i)\, \boldsymbol{u}_i \tag{11b}$$

and $\alpha_i = \alpha_i(\tilde{\boldsymbol{\xi}}(t), \boldsymbol{u}(t))$ the obtained differential equation

$$\dot{\tilde{\boldsymbol{\xi}}}(t) = \sum_{i=1}^{N_r} \alpha_i\, \boldsymbol{A}_i\, \tilde{\boldsymbol{\xi}}(t) + \sum_{i=1}^{N_r} \alpha_i\, \boldsymbol{B}_i\, \boldsymbol{u}(t) + \sum_{i=1}^{N_r} \alpha_i\, \boldsymbol{a}_i \tag{11c}$$

has the same form as (1). The dynamic linearization of (4) about the time-varying point $(\boldsymbol{x}_i, \boldsymbol{y}_i) \in \{(\boldsymbol{x}_s(t), \boldsymbol{y}_s(t)),\ t \in [0, \infty)\}$ is so given by the TS fuzzy systems (1).

C. Error Analysis of Approximation

We investigate now the approximation accuracy bounds of the TS fuzzy system (11). The following results are constructive and form the basis of some of the modeling methods in Section III.
It is obvious that a useful model for approximate modeling has to be close to the system, in the sense that explains the behavior of the original system inside predefined accuracy bounds. One possible choice for measuring the accuracy is to consider the Euclidean distance between the ride-hand side of the original system and the TS fuzzy system. At first, it is helpful to introduce the following *definitions*:

1) ceil$(\cdot)$ is a function

$$\text{ceil} \ : \quad \mathbb{R} \to \mathbb{N}\,, \qquad x \mapsto \text{ceil}(x)\,, \tag{12}$$

that returns the least integer which is not less than its argument, for instance ceil$(2.567) = 3$.

2) $\boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u})$ is a (time-variable) weighted sum

$$\boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u}) := \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{x}, \boldsymbol{u})\, \boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}) \tag{13}$$

of the affine functions

$$\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{A}_i\, \boldsymbol{x} + \boldsymbol{B}_i\, \boldsymbol{u} + \boldsymbol{a}_i\,, \qquad i = 1, \ldots, r \tag{14}$$

with $\boldsymbol{x} \in X \subset \mathbb{R}^n$ and $\boldsymbol{u} \in Y \subset \mathbb{R}^m$.

3) The distance $d(\boldsymbol{f}, \boldsymbol{f}_{WA})$ between $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ and $\boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u})$ is defined by the supremum of the Euclidean error norm on $\mathbb{G} \subseteq X \times U$:

$$d(\boldsymbol{f}, \boldsymbol{f}_{WA}) := \sup_{(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{G}} \|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u})\|_2 \tag{15}$$

4) λ_H represents the maximum absolute Eigenvalue of the Hessian matrices associated with the Taylor remainder: $\lambda_H = \max\limits_{i,j} \left[\lambda_{H_{ij}}\right]$ with $\lambda_{H_{ij}}$ = Eig$[\frac{\partial^2 f_j(\vartheta, \boldsymbol{\psi})}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}}]$

Theorem 1: Let $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ be a smoothed[1] function (right-hand side of (4a)) on a compact region $\mathbb{G} \subseteq X \times U$ whereby

$$\begin{aligned} \mathbb{G} = \{ & \boldsymbol{\psi} = [\psi_1\ \psi_2\ \ldots\ \psi_{n+m}]^T \in \mathbb{R}^{n+m} \mid \\ & |\psi_j - \zeta_j| \leqslant \frac{\beta_j}{2},\ j = 1, \ldots, n+m \} \end{aligned} \tag{16}$$

with

$$\boldsymbol{\psi} := \left[\ \boldsymbol{x}^T\ \boldsymbol{u}^T\ \right]^T \,,$$

and let $\varepsilon \in \mathbb{R}_+$ be an error bound. Then satifies $\boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u})$ (13) with N_r local models

$$N_r = \prod_{i=1}^{n+m} \text{ceil} \left(\frac{\beta_i}{2\sqrt{2\varepsilon}} \sqrt{\lambda_H\,(n+m)\sqrt{n}}\ \right) \tag{17}$$

the inequation

$$d(\boldsymbol{f}, \boldsymbol{f}_{WA}) \ \leqslant\ \varepsilon\,,$$

whereby $\lambda_H = \max\limits_{i,j} \left[\lambda_{H_{ij}}\right]$ (see the above definition).
When the operating space is high dimensional the number of local models N_r increases exponentially with the dimension dim$(\boldsymbol{\psi})$ = $n + m$ of the operating space. However the nonlinear model can be scheduled on a space of lower dimension, if the system is linear with respect to some state and input-variables. The following theorem exploits this circumstances.

Theorem 2: Let $\boldsymbol{f}(\boldsymbol{\psi}) = \boldsymbol{F}\,\boldsymbol{\psi}_L + \boldsymbol{f}_{nl}(\boldsymbol{\psi}_N)$ be a smoothed function with $\boldsymbol{\psi}_L \in \mathbb{G}_L$, $\boldsymbol{\psi}_N \in \mathbb{G}_N$ on the compact region $\mathbb{G} = \mathbb{G}_L \times \mathbb{G}_N$, and let $\varepsilon \in \mathbb{R}_+$ be an error bound. The vectors $\boldsymbol{\psi}_L$ and $\boldsymbol{\psi}_N$ consist of some components of $\boldsymbol{x}$ and $\boldsymbol{u}$ with $\boldsymbol{\psi}_L := [\boldsymbol{x}_L^T\ \boldsymbol{u}_L^T]^T$ and $\boldsymbol{\psi}_N := [\boldsymbol{x}_N^T\ \boldsymbol{u}_N^T]^T$. The nonlinear portion of $\boldsymbol{f}(\boldsymbol{\psi})$ is $\boldsymbol{f}_{nl} : \boldsymbol{\psi}_N \to \mathbb{R}^n$ and the remaining variables are linear in $\boldsymbol{\psi}_L$ with $\boldsymbol{F} \in \mathbb{R}^{n \times n_L}$, $n_L = \text{dim}(\boldsymbol{\psi}_L)$, then the inequation

$$d(\boldsymbol{f}, \boldsymbol{f}_{WA}) \ \leqslant\ \varepsilon$$

is satisfied by

$$\boldsymbol{f}_{WA}(\boldsymbol{\psi}) = \boldsymbol{F}\,\boldsymbol{\psi}_L + \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{\psi}_N)\, \boldsymbol{f}_i(\boldsymbol{\psi}_N)\,, \tag{18a}$$

$$\boldsymbol{f}_i(\boldsymbol{\psi}_N) = \boldsymbol{f}_i(\boldsymbol{x}_N, \boldsymbol{u}_N) = \boldsymbol{A}_i\, \boldsymbol{x}_N + \boldsymbol{B}_i\, \boldsymbol{u}_N + \boldsymbol{a}_i \tag{18b}$$

with $\boldsymbol{\psi}_N = \boldsymbol{z} \in \mathbb{Z} = \mathbb{G}_N$ and N_r local models

$$N_r = \prod_{i=1}^{n_N} \text{ceil} \left(\frac{\beta_i}{2\sqrt{2\,\varepsilon}} \sqrt{\lambda_H\, n_N \sqrt{n}} \right) \tag{19}$$

[1]There exist a zero-th and first order Taylor series expansion of $\boldsymbol{f}$.

whereby $n_N = \dim(\boldsymbol{\psi}_N)$ and $\lambda_H = \max_{i,j} \left[\lambda_{H_{ij}}\right]$.

Comment: The proofs of theorem 1 and 2 are carried out in [6] based on the results in [1].

III. Approximation Modeling of Nonlinear Mechanical Oscillators

On the basis of the approximation results that were derived in the previous section two case studies of 1-DOF nonlinear mechanical oscillators will be investigated.

A. Mechanical Oscillators with a Nonlinear Spring

We consider the physical model of an oscillator with a nonlinear spring

$$m\,\ddot{x} + d\,\dot{x} + k(x) = F\,, \quad x(t_0) = 0\,, \quad \dot{x}(t_0) = 0 \tag{20}$$

with x as the displacement of the rigid body and an extended force $F(t)$. The nonlinear characteristic of the spring is represented by

$$k(x) = c_0\,x + c_1\,x^3\,, \qquad c_0 > 0\,, \quad c_1 > 0\,. \tag{21}$$

We assume, *first*, that the parameters in (20), (21) are given:

$m = 2$ (kg) as mass of the rigid body,
$d = 4$ (N/m/s) as the damping coefficient,
$c_0 = 20$ (N/m) as the linear stiffness coefficient in (21),
$c_1 = 800$ (N/m^3) as the cubic stiffness coefficient in (21).

and *second*, that the working space is bounded

$$|F(t)| \le F_{max} = 4.5 \text{ (N)},\ |x(t)| \le x_{max} = 0.15 \text{ (m)},\\ |\dot{x}(t)| \le \dot{x}_{max} = 0.1 \text{ (m/s)}\,. \tag{22}$$

Using the state-variables

$$x_1 := x\,, \qquad x_2 := \dot{x}\,,$$

and selecting the input and the output variables as

$$u := F\,, \qquad y := x\,,$$

we get a state-space representation of (20):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{c_0}{m} & -\frac{d}{m} & \frac{1}{m} \end{bmatrix}}_{\boldsymbol{F}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}}_{\boldsymbol{\psi}_L} + \underbrace{\begin{bmatrix} 0 \\ -\frac{c_1}{m}\,x_1^3 \end{bmatrix}}_{\boldsymbol{f}_{nl}(\boldsymbol{\psi}_N)}\,. \tag{23}$$

The goal now is to find a TS fuzzy system (1) as an approximator of (23) with an ε-accuracy for a given $\varepsilon = 0.25$. Then it suffices to construct a TS fuzzy system using (19) with

$$N_r = \prod_{i=1}^{n_N} \text{ceil}\left(\frac{\beta_i}{2\sqrt{2\,\varepsilon}}\sqrt{\lambda_H\,n_N\,\sqrt{n}}\right)$$

local models with $\beta_1 = 2 \cdot x_{max}$ (size of the operating region of the nonlinear part), because the number of variables in the nonlinear term is only $n_N = 1$ and the system order is $n = 2$. So it is also sufficient to determine solely the Hessian matrices from the nonlinear term $\boldsymbol{f}_{nl}$

$$\boldsymbol{H}_{ij} = \left[\ \frac{\partial^2 f_{nl_j}}{\partial x_1^2}\ \right] \qquad \text{for} \quad j = 1, 2\,.$$

The corresponded Eigenvalues are

$$\lambda_{H_{i1}} = \text{Eig}[\boldsymbol{H}_{i1}] = 0\,, \qquad \lambda_{H_{i2}} = \text{Eig}[\boldsymbol{H}_{i2}] = -6\,\frac{c_1}{m}\,x_{1_i}\,.$$

Based on the fact that the displacement x is bounded the maximal Eigenvalue is

$$\lambda_H = \max_i \left[-6\,\frac{c_1}{m}\,x_{1_i}\right] = -6\,\frac{c_1}{m}\,(-x_{max}) = 360\,,$$

so the number of local models can be obtain

$$N_r = \text{ceil}\left(\frac{0.3}{2\sqrt{2 \cdot 0.25}}\sqrt{360\sqrt{2}}\right) = 5\,.$$

That is, to attain a desired ε-accuracy

$$d(\boldsymbol{f}, \boldsymbol{f}_{WA}) \leqslant \varepsilon = 0.25\,,$$

the TS fuzzy system (1) must be a combination of at least $N_r = 5$ linear models. In the next step the matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ and the vector $\boldsymbol{a}_i$ for $i = 1, \ldots, 5$ will be calculated by (6) at the operating points $\mathbb{G}_s = \left\{[\,x_{1_i}\ x_{2_i}\ u_i\,]^T\right\}$ with

$$x_{1_i} \in \{-0.12\,,\ -0.06\,,\ 0.0\,,\ 0.06\,,\ 0.12\}\,. \tag{24}$$

The operating points are chosen equidistantly since the upper bound for the number of models N_r is based on a worst case scenario, namely the maximum nonlinearity measured with the maximum Eigenvalue λ_H of the Jacobian that can occur all over the predefined operating region (22). Further on, it is obvious that the only variable in the scheduling-vector $\boldsymbol{z} \in \mathbb{R}^1$ is the state x_1

$$\boldsymbol{z} = z_1 := x_1\,, \tag{25}$$

because all other variables, in this case x_2 and u, appear linear in $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, see (23). After these preliminary considerations we obtain

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m}\left(c_0 + 3\,c_1\,x_{1_i}^2\right) & -\frac{d}{m} \end{bmatrix},$$

with (6a) and (11a) based on the original system (23) and with (6b) and (11b) we get

$$\boldsymbol{B}_i = \boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{a}_i &= \boldsymbol{f}(\boldsymbol{x}_i, u_i) - \boldsymbol{A}_i\,\boldsymbol{x}_i - \boldsymbol{B}\,u_i \\ &= \begin{bmatrix} 0 \\ f_2(\boldsymbol{x}_i, u_i) + \frac{1}{m}\left(c_0 x_{1_i} + 3 c_1 x_{1_i}^3\right) + \frac{d}{m} x_{2_i} - \frac{1}{m}\,u_i \end{bmatrix}. \end{aligned}$$

Additional to the previous choice of x_{1_i} we recognized that x_{2_i} and u_i must be determined in $\boldsymbol{a}_i$. In this case it is possible to define the operating points at the equilibria points of the nonlinear system (23). Let $\boldsymbol{f}(\boldsymbol{x}_i, u_i) \equiv \boldsymbol{0}$ then it follows that

$$u_i = c_0\,x_{1_i} + c_1\,x_{1_i}^3\,, \qquad x_{2_i} = 0\,. \tag{26}$$

for $i = 1, \ldots, 5$. Using the predefined values (24) we get from (26)

$$u_i \in \{-3.7824\,,\ -1.3728\,,\ 0.0\,,\ 1.3728\,,\ 3.7824\}\,.$$

Note that the nonlinear system (23) is linear in the output equation, so it implies that $\boldsymbol{C}_i = \boldsymbol{C} = [\,1\;0\,]$ for $i = 1, \ldots, 5$. Based on the previous results we get now the approximate model

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{5} \alpha_i(x_1)\, \boldsymbol{A}_i\, \boldsymbol{x} + \boldsymbol{B}\, u + \sum_{i=1}^{5} \alpha_i(x_1)\, \boldsymbol{a}_i \;, \\ \boldsymbol{y} &= \boldsymbol{C}\, \boldsymbol{x} \;. \end{aligned} \tag{27}$$

of the form (1). Last we have to determine the position and the shape of the weighting functions $\alpha_i(x_1)$. It is obvious that the position should correlate with the operating points (24). Assuming that the models with $\{\boldsymbol{A}_i, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{a}_i\}$ are locally valid, five equally spaced triangular functions are defined in the domain $x_1 \in [-0.15, 0.15]$ as shown in Figure 1.

Finally, in this case study, some simulation results are

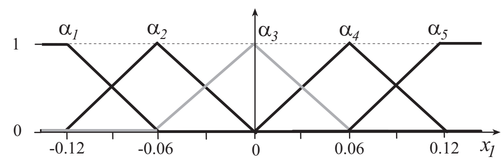

Fig. 1. Distribution of the weighting functions $\alpha_i(x_1)$ in (27)

presented. *First*, a comparison of phase portraits of the approximate model (27) and the original model is shown in Figure 2. Both simulations are carried out under the same initial conditions and with the same input signal $u(t) = 4.5 \cdot \sin(2\,\pi f\, t)$ with $f = 0.05$ Hz for $t \in [0, 20]$ s. *Second*,

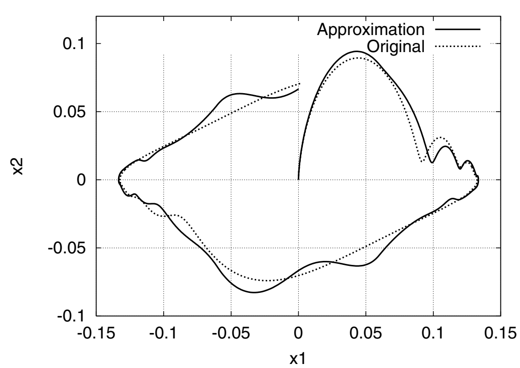

Fig. 2. Comparison of the phase portraits of the approximate system (27) and the original system (23) with common sinusoidal input

Figure 3 clearly presented that the Euclidean error norm

$$E(t) = \|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \boldsymbol{f}_{WA}(\boldsymbol{x}, \boldsymbol{u})\|_2 \tag{28}$$

is less than the predefined value $\varepsilon = 0.25$ relate to $t \in [0, 20]$ s.

B. Mechanical Oscillators with a Nonlinear Spring and Damper

We consider now the physical model of an oscillator

$$m\, \ddot{x} + d(\dot{x}) + k(x) = F\,, \quad x(t_0) = 0\,, \quad \dot{x}(t_0) = 0 \tag{29}$$

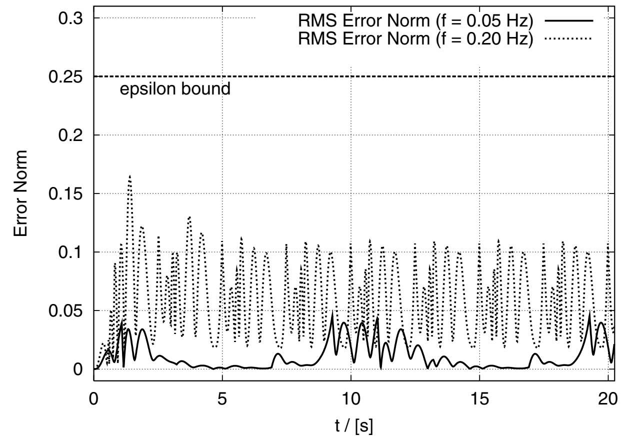

Fig. 3. Calculated error norm $\|\boldsymbol{f}(\boldsymbol{x}, u) - \boldsymbol{f}_{WA}(\boldsymbol{x}, u)\|_2$ for two sinusoidal input signals $u(t) = 4.5 \cdot \sin(2\,\pi f\, t)$ with $f = 0.05$ Hz and $f = 0.2$ Hz

with a nonlinear spring (21) and as an extension to the previous case with a nonlinear damper. The behavior of the damper is described by

$$\begin{aligned} d(\dot{x}) &= d_0\, \dot{x} + \mathrm{sgn}(\dot{x})\, d_1 \sqrt{|\dot{x}|} \\ &= \begin{cases} d_0\, \dot{x} \,-\, d_1 \sqrt{|\dot{x}|} & \text{if} \quad \dot{x} < 0 \\ d_0\, \dot{x} \,+\, d_1 \sqrt{\dot{x}} & \text{else} \end{cases} \;. \end{aligned} \tag{30}$$

We assume, *first*, that the parameters in (21), (29) and (30) are given with

$$\begin{aligned} d_0 = 0.5 \text{ (N/m/s)} \;, \qquad d_1 = 0.8 \text{ (N/(m/s)}^{1/2}) \\ c_0 = 20 \text{ (N/m)} \;, \quad c_1 = 215 \text{ (N/m}^3) \;, \quad m = 2 \text{ (kg)} \;. \end{aligned}$$

and *second*, that the working space is bounded

$$\begin{aligned} |F(t)| \le F_{max} = 3.5 \text{ (N)} \;, |x(t)| \le x_{max} = 0.19 \text{ (m)} \;, \\ |\dot{x}(t)| \le \dot{x}_{max} = 0.3 \text{ (m/s)} \;. \end{aligned} \tag{31}$$

Using the same state-, input- and output-variables as in the previous case (29) can be written in the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{c_0}{m} & -\frac{d_0}{m} & \frac{1}{m} \end{bmatrix}}_{\boldsymbol{F}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}}_{\boldsymbol{\psi}_L} \\ &+ \underbrace{\begin{bmatrix} 0 \\ -\frac{c_1}{m}\, x_1^3 - \frac{d_1}{m}\, \mathrm{sgn}(x_2) \sqrt{|x_2|} \end{bmatrix}}_{\boldsymbol{f}_{nl}(\boldsymbol{\psi}_N)} \end{aligned} \tag{32}$$

with $\boldsymbol{\psi}_N = [\,x_1\; x_2\,]^T$.

The goal here is to find a TS fuzzy system (1) as an approximator of (32) with an ε-accuracy for a given $\varepsilon = 0.2$. Then it suffices to construct a TS fuzzy system using (19) with

$$\begin{aligned} N_r = \mathrm{ceil}\left(\frac{0.38}{2\,\sqrt{2} \cdot 0.2} \sqrt{\lambda_H \cdot 2\,\sqrt{2}} \right) \\ \cdot \mathrm{ceil}\left(\frac{0.6}{2\,\sqrt{2} \cdot 0.2} \sqrt{\lambda_H \cdot 2\,\sqrt{2}} \right) \end{aligned} \tag{33}$$

local models where $\beta_1 = 2 \cdot x_{max} = 0.38$ and $\beta_2 = 2 \cdot \dot{x}_{max} = 0.6$ (size of the operating region of the nonlinear part), because the order of the nonlinear part in (32) is $n_N = 2$. So, the Hessian matrices from the term $\boldsymbol{f}_{nl} = [\,f_{nl_1}\ f_{nl_2}\,]^T$ are calculated as in the previous case:

$$\boldsymbol{H}_{ij} = \left[\begin{array}{cc} \frac{\partial^2 f_j}{\partial x_1^2} & \frac{\partial^2 f_j}{\partial x_2\,\partial x_1} \\ \frac{\partial^2 f_j}{\partial x_1\,\partial x_2} & \frac{\partial^2 f_j}{\partial x_2^2} \end{array} \right]$$

for $j = 1, 2$ it follows that $\boldsymbol{H}_{i1} = \boldsymbol{0}_{2\times 2}$ and

$$\boldsymbol{H}_{i2} = \left[\begin{array}{cc} -6\,\frac{c_1}{m}\,x_{1_i} & 0 \\ 0 & \frac{d_1}{4}\,\mathrm{sgn}(x_{2_i})\,|x_{2_i}|^{-\frac{3}{2}} \end{array} \right] .$$

Based on the fact that the displacement x and the velocity $\dot{x}$ of the mass are bounded (31) the maximal Eigenvalue is

$$\lambda_H = \max \left[[0,0]\,;\, \max_i \left[-6\frac{c_1}{m}\,x_{1_i}, \frac{d_1}{4}\mathrm{sgn}(x_{2_i})\,|x_{2_i}|^{-\frac{3}{2}} \right] \right] .$$

It follows that the number of local models can be obtain using (33) with $\lambda_H = 122.55$, so

$$N_r = 6 \cdot 9 = 54\ .$$

We get now the TS fuzzy system of the nonlinear oscillator (29)

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{54} \alpha_i(x_1, x_2)\,\boldsymbol{A}_i\,\boldsymbol{x} + \boldsymbol{B}\,u + \sum_{i=1}^{54} \alpha_i(x_1, x_2)\,\boldsymbol{a}_i\ , \\ \boldsymbol{y} &= \boldsymbol{C}\,\boldsymbol{x}\ , \end{aligned} \tag{34a}$$

where $\boldsymbol{z} = [\,x_1\ x_2\,]^T$ and where the not yet calculated $\boldsymbol{A}_i$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are determined by the Jacobians $(6a)$, $(6b)$ and $(6c)$ (for detailed results see [6]). Last we have to determine the position and the shape of the weighting functions $\alpha_i(x_1, x_2)$ in (34a). Assuming that the state-space models with $\{\boldsymbol{A}_i, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{a}_i\}$ are locally valid 54 triangular functions are defined in the domain $(x_1, x_2) \in [-0.19, 0.19] \times [-0.3, 0.3]$.

IV. Application to Model-based Nonlinear State Feedback Controller

Last, we briefly describe the application of the previously derived TS fuzzy systems (27) and (34a) in the framework of model-based control design. Control laws for TS fuzzy systems are frequently put in the form called parallel distributed compensation (PDC) [8]. A survey is given e.a. in [2]. For a PDC synthesis, the control law is obtained according to the fuzzy model (1) in the following way

$$\boldsymbol{u}_{fb}(\boldsymbol{z}) = \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\,\boldsymbol{F}_i\,\boldsymbol{x} \tag{35}$$

with the same $\alpha_i(\boldsymbol{z})$ as (1). We have got to be aware that the weighted combination of local linear state-feedback gains $\boldsymbol{F}_i$ holds only for the linear part without the offset vectors $\boldsymbol{a}_i$ for $i = 1, \ldots, N_r$. For the dynamic compensation of these offsets a quasi feedforward part (it is not a feedforward in the strict sense, since it depends also on $\boldsymbol{z}$ that may be consists of measured states) can be determined by

$$\boldsymbol{u}_{ff}(\boldsymbol{z}) = -\left[\boldsymbol{B}(\boldsymbol{z})^T\boldsymbol{B}(\boldsymbol{z})\right]^{-1}\boldsymbol{B}(\boldsymbol{z})^T \sum_{i=1}^{N_r} \alpha_i(\boldsymbol{z})\,\boldsymbol{a}_i\ , \tag{36}$$

where $\left[\boldsymbol{B}(\boldsymbol{z})^T\boldsymbol{B}(\boldsymbol{z})\right]^{-1}\boldsymbol{B}(\boldsymbol{z})^T$ is the peudo-inverse with

$$\boldsymbol{B}(\boldsymbol{z}) := \sum_{j=1}^{N_r} \alpha_j(\boldsymbol{z})\,\boldsymbol{B}_i. \tag{37}$$

Therefore, the control law based on a superposition of both, the feedforward part (36) and feedback part (35) with $\boldsymbol{u}(\boldsymbol{z}) = \boldsymbol{u}_{ff}(\boldsymbol{z}) + \boldsymbol{u}_{fb}(\boldsymbol{z})$ that represents a gain-scheduling controller [4] where $\boldsymbol{z}$ is the scheduling-vector. It is assumed that the states $\boldsymbol{x}$ and the states in $\boldsymbol{z}$, see (25) for case study one and $\boldsymbol{z} = [\,x_1\ x_2\,]^T$ for case study two, are available for control.

V. Conclusions

In this paper an approximate modeling approach of nonlinear systems using Takagi-Sugeno fuzzy models was proposed.

It was shown that a estimated number of models calculated by analytically derived relations is sufficient to construct a TS fuzzy system with predefined accuracy. These relations depend on the (predefined) upper bound of the error, the size of the compact operating space and the estimation of the largest nonlinearity of the original system. The nonlinearity was measured by the maximum absolute Eigenvalue of the Hessian matrices associated with the Taylor remainder. The approximate modeling approach was applied to two different models of 1-DOF nonlinear mechanical oscillators.

Finally, a feasible application of the above derived models as a framework for fuzzy gain-scheduling control design was briefly described.

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