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Design of normalizing precompensators via alignment of output-input principal directions

Marcos V. Moreira and João C. Basilio

***Abstract*— Normalization is a crucial requirement for the effectiveness of multivariable control system design within the Characteristic Locus Method. Previous work addresses this problem by solving an optimization problem formulated in order to increase normality; its formulation, however, do not consider the closed-loop system robustness with respect to perturbations at the plant input. In this paper a different approach to the design of normalizing precompensators will be proposed. It is based on the minimization of a cost function representing the measure of misalignment between the output and input principal directions of the precompensated system. The main advantage of this approach is that, since normalization is obtained via alignment, the sensitivity of the characteristic loci to perturbations at both the plant input and output is reduced.**

I. INTRODUCTION

The design of a multivariable control system within the Characteristic Locus Method (CLM) [10] is carried out using the eigenfunctions of the open-loop transfer matrix which, according to the generalized Nyquist stability criterion [12], define the stability of the closed-loop system. Its essence is to construct a commutative controller, *i.e.*, a controller with the same eigenvector and dual-eigenvector matrices (frames) as the plant and to manipulate the controller eigenfunctions so as to achieve closed-loop stability and to satisfy the usual performance requirements. This poses two serious problems: (i) except in special cases, the eigenvector and dual-eigenvector matrices of the plant are irrational; (ii) for plants whose frequency responses are far from normal at a certain frequency band, the characteristic loci of the open-loop system are very sensitive to perturbations at the plant input and output at these frequencies [5], [15], [16].

Problem (i) can be circumvented by using, as the controller frame, some approximation of the plant frame [1], [4], [9], [11], or using the parameterization presented in [14] of all proper and rational controllers that exactly commute with the plant and stabilize the closed-loop system.

To overcome problem (ii), it is proposed the design of the so called reversed-frame-normalizing-controllers (RFNC) [2], [8]. Although the design of RFNC improves the closed-loop system robustness with respect to perturbations at the plant input and output, it is based on the quasi-Nyquist loci of the plant, which cannot replace the characteristic loci as an analysis tool for general systems. Therefore, a more interesting synthesis method should be based on the CLM, being composed of two stages: (i) first, precompensate the plant in order to approximately normalize it in the necessary frequency range and then, (ii) apply the CLM to the precompensated system.

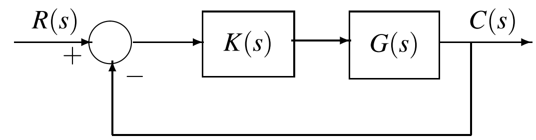

Fig. 1. Closed-loop feedback control system

A precompensation scheme with the view to making the precompensated plant as normal as possible has recently been presented [3], where it was proposed a normalizing precompensator with maximum singular value less than or equal to one at almost all frequencies in order to avoid the amplification of the radii of the characteristic locus band. The main drawbacks of this normalization technique are: (i) the phases of the entries of the rational precompensator (the one which will be actually implemented) are obtained without concerning with possible degradation of the optimization cost and, (ii) in the formulation of the problem, it is not considered the characteristic locus sensitivity with respect to perturbations at the plant input.

In this paper, a new precompensation method to normalize a plant and, at the same time, to reduce the sensitivity of the characteristic loci with respect to perturbations at the plant input and output, when an exact commutative controller is considered, is presented. The key to this new formulation is the search of a precompensator that reduces the misalignment between the output and input principal directions of the precompensated plant, leading, therefore, to an approximately normal precompensated system.

II. MAIN CONCEPTS

Let $G(s)$ and $K(s)$ be the $m \times m$ transfer matrices of the plant and controller, respectively. According to the generalized Nyquist stability criterion, the feedback system of Fig. 1 will be stable if and only if the net sum of anti-clockwise encirclements of the critical point $-1+j0$, by the characteristic loci of $G(s)K(s)$, equals the number of unstable poles of $G(s)$ and $K(s)$.

In order to be able to use the generalized Nyquist stability criterion as a design tool, a controller $K(s)$ such that $G(s)K(s) = K(s)G(s)$ is sought. This condition is satisfied providing $G(s)$ and $K(s)$ share the same eigenvector and dual eigenvector frames and, therefore, the eigenvalues of the product $G(s)K(s)$ are equal to the product of the eigenvalues of $G(s)$ and $K(s)$.

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M. V. Moreira and J. C. Basilio are with COPPE - Programa de Engenharia Elétrica, Universidade Federal do Rio de Janeiro, 21.949-900, Rio de Janeiro, R.J., Brazil. moreira@pee.coppe.ufrj.br, basilio@dee.ufrj.br

Besides the unavoidable problem of irrational eigenvector matrices, the CLM may also suffer from sensitivity problem when the plant transfer matrix is far from normal[1] [5]. This means that when the plant is far from normal at a certain frequency band, it is necessary to design a precompensator $K_p(s)$ in order to normalize the precompensated plant $G(j\omega)K_p(j\omega)$. Once $G(j\omega)K_p(j\omega)$ is approximately normal at the necessary frequency range, then a commutative controller $K_c(s)$ can be designed effectively via the CLM. Therefore, the design of a normalizing precompensator must be the first stage of the design of multivariable control systems within the CLM [3].

The importance of normality for the sensitivity of the characteristic loci is presented in [5] and [15]. Supposing a stable matrix perturbation $I+M_G(s)$, at any point in the configuration of Fig. 1, it is shown in [5] that the closed-loop system remains stable if the maximum singular value of $M_G(j\omega)$ multiplied by the maximum singular value of $[I+T(j\omega)^{-1}]^{-1}$ is less than 1, for all frequencies, where $T(s)$ is the return ratio matrix for the point where the loop was broken. The consequence of this fact is that [15] the characteristic loci are least sensitive to perturbations at the plant output and input, if the return ratio matrices for those points, namely $G(s)K(s)$ and $K(s)G(s)$, respectively, are both normal at the necessary frequency range.

Notice that if $G(j\omega)$ is normal for all frequencies and if $K(s)$ is an exact commutative controller, then $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$ are normal. However, if $G(j\omega)$ is not normal at a certain frequency band, and a normalizing precompensator $K_p(s)$ is designed, then it is necessary to design a controller $K_c(s)$ which commutes with $G(s)K_p(s)$, and thus the controller to be implemented

$$K(s) = K_p(s)K_c(s) \tag{1}$$

does not necessarily lead to robustness with respect to small perturbations at the plant input. To overcome this problem, the design of a precompensator $K_p(s)$ that makes both $K_p(j\omega)K_c(j\omega)G(j\omega)$ and $G(j\omega)K_p(j\omega)K_c(j\omega)$ approximately normal is proposed in this paper.

III. A NORMALIZING PRECOMPENSATOR

A. Problem formulation

A direct approach to the problem of designing a normalizing precompensator for a plant has been presented in [3], where a precompensator $\bar{K}_p(s)$ has been designed to make the transfer matrix $G(s)\bar{K}_p(s)$ approximately normal in the necessary frequency range. The measure of the deviation from normality of a complex matrix G, used in [3], has been defined as:

$$\delta(G) = \frac{\|G^\star G - GG^\star\|_{\mathscr{F}}^2}{\|G^\star G\|_{\mathscr{F}}^2}, \tag{2}$$

where $\|.\|_{\mathscr{F}}$ denotes the Frobenius norm, which, for a matrix $E \in \mathbb{C}^{m\times m}$ is defined as:

$$\|E\|_{\mathscr{F}}^2 = tr(E^\star E), \tag{3}$$

[1]A matrix $G \in \mathbb{C}^{m\times m}$ is normal if it commutes with its conjugate transpose, $G^\star$, *i.e.* $GG^\star = G^\star G$.

where $tr(.)$ denotes the trace of a matrix. The ideal frequency response of the precompensator ($\bar{K}_p(s)$), here denoted as $K_p(j\omega_k)$, has been obtained, by solving, for a finite number of frequencies ω_k, $k=1,\ldots,n$, the optimization problem $\min\delta[G(j\omega_k)K_p(j\omega_k)]$, subject to constraints on the precompensator structure and on the modulus of its entries. In the sequel, rational and stable transfer functions have been obtained for the elements of $\bar{K}_p(s)$ with the view to approximating the frequency response of its elements to those obtained for $K_p(j\omega_k)$. The structure for $K_p(j\omega_k)$ used in [3] is of a permuted diagonal matrix with constraints on the modulus of its entries to guarantee that the maximum singular value of the precompensator be less than or equal to 1.

One of the main drawbacks of this precompensation method is that the sensitivity of the characteristic loci to perturbations at the plant input, when $K(s) = K_p(s)K_c(s)$, has not been considered. Indeed, if $G(j\omega)K_p(j\omega)$ is normal at a certain frequency $\omega=\omega_0$, then $G(j\omega_0)K(j\omega_0)$ is normal, but $K(j\omega_0)G(j\omega_0)$ is not necessarily normal and the characteristic loci can be very sensitive to perturbations at the plant input.

With the view to considering the normalization of both $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$ it was introduced in [8] the so called reversed-frame-normalizing-controllers (RFNC), whose theoretical justification is given below.

Lemma 1: Suppose G and $K \in \mathbb{C}^{m\times m}$ are both of rank m and let

$$G = Y\Sigma U^\star \tag{4}$$

be a singular value decomposition of G, where $\Sigma = diag\{\sigma_i, i=1,\ldots,m\}$. Then GK and KG are both normal if and only if

$$K = U\Gamma_K Y^\star \tag{5}$$

for some nonsingular diagonal matrix $\Gamma_K \in \mathbb{C}^{m\times m}$.

Proof: See [8]. □

According to lemma 1, the characteristic loci are at their least sensitive to small perturbations at the plant input and output if and only if the singular vector frames of $K(s)$ are those of $G(s)$ taken in reversed order. However, in this paper, the controller is defined by Eq. (1), which implies that $K_p(s)$ must have a specific structure so that $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$ be both normal at the frequencies of interest.

Theorem 1: Suppose G and $K \in \mathbb{C}^{m\times m}$ are both of rank m and let the singular value decomposition of G be given by Eq. (4). In addition let $K = K_pK_c$, where $K_c \in \mathbb{C}^{m\times m}$ commutes exactly with GK_p. Then, GK and KG are both normal matrices if and only if $K_p = U\Phi Y^\star$, for some nonsingular diagonal matrix Φ.

Proof: ($\Rightarrow$) If GK and KG are normal, then, according to lemma 1,

$$K = K_pK_c = U\Gamma_K Y^\star. \tag{6}$$

Therefore, from Eqs. (4) and (5), $GK = Y\Sigma U^\star U\Gamma_K Y^\star = Y\Sigma\Gamma_K Y^\star$, which is a spectral decomposition for GK. Supposing that K_c is an exact commutative controller, then GK and K_c share the same eigenvector matrices, which means that a spectral decomposition of K_c can be written as $K_c = Y\Lambda_c Y^\star$, where Λ_c is a diagonal matrix, and thus, Eq. (6) becomes

$$K_p Y \Lambda_c Y^\star = U\Gamma_K Y^\star \Leftrightarrow K_p = U\Gamma_K \Lambda_c^{-1} Y^\star. \qquad (7)$$

Therefore, defining $\Phi = \Gamma_K \Lambda_c^{-1}$, yields the result.

($\Leftarrow$) The proof is straightforward and will be omitted. □

Theorem 1 shows that the least sensitivity of the characteristic loci with respect to perturbations at the plant input and output, at a given frequency ω_0, is achieved when the normalizing precompensator $K_p(j\omega_0)$ is such that the matrix $U(j\omega_0)^\star K_p(j\omega_0)Y(j\omega_0)$ is diagonal. However, since normal matrices are a relatively small set compared to approximately normal matrices, then instead of trying to achieve the exact normality of the precompensated plant, it is more realistic to find ways to approximately normalize both return ratio matrices, $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$, with a precompensator $K_p(j\omega)$, at the frequencies of interest. To do so, the following result is needed.

Lemma 2: Let $G \in \mathbb{C}^{m\times m}$. Then G is normal if and only if G has a complete orthonormal set of eigenvectors.

Proof: See [7]. □

Lemma 2 suggests a way of measuring how close to normal a given matrix G is [8], [3], namely G is approximately normal if the condition number of its eigenvector matrix is approximately equal to one.

In this paper, it will be shown that with the precompensator structure given by

$$K_p(j\omega) = \alpha(j\omega)K_u(j\omega), \qquad (8)$$

where $\alpha(j\omega) \in \mathbb{C}$ and $K_u(j\omega)$ is a unitary matrix, it is possible to achieve the same degree of normality for $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$, in the sense that the condition number of their eigenvector matrices are equal, as shown in the sequel.

Lemma 3: Let $G \in \mathbb{C}^{m\times m}$ have the singular value decomposition given by Eq. (4), and define the complex matrix

$$M = U^\star K_p Y. \qquad (9)$$

Then, GK_p is normal if and only if ΣM is normal.

Proof: The proof is straightforward and will be omitted.

Theorem 2: Let K and K_p be given by Eqs. (1) and (8) for a given frequency, respectively, and assume that K_c commutes exactly with GK_p. If GK_p is approximately normal, in the sense that its eigenvector matrix has condition number approximately equal to one, then GK and KG are also approximately normal.

Proof: Suppose that ΣM has the following spectral decomposition:

$$\Sigma M = W_{\Sigma M}\Lambda_{\Sigma M} V_{\Sigma M}, \qquad (10)$$

where $\Lambda_{\Sigma M}$ is a diagonal matrix, $W_{\Sigma M}$ is the eigenvector matrix of ΣM and $V_{\Sigma M} = W_{\Sigma M}^{-1}$. Then, GK_p can be written as:

$$GK_p = Y W_{\Sigma M}\Lambda_{\Sigma M} V_{\Sigma M} Y^\star, \qquad (11)$$

and since Y is a unitary matrix, the condition number of its eigenvector matrix is equal to the condition number of the eigenvector matrix of ΣM, *i.e.*, $\mathscr{C}[YW_{\Sigma M}] = \mathscr{C}[W_{\Sigma M}]$, where $\mathscr{C}[.]$ denotes condition number. Therefore, it suffices to prove that the condition numbers of the eigenvector matrices of GK and KG are both equal to $\mathscr{C}[W_{\Sigma M}]$.

Since K_c commutes exactly with GK_p, it has the same eigenvector matrix as GK_p and thus:

$$K_c = Y W_{\Sigma M}\Lambda_c V_{\Sigma M} Y^\star. \qquad (12)$$

Thus a spectral decomposition for GK can be given as:

$$GK = Y W_{\Sigma M}\Lambda_{\Sigma M}\Lambda_c V_{\Sigma M} Y^\star. \qquad (13)$$

To show that KG is also approximately normal when GK_p is approximately normal, notice, initially that since $K_p = \alpha K_u$, where K_u is a unitary matrix, then

$$M = U^\star K_p Y = \alpha U^\star K_u Y = \alpha M_u, \qquad (14)$$

where $M_u = U^\star K_u Y$ is a unitary matrix. Therefore, using Eqs. (9), (12) and (4), one obtains

$$KG = UMW_{\Sigma M}\Lambda_c V_{\Sigma M}\Sigma U^\star. \qquad (15)$$

According to (10), $V_{\Sigma M}\Sigma M = \Lambda_{\Sigma M} V_{\Sigma M}$ and thus:

$$V_{\Sigma M}\Sigma = \Lambda_{\Sigma M} V_{\Sigma M} M^{-1}. \qquad (16)$$

Substituting Eq. (16) in Eq. (15) and making $M = \alpha M_u$, yields:

$$KG = UM_u W_{\Sigma M}\Lambda_c \Lambda_{\Sigma M} V_{\Sigma M} M_u^\star U^\star. \qquad (17)$$

From Eqs. (13) and (17) it can be easily seen that the condition numbers of the eigenvector matrices of GK and KG are both equal to $\mathscr{C}[W_{\Sigma M}]$. □

Theorem 2 shows that with the precompensator structure given by Eq. (8) it is possible to approximately normalize $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$, simply by seeking a precompensator $K_p(j\omega)$ that approximately normalizes $G(j\omega)K_p(j\omega)$ at a given frequency ω. Thus, in this paper, $K_p(j\omega)$ will have the structure given by Eq. (8).

Remark 1: Notice that if $G(j\omega)K_p(j\omega)$ is approximately normal, for $K_p(j\omega) = \alpha(j\omega)K_u(j\omega)$, then $G(j\omega)K_u(j\omega)$ is also approximately normal and vice-versa. Therefore, although the complex number $\alpha(j\omega)$ may not be equal to one, for simplicity, in the rest of this section it will be assumed that $K_p(j\omega) = K_u(j\omega)$. As it will be seen in the next section, the choice of $\alpha(j\omega)$ plays a key role in the approximation of the desired frequency response of $K_p(j\omega)$ by a rational and stable transfer matrix. □

Supposing that $K_p(j\omega) = K_u(j\omega)$ then, according to theorem 1, the normality of $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$ is achieved if and only if $U(j\omega)^\star K_u(j\omega)Y(j\omega)$ is diagonal. Moreover, it can be proven that the approximate normality of $G(j\omega)K_u(j\omega)$ is related to an approximate diagonal form of $U(j\omega)^\star K_u(j\omega)Y(j\omega)$. In order to show this, it is first necessary to introduce the concept of an aligned matrix, as follows.

Definition 1: Let G be a complex matrix. If all possible singular value decompositions of $G = Y\Sigma U^\star$, are such that Y, U are aligned, *i.e.*

$$U^\star Y = e^{j\Theta} \qquad (18)$$

where $\Theta = diag\{\theta_i, i = 1,\ldots,m\}$, then, G is said to be aligned. □

Definition 1 leads to the following result.

Lemma 4: Let G be a complex matrix with distinct singular values. Then, G is aligned if and only if G is normal.

Proof: See [8]. □

From lemma 4, it is possible to state the following theorem.

Theorem 3: Let K_u be a unitary matrix and let $G = Y\Sigma U^\star$ be a singular value decomposition of G, where all singular values are distinct. Then, the product GK_u is normal, if and only if there exists $\Theta = diag\{\theta_1, \ldots, \theta_m\}$, such that

$$U^\star K_u Y = e^{j\Theta}. \tag{19}$$

Proof: Let the product GK_u be a normal matrix. Then $GK_u = Y\Sigma U^\star K_u$ and, since by assumption K_u is a unitary matrix, $U^\star K_u$ is also a unitary matrix. Therefore $GK_u = Y\Sigma\bar{U}^\star$, where $\bar{U}^\star = U^\star K_u$, may be seen as a singular value decomposition of GK_u. According to lemma 4, since the matrix GK_u is normal and since G has distinct singular values, it is also aligned. Thus, $\bar{U}^\star Y = e^{j\Theta}$, namely, $U^\star K_u Y = e^{j\Theta}$.

The converse can be easily proved by noticing that if equality $U^\star K_u Y = e^{j\Theta}$ is satisfied then GK_u is normal. □

Theorem 3 shows that, when all singular values of $G(j\omega)$ are distinct at a given frequency, the diagonal form of $U(j\omega)^\star K_u(j\omega)Y(j\omega)$ is also a necessary and sufficient condition for $G(j\omega)K_u(j\omega)$ to be normal, *i.e.*, $G(j\omega)K_u(j\omega)$ is normal if and only if it is aligned. Therefore, if $K_u(j\omega) = U(j\omega)e^{j\Theta}Y(j\omega)^\star$ for any diagonal matrix Θ, the static precompensator $K_p(j\omega) = \alpha(j\omega)K_u(j\omega)$, normalizes the plant. However, this precompensator cannot be, in general, approximated at the necessary frequency range by a dynamic precompensator $\bar{K}_p(s)$, with all entries being chosen as stable and proper rational transfer functions as is done in [3]. This suggests that the search of an exact alignment between $K_u(j\omega)^\star U(j\omega)$ and $Y(j\omega)$ should be replaced by the search of a unitary matrix $K_u(j\omega)$ that makes $K_u(j\omega)^\star U(j\omega)$ and $Y(j\omega)$ approximately aligned. Therefore, it is necessary to define a measure of the deviation of $G(j\omega)$ from alignment. From Eq. (18) a natural definition of a measure of deviation from alignment is as follows[2]:

$$m(G) = \min_{\Theta} \|U^\star Y - e^{j\Theta}\|^2_{\mathscr{F}}, \tag{20}$$

where $m(G) = 0$ when $G(j\omega)$ is aligned. The following result shows that if $G(j\omega)$ is approximately aligned, *i.e.* if $m(G) \to 0$, then $G(j\omega)$ is approximately normal, in the sense that $\delta(G) \to 0$.

Theorem 4: According to the measures defined in Eqs. (2) and (20) then, if $m(G) \to 0$ then $\delta(G) \to 0$.

Proof: Let $G = Y\Sigma U^\star$ be a singular value decomposition of G. Therefore,

$$\begin{aligned}\|GG^\star - G^\star G\|_{\mathscr{F}} &= \|Y\Sigma U^\star U\Sigma Y^\star - U\Sigma Y^\star Y\Sigma U^\star\|_{\mathscr{F}} \\ &\leq 2\|\Sigma^2\|_{\mathscr{F}}\|U^\star Y - e^{j\Theta}\|_{\mathscr{F}}.\end{aligned}$$

Dividing both sides of the inequality above by $\|G^\star G\|_{\mathscr{F}}$ and using the fact that $\|\Sigma^2\|_{\mathscr{F}} = \|G^\star G\|_{\mathscr{F}}$ yields $\delta(G) \leq 4m(G)$, which completes the proof. □

Theorem 4 shows that if there exists a unitary matrix $K_u(j\omega)$ that makes $G(j\omega)K_u(j\omega)$ approximately aligned, then $G(j\omega)K_u(j\omega)$ is also approximately normal. Therefore, from theorems 1, 2, 3 and 4, the problem of designing a precompensator that approximately normalizes a plant, at a given frequency, can be formulated as follows:

$$\textbf{Prob. 1: } \min_{K_u}\min_{\Theta} J(K_u, \Theta), \tag{21}$$

where

$$J(K_u, \Theta) = \|U^\star K_u Y - e^{j\Theta}\|^2_{\mathscr{F}}, \tag{22}$$

subject to K_u be a unitary matrix and each entry of the main diagonal of Θ, $\theta_i \in (0\,, 2\pi]$, for $i = 1, \ldots, m$.

B. Solution of the optimization problem

Using the definition of Frobenius norm and after some straightforward manipulation, Eq. (22) can be written as:

$$J(K_u, \Theta) = 2m - 2Re\{tr(U^\star K_u Y e^{-j\Theta})\}. \tag{23}$$

Defining the unitary matrices $T = U^\star K_u Y$ and $H = U^\star K_u Y e^{-j\Theta}$ and denoting each element of H by h_{ij}, then Eq. (23) can be re-written as:

$$J(K_u, \Theta) = 2m - 2\left[Re(h_{11}) + Re(h_{22}) + \ldots + Re(h_{mm})\right]. \tag{24}$$

From Eq. (23), it can be easily seen that each element $e^{-j\theta_i}$ of the main diagonal of $e^{-j\Theta}$ multiplies the *i-th* column of T. Denoting each element of T by t_{ij}, then Θ that minimizes the cost function $J(K_u, \Theta)$ given by Eq. (24) is such that each complex number $h_{ii} = t_{ii}e^{-j\theta_i}$ must be real and positive. Thus, defining $t_{ii} = |t_{ii}|e^{j\alpha_i}$, then the optimum θ_i, $i = 1, 2, \ldots, m$ will be given by $\theta_i = \alpha_i + 2\kappa\pi$, where $\kappa \in \mathbb{Z}$. Therefore, the minimum value for $J(K_u, \Theta)$ depends only on K_u and is given by:

$$J_{min}(K_u) = 2m - 2\left[|t_{11}| + |t_{22}| + \ldots + |t_{mm}|\right]. \tag{25}$$

Consequently, according to Eq. (25), the optimization problem 1, is equivalent to:

$$\textbf{Prob. 2: } \max_{K_u} J_{max}(K_u). \tag{26}$$

where

$$J_{max}(K_u) = \sum_{i=1}^{m} |t_{ii}|. \tag{27}$$

Notice, according to Eq. (27) and the definition of T, that the precompensator K_u that solves problem 2 makes the unitary matrix T as diagonally dominant as possible. In this paper, the structure adopted for K_u will be of a permuted diagonal matrix as done in [3] and will be given as:

$$K_u = P_l K_d, \tag{28}$$

where P_l ($l = 1, \ldots, m!$) is a matrix formed with all possible permutation of the columns of the identity matrix (permutation matrix) and K_d is a diagonal matrix where its main diagonal entries are equal to 1 or -1 to guarantee that K_u is unitary; notice that, since multiplication by -1 does not change the unitary nature of a matrix, it is possible to form

[2]This measure of alignment is the same as that used in [8] with a different norm.

2^{m-1} matrices K_d. This choice of K_u is motivated by the fact that precompensation using permutation matrices is usual in the design of multivariable controllers via Nyquist Array Methods [13], [6] in the attainment of diagonal dominance. The use of other structures for K_u will be the subject of future research.

C. Precompensator implementation

Once $K_u(j\omega)$ has been computed for each frequency ω in the necessary frequency range, then the next step is the design of a dynamic normalizing precompensator $\bar{K}_p(s)$. Notice that $K_u(j\omega)$ is real and unitary, while $\bar{K}_p(s)$ must have as entries only rational and stable transfer functions. This shows the need for adding phase and modulus to each nonzero entry of $K_u(j\omega)$. Notice, however, that the static precompensator K_p has been made equal to K_u only for simplicity (remark 1), being actually equal to $K_p(j\omega) = \alpha(j\omega)K_u(j\omega)$. Therefore, choosing appropriately the values of $\alpha(j\omega)$, it is possible to add the same phase and modulus to each nonzero entry of $K_u(j\omega)$ with the view to approximating them to the frequency response of a rational and stable transfer function. In practice, however, it is not necessary to choose $\alpha(j\omega)$ in order to compute the entries of $\bar{K}_p(s)$. Notice that this can be done by choosing, for a certain frequency band, the same transfer functions for the entries of $\bar{K}_p(s)$ associated with the nonzero entries of $K_u(j\omega)$ at this frequency band, such that their moduli approximately match the moduli of the nonzero entries of $K_u(j\omega)$. This procedure leads to a dynamic precompensator $\bar{K}_p(s)$ that approximately aligns the precompensated plant $G(j\omega)\bar{K}_p(j\omega)$ providing that $K_u(j\omega)$ also aligns $G(j\omega)K_u(j\omega)$. An exception is made at the vicinity of the frequencies where the frequency response moduli of K_u jump from 1 to 0 or from 0 to 1, since at these frequencies the moduli of the entries of $\bar{K}_p(j\omega)$ are different from those of $K_u(j\omega)$. This problem can be overcome by increasing the order of the pole or zero associated with the frequency where the jump occurs; although at the expenses of an increase in the order of the precompensator. In this paper, with the view to making the precompensated system with the lowest possible order, only approximations by lead/lag transfer functions will be used to obtain the rational precompensator $\bar{K}_p(s)$.

The procedure to obtain the normalizing precompensator can be summarized in the following algorithm.

Algorithm 1:

1) Form the 2^{m-1} diagonal matrices K_d of dimension m with either 1 or -1 in its main diagonal.
2) Select a finite number of frequencies ω_k, $k = 1,2,\ldots,n$ and set $l = 1$ and $k = 1$.
3) If $l = 1$, choose a permutation matrix P_l. If $l > 1$ form a different permutation matrix P_l from the other permutation matrices already formed.
4) Using P_l defined in step 3, compute, for each one of the 2^{m-1} matrices K_d, defined in step 1, $T = U^\star P_l K_d Y$ and the cost function $J_{max}(P_l, K_d) = \sum_{i=1}^{m} |t_{ii}|$. Find $J_{max}(l) = \max_{K_d} J_{max}(P_l, K_d)$ and the matrix K_d^{max} which leads to $J_{max}(l)$.
5) Make $l = l+1$ and repeat steps 3 and 4 until $l = m!$.
6) Among all l values of $J_{max}(l)$, computed in step 4, choose $J_{max} = \max_l J_{max}(l)$ and select the matrices K_d^{opt} and P_l^{opt} which leads to J_{max}. Form $K_u(j\omega_k) = P_l^{opt} K_d^{opt}$.
7) Set $k = k+1$ and $l = 1$ and go back to step 3. Repeat steps 3 to 7 until $k = n$.
8) Find rational and stable transfer functions for each entry of $\bar{K}_p(s)$, such that the magnitude of the frequency response of its entries approximately match those of $K_u(j\omega_k)$, for $k = 1,\ldots,n$. □

IV. EXAMPLE

Let the transfer function matrix of the linearized model of the vertical plane dynamics of an aircraft be given by [13]:

$$G(s) = \frac{1}{d(s)}N(s), \tag{29}$$

where $N(s) = [n_{ij}(s)]$, $i,j = 1,2,3$, and $d(s)$ are given as:

$$\begin{aligned}
n_{11}(s) &= -1.5750s^3 - 1.1190s^2 + 1.5409s - 0.0816\\
n_{12}(s) &= 0.2909s^2 + 0.2527s + 0.3712\\
n_{13}(s) &= 0.0732s^3 - 0.0646s^2 - 1.2125s - 0.0204\\
n_{21}(s) &= -0.12s^4 - 0.0739s^3 - 0.5319s^2 - 0.2458s\\
n_{22}(s) &= s^4 + 1.5415s^3 + 1.6537s^2\\
n_{23}(s) &= -0.0052s^3 + 0.1570s^2 + 0.1828s\\
n_{31}(s) &= 4.419s^3 + 1.6674s^2 + 0.1339s\\
n_{32}(s) &= 0.0485s^2 + 0.3279s\\
n_{33}(s) &= -1.6650s^3 - 1.1574s^2 - 0.0918s\\
d(s) &= s^5 + 1.5953s^4 + 1.7572s^3 + 0.1112s^2 + 0.0561s.
\end{aligned}$$

In order to use the CLM to design a commutative controller for $G(s)$, it is first necessary to verify if $G(s)$ is close to normal in the necessary frequency range. This can be done by computing the measure of normality, defined in Eq. (2), and the condition number of the eigenvector matrix of G. It can be seen from Figs. 2 (a) and (b) (dashed lines) that $G(j\omega)$ is far from normal at low and high frequencies. It is important also to note that at very high frequencies G becomes normal, which occurs because $G(j\omega) \to O$ when $\omega \to \infty$. Moreover, in Fig. 2 (c) (dashed-line) it can also be seen that $G(j\omega)$ is far from aligned at all frequencies. Therefore, it is necessary to design a normalizing precompensator for $G(s)$.

The precompensator design is carried out in accordance with algorithm 1. Since $m = 3$ (the dimension of $G(s)$), the first step is to form $2^{m-1} = 4$ diagonal matrices K_d with 1 and -1 in its main diagonal. In the sequel, it is necessary to form $l = m! = 6$ permutation matrices P_l; thus steps 3 and 4 of algorithm 1 will be repeated 6 times for each matrix K_d obtained in step 1. The next step is to obtain K_d^{opt} and P_l^{opt} for each frequency point in the frequency range, leading to the desired unitary matrix $K_u(j\omega)$. Fig. 3 (x-marked lines) shows the magnitude of each entry of $K_u(j\omega)$. Notice that, for such a $K_u(j\omega)$ the precompensated plant GK_u is approximately aligned for almost all frequencies as can be seen from Fig. 2 (c) (dash-dotted line). This implies, according to theorem 4, that GK_u is also approximately normal at the same frequencies as can be seen from Figs. 2 (a) and (b) (dash-dotted line). It is also important to remark that the condition number of the eigenvector matrix of GK_u

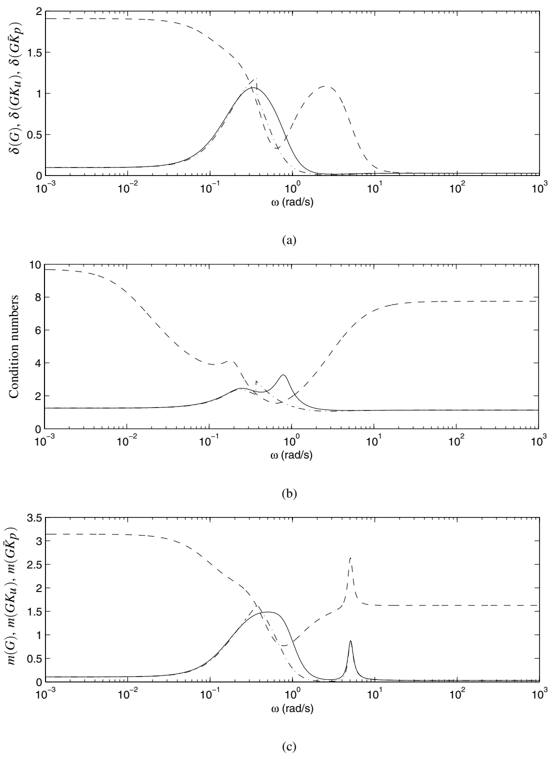

Fig. 2. (a) Measures of normality: $\delta(G)$ (dashed line), $\delta(GK_u)$ (dash-dotted line) and $\delta(G\bar{K}_p)$ (solid line); (b) Condition numbers of the eigenvector matrices of $G(j\omega)$ (dashed line), $G(j\omega)K_u(j\omega)$ (dash-dotted line) and $G(j\omega)\bar{K}_p(j\omega)$ (solid line); and (c) Measures of alignment: $m(G)$ (dashed line), $m(GK_u)$ (dash-dotted line) and $m(G\bar{K}_p)$ (solid line).

is for most of the frequency range smaller than 1.3, which represents a significant improvement on the normality of G. Similar conclusions could be drawn from the analysis of $\delta(GK_u)$ according to Fig. 2 (a) (dash-dotted line).

The final step in the design (step 8) is to find stable transfer functions for each entry of $\bar{K}_p(s)$, such that the frequency response magnitude of its entries approximately match the nonzero entries of K_u shown in Fig. 3 (x-marked lines) for each frequency. Notice that, in this example, the entries of $\bar{K}_p(s)$ can be chosen to be first order transfer functions, such that the entries of $\bar{K}_p(j\omega)$ approximately match the nonzero entries of K_u at low frequencies, and vanish at high frequencies, and another transfer function that approximately match the nonzero entries of K_u at high frequencies, and vanishes at low frequencies. A dynamic precompensator that satisfies these requirements is given by:

$$\bar{K}_p(s) = \begin{bmatrix} 0 & \frac{0.1}{s+0.1} & \frac{s}{s+5} \\ \frac{0.1}{s+0.1} & \frac{s}{s+5} & 0 \\ \frac{s}{s+5} & 0 & \frac{0.1}{s+0.1} \end{bmatrix}. \tag{30}$$

Notice that there is a close agreement between the magnitudes of each entry of $\bar{K}_p(j\omega)$ and $K_u(j\omega)$, which leads to a $\bar{K}_p(j\omega)$ approximately unitary at low and high frequencies.

Fig. 2 (c) shows that $G\bar{K}_p$ is also approximately aligned at almost all frequencies (solid line) and therefore $G\bar{K}_p$ is, as expected, approximately normal at these frequencies; the same conclusion can be drawn from Figs. 2 (a) and (b).

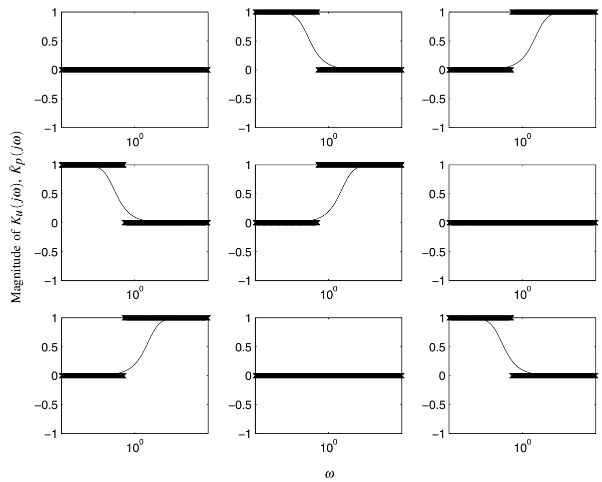

Fig. 3. Magnitude for the optimum $K_u(j\omega)$ (x-marked line) and for the rational approximation $\bar{K}_p(j\omega)$ (solid line).

It is also important to remark that, except at intermediate frequencies (at the vicinity of the frequency where there are jumps in the elements of K_u), the measures of normality and misalignment of $G(j\omega)\bar{K}_p(j\omega)$ are very close to the desired one.

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