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Mechanical systems and rendez-vous controllability

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Abstract— **Motivated by the study of the controlled Kepler problem, we analyze the controllability properties of some classes of mechanical systems. Consider a control system with drift. Our aim is to determine if, given a pair of initial states at a fixed initial time, assuming that the controller acts only on one of the two corresponding trajectories, it is possible for the controlled trajectory to reach the uncontrolled one in finite time. We prove that this is the case for the controlled Kepler problem, taking into account both elliptic and non-elliptic orbits. We extend the result to other classes of mechanical controlled systems.**

I. Introduction

The study of low-thrust transfer between elliptic orbits of the Kepler problem stimulates a great attention, between specialists and non-specialists, due mainly to its potential applications to the space industry. For many practical satellite-based technologies, including telecommunications, the actual goal of the orbit transfer is to join with a satellite endowed with electro-ionic engines an assigned geostationary orbit *at a precise longitude*. This special kind of transfer is usually called *rendez-vous*. Similar problems, where the rendez-vous is targeted at elliptic orbits which are not necessarily geostationary, are frequently met in the domain of formation flying.

The motivation of the present work is to better frame the concept of rendez-vous and to extend it to more general control systems. The notion of *rendez-vous controllability* is easily formulated: Given a control system $\dot{q} = f(q, u)$, $q \in M$, $u \in U$, and fixed a control $\bar{u} \in U$ which plays the role of *basic* dynamics, we say that the system is rendez-vous controllable if, for every pair of states (q_0, q_1), there exists an admissible trajectory $q(\cdot)$, defined on an interval $[0, T]$, T being free, such that $q(0) = q_0$ and $q(T) = \phi(T, q_1, \bar{u})$, where $t \mapsto \phi(t, q_1, \bar{u})$ is the solution of $\dot{q} = f(q, \bar{u})$ such that $\phi(0, q_1, \bar{u}) = q_1$.

Control problems for which the controller's goal is to perform such kind of rendez-vous arise naturally, and are in fact currently handled. We can think, for instance, of any control system where uncontrolled trajectories are determined by some drift. The problem of reaching an "object" moving according to the dynamics determined by such drift is of intrinsic rendez-vous nature.

Rendez-vous controllability problems can be seen as classical controllability problems by adding the time as an extra variable to the original system. However, we belive it useful to acknowledge the specificity of such kind of problems and to treat them as elements of a common class.

Our attention to rendez-vous controllability moved from a quite natural question about the controlled Kepler problem: Is the Kepler problem rendez-vous controllable on the totality of elliptic and non-elliptic orbits? The non-elliptic part of the question becomes relevant when we consider, for instance, the problem of sending an exploratory device to an asteroid passing close to the Earth. As long as the gravitational field of the Earth is predominant, the asteroid moves approximatively on an hyperbolic trajectory for the Kepler problem centered at the Earth.

The answer to the rendez-vous controllability issue is positive, as it is shown in Section III. We stress that the proof follows constructive guidelines. Indeed, an explicit strategy is proposed to join the domain of elliptic orbits to a target hyperbolic one and then to reach the desired "longitude" on it. Efficient transfer strategies between elliptic orbits are not discussed here. For a feedback control aimed at the rendez-vous to a geostationary orbit, we refer to the recent work by Kellett and Praly [1].

In Section IV we generalize the results obtained for the Kepler problem to more general controlled problems. Such problems are of the same mechanical nature as the Kepler one, in the sense that the control is assumed to act as an external acceleration. The aim is less to provide the widest possible result than to suggest the possible applications of the method. The recurrence hypothesis which is asked on bounded trajectories of the uncontrolled system is suggested by the well-known properties of Hamiltonian systems (namely, the fact that Hamiltonian flows are volume-preserving).

An important feature of the method, which reflects the original low-thrust assumption, is that it is independent on the maximal size of admissible controls. Physically, this corresponds to the assumption that the controller has an arbitrarily small allowed acceleration capacity, which is independent, however, of the time and of the state. Such feature gives rise to the notion of *unrestricted rendez-vous controllability*, and explains the qualitative/geometrical, rather than quantitative, arguments which are presented. The corresponding notion of *unrestricted complete controllability* was introduced in [2] for a class of Dubins'-like control problems on Riemannian manifolds. Similarly to the problems treated here, Dubins'-like control problems are characterized by the fact that the control plays the role of an external acceleration.

Future extensions of the present work are expected to go in the direction of unifying the results presented here

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with those of [2] and [3], considering systems defined on submanifolds of tangent bundles of Riemannian manifolds. In particular, results as [3, Proposition 4.1] suggest that controllability can be recovered from recurrence assumptions on the projection of uncontrolled trajectories on the base manifold, instead that on trajectories on the tangent bundle (as done in Section IV). Such kind of extensions would surely enrich the theory and definitely enlarge its horizons beyond Kepler-like applications.

II. Definitions and basic facts

Let M be a smooth ($\mathcal{C}^\infty$) manifold and U be a measurable subset of $\mathbf{R}^m$, $m \geq 1$. Consider a map $f : M \times U \to TM$ and assume that f is the restriction on $M \times U$ of a smooth map from $M \times \mathbf{R}^m$ to TM. Let, for every $u \in U$, $f_u(\cdot) = f(\cdot, u)$ be a vector field on M.

Recall that a vector field $g : M \to TM$ is called *complete* if all the solutions of the dynamical system $\dot{q} = g(q)$ are defined on the entire line $\mathbf{R}$. For every complete vector field g on M, we write $e^{tg} : M \to M$ to denote the flow associated with g at time t, that is, $t \mapsto e^{tg}(q)$ is the solution to $\dot{q} = g(q)$ passing through the point q at time $t = 0$.

An *admissible control* for the control system

$$\dot{q} = f(q, u) \tag{1}$$

is, by definition, a measurable essentially bounded function $u : [0, +\infty) \to U$. An *admissible trajectory* for (1) is a solution of (1) corresponding to an admissible control.

A classical notion of controllability for (1) is given by complete controllability: We say that (1) is *completely controllable* if, for every $q_0, q_1 \in M$, there exist $T \geq 0$ and an admissible trajectory $q : [0, T] \to M$ such that $q(0) = q_0$ and $q(T) = q_1$.

A controllability notion focused more on orbits, instead of points, is introduced by the following definition.

Definition 2.1: We say that (1) is *rendez-vous controllable* with respect to $\bar{u} \in U$ if $f_{\bar{u}}$ is complete and, for every $q_0, q_1 \in M$, there exist $T \geq 0$ and an admissible trajectory $q : [0, T] \to M$ such that $q(0) = q_0$ and $q(T) = e^{T f_{\bar{u}}}(q_1)$.

Rendez-vous controllability does not imply, in general, complete controllability. A simple counterexample is given by the control system $\dot{q} = 1 + u$, $q \in \mathbf{R}$, $|u| < 1$. As $\bar{u}$ one can take any of the admissible controls.

The definition of rendez-vous controllability makes sense particularly when there exists a control $\bar{u}$ which corresponds to a physical drift, as it is the case for controlled mechanical systems. Notice, by the way, that it can be useful for applications – and completely straightforward – to extend the definition of rendez-vous controllability to the case in which $\bar{u}$ is replaced by a non-constant admissible control.

In the present work, we restrict our attention to systems of the type

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\,, \\ \dot{v} & = & \psi(x, v) + \Gamma(x, v, u)\,, \end{array} \right. \tag{2}$$

where (x, v) belongs to an open subset Ω of $\mathbf{R}^{2n}$.

We assume, moreover, that there exists a positive real number ε such that the closed ball in $\mathbf{R}^n$ centered at the origin of radius ε, denoted by B^n_ε, is contained in $\Gamma(x, v, U)$ for every $(x, v) \in \Omega$. In order to prove that (2) is rendez-vous controllable, we simplify the notations by assuming that $U = B^n_\varepsilon$ and $\Gamma(x, v, u) = u$, that is, we focus our attention on control systems of the form

$$(\Sigma_\varepsilon) \ : \qquad \left\{ \begin{array}{rcl} \dot{x} & = & v\,, \\ \dot{v} & = & \psi(x, v) + u\,, \end{array} \right. \qquad u \in B^n_\varepsilon\,.$$

An important feature of the arguments developed in the next two sections is that they do not depend on the size of ε. This leads to the notion of *unrestricted* rendez-vous controllability, in analogy with the corresponding notion of unrestricted complete controllability, introduced in [2], [3] and recalled below.

Definition 2.2: We say that $\varepsilon \mapsto (\Sigma_\varepsilon)$ has the *unrestricted rendez-vous controllability property* (equivalently, that it is URVC) if, for every $\varepsilon > 0$, (Σ_ε) is rendez-vous controllable with respect to 0. Similarly, we say that $\varepsilon \mapsto (\Sigma_\varepsilon)$ has the *unrestricted complete controllability property* if, for every $\varepsilon > 0$, (Σ_ε) is completely controllable.

Let us recall the classical notion of recurrence. We say that a vector field g on M is *recurrent at* $q \in M$ if there exists a sequence of positive times $\{t_n\}_{n \in \mathbf{N}}$ converging to infinity and such that $e^{t_n f}(q) \to q$ as n tends to infinity. The vector field g is called *recurrent* on M, if it is recurrent at every point in a dense subset of M.

III. The controlled Kepler system is URVC

Let $\mathbf{R}^3_0 = \mathbf{R}^3 \setminus \{0\}$. Define $L : T\mathbf{R}^3_0 = \mathbf{R}^3_0 \times \mathbf{R}^3 \to \mathbf{R}^3$ by

$$L(x, v) = x \times v\,,$$

where by " $\times$ " we denote the vector product in $\mathbf{R}^3$.

Define

$$\Omega = \{(x, v) \in T\mathbf{R}^3_0 \mid L(x, v) \neq 0\}\,.$$

The *controlled Kepler system* is the control system, defined on Ω,

$$(K_\varepsilon) \ : \qquad \left\{ \begin{array}{rcl} \dot{x} & = & v\,, \\ \dot{v} & = & -\mu \frac{x}{\|x\|^3} + u \end{array} \right. \qquad u \in B^3_\varepsilon\,,$$

where μ is a positive constant.

In this section we prove that

Proposition 3.1: The Kepler system $\varepsilon \mapsto (K_\varepsilon)$ is URVC.

Let $\psi(x) = -\mu \frac{x}{\|x\|^3}$ and define, for every $(x, v) \in \Omega$,

$$f_0(x, v) = (v, \psi(x))\,, \tag{3}$$

$$E(x, v) = \frac{1}{2}\|v\|^2 - \frac{\mu}{\|x\|}\,. \tag{4}$$

Notice that f_0 is a complete vector field on Ω (a necessary condition for the URVC of the Kepler system).

We can restrict (K_ε) to $\mathcal{D} = \{(x, v) \in \Omega \mid E(x, v) < 0\}$, the union of the supports of all elliptic non-degenerate trajectories. Notice that also the restriction of f_0 to $\mathcal{D}$ is complete.

Let us recall a classical controllability result related to recurrence (see [4], [5]). Let $X_0,\dots,X_m$ be $m+1$ vector fields on a smooth manifold M. The *Lie algebra generated by* $X_0,\dots,X_m$ is the family of vector fields

$$\begin{aligned}\mathrm{Lie}(X_0,\dots,X_m) = \\ \quad \mathrm{span}\{[X_{j_k},[X_{j_{k-1}},[\dots,X_{j_1}]\cdots] \mid \\ \quad k\geq 1,\ j_1,\dots,j_k\in\{0,\dots,m\}\}\end{aligned}$$

where $[X_i,X_j]$ denotes the Lie bracket between X_i and X_j. Assume that $\mathrm{Lie}(X_0,\dots,X_m)$ spans T_qM at every point $q\in M$. (In this case, we say that the family of vector fields $\{X_0,\dots,X_m\}$ is *bracket generating*.) Assume, moreover, that X_0 is recurrent on M. Then the control system $\dot q = X_0(q)+\sum_{i=1}^m u_iX_i(q)$, $u\in B_\varepsilon^m$, is completely controllable for every $\varepsilon>0$.

Caillau, in his PhD thesis [6], noticed that such result implies the complete controllability of (K_ε) restricted to $\mathcal{D}$, for every $\varepsilon>0$. The proof of the rendez-vous controllability of the restricted system needs only some minor modifications. First, consider as X_0 the vector field

$$\begin{array}{ccc}\mathbf{R}\times\mathcal{D} & \longrightarrow & \mathbf{R}\times\mathbf{R}^3\times\mathbf{R}^3\\ (t,x,v) & \longmapsto & (1,v,\psi(x))\end{array}$$

and, for $i=1,2,3$, let $X_i(t,x,v)=(0,0,e_i)$, where $\{e_1,e_2,e_3\}$ is an orthonormal basis of $\mathbf{R}^3$.

Let $\tau:\mathcal{D}\to(0,+\infty)$ be the map associating with $(x,v)\in\mathcal{D}$ the minimal period of the uncontrolled trajectory of (K_ε) passing through (x,v). The precise expression for τ is given by the third Kepler law, that is,

$$\tau(x,v)=2\pi\sqrt{\frac{a^3}{\mu}}\,,$$

where a is the semi-major axis of the elliptic trajectory with initial conditions (x,v), that is,

$$a=-\frac{2}{\mu}E(x,v)=\frac{2}{\|x\|}-\frac{\|v\|^2}{\mu}\,.$$

What matters most to our approach is the smoothness of τ with respect to (x,v), which guarantees that the quotient of $\mathbf{R}\times\mathcal{D}$ by the equivalence relation

$$\begin{aligned}&(t_1,x_1,v_1)\sim(t_2,x_2,v_2)\Longleftrightarrow\\ &\quad t_1-t_2\in\tau(x_1,v_1)\mathbf{Z}\,,\quad x_1=x_2\,,\quad v_1=v_2\end{aligned} \tag{5}$$

is a well-defined smooth manifold M. Moreover, the vector fields $X_0,\dots,X_3$ are constant on each equivalence class defined by $\sim$. Therefore, with a slight abuse of notation, we can consider $X_0,\dots,X_3$ as well-defined smooth vector fields on M. It is easy to check that the Lie algebra generated by $X_0,\dots,X_3$ spans T_qM at every point $q\in M$. The periodicity (and, a fortiori, recurrence) of X_0 implies the complete controllability of the control-affine control system defined by $X_0,\dots,X_3$ on M.

The rendez-vous controllability of (K_ε) restricted to $\mathcal{D}$ follows: Indeed, given $(x_0,v_0),(x_1,v_1)\in\mathcal{D}$, the existence of an admissible trajectory $(x,v):[0,T]\to\mathcal{D}$ such that $(x,v)(0)=(x_0,v_0)$ and $(x,v)(T)=e^{Tf_0}(x_1,v_1)$ is equivalent to the existence of an admissible trajectory in M from $[(0,x_0,v_0)]$ to

$$\begin{aligned}&\left\{\left[\left(t,e^{tf_0}(x_1,v_1)\right)\right]\,\middle|\, t\geq 0\right\}=\\ &\qquad\left\{\left[\left(t,e^{tf_0}(x_1,v_1)\right)\right]\,\middle|\, t\in[0,\tau(x_1,v_1)]\right\}\,,\end{aligned}$$

where square brackets denote equivalence classes for the equivalence relation defined by (5).

Let us step back for the moment from rendez-vous controllability, and prove that (K_ε) is completely controllable on Ω. Fix $(x_0,v_0),(x_1,v_1)\in\Omega$. We want to show that there exists an admissible trajectory $(x,v):[0,T]\to\Omega$ such that $(x,v)(0)=(x_0,v_0)$ and $(x,v)(T)=(x_1,v_1)$.

Consider the case where $(x_1,v_1)\in\mathcal{D}$. Due to the complete controllability of (K_ε) on $\mathcal{D}$, it is enough to prove that there exists an admissible trajectory $(x,v):[0,T]\to\Omega$ such that $(x,v)(0)=(x_0,v_0)$ and $(x,v)(T)\in\mathcal{D}$. The idea is to apply the feedback

$$u=-\varepsilon\frac{v}{\|v\|}\,. \tag{6}$$

Denote by $(x,v)(\cdot)$ the trajectory in Ω corresponding to (6) and such that $(x,v)(0)=(x_0,v_0)$. Let $[0,T)$ be the largest interval of definition of $(x,v)(\cdot)$, $T\in(0,+\infty]$. If T is finite, then $L(x(t),v(t))\to 0$ as t tends to T, since neither $x(t)$ nor $v(t)$ can explode in finite time. Since

$$\frac{d}{dt}L(x(t),v(t))=-\frac{\varepsilon}{\|v(t)\|}L(x(t),v(t))\,,$$

then the only possibility is that $v(t)\to 0$ as t tends to T. Then, for t close to T, $E(x(t),v(t))$ is negative, which means that $(x(t),v(t))$ belongs to $\mathcal{D}$, and we are done. As for the case $T=+\infty$, notice that

$$\frac{d}{dt}E(x(t),v(t))=-\varepsilon\|v(t)\|\,.$$

Assume by contradiction that $(x,v)(\cdot)$ never enters $\mathcal{D}$. Hence, $\lim_{t\to\infty}E(x(t),v(t))\geq 0$, which implies that $\liminf_{t\to\infty}\|v(t)\|=0$. But, if $\|v(t)\|\ll\varepsilon$, then $E(x(t),v(t))\ll\varepsilon$ and $1/\|x(t)\|\ll\varepsilon$. This last inequality, in turns, implies that

$$\begin{aligned}\frac{d^2}{dt^2}E(x(t),v(t)) &= -\varepsilon\frac{v(t)\cdot\dot v(t)}{\|v(t)\|}\\ &= \varepsilon\mu\frac{x(t)\cdot v(t)}{\|x(t)\|^3\|v(t)\|}-\varepsilon^2\simeq-\varepsilon^2\,,\end{aligned}$$

where by " $\cdot$ " we denote the scalar product. It is easy to conclude that the assumption that $E(x(t),v(t))\geq 0$ for every t leads to a contradiction.

We proved that, for every $(x_0,v_0)\in\Omega$, there exists an admissible trajectory for (K_ε) which steers (x_0,v_0) to $\mathcal{D}$.

Notice now that, if $t\mapsto(x(t),v(t))$ is an admissible trajectory for (K_ε), then $t\mapsto(x(-t),-v(-t))$ also is. Therefore, reversing the time in the above argument, we can conclude that for every $(x_1,v_1)\in\Omega$ there exists an admissible trajectory of (K_ε) which joins an element of $\mathcal{D}$

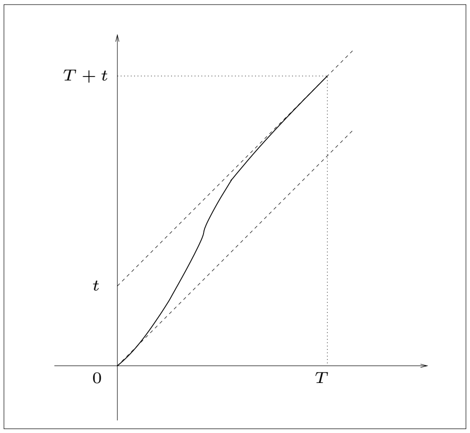

Fig. 1. The graph of λ

to (x_1, v_1). This completes the proof that $\varepsilon \mapsto (K_\varepsilon)$ has the unrestricted controllability property.

The final step in order to prove unrestricted rendez-vous controllability is to show that, given $\varepsilon > 0$, $q_0 = (x_0, v_0) \in \Omega$ and $t \in \mathbf{R}$, there exists an admissible trajectory $q : [0,T] \to \Omega$ such that $q(0) = q_0$ and $q(T) = e^{(T+t)f_0}(q_0)$. If $q_0 \in \mathcal{D}$, this has already been proved to follow from the recurrence of X_0. Let then $q_0 \in \Omega \setminus \mathcal{D}$. In particular, for every $c > 0$, we can assume that $\|\psi(e^{\tau f_0}(q_0))\| \leq c$ for every $\tau \geq 0$ (simply replace q_0 by $e^{\bar{\tau} f_0}(q_0)$ with $\bar{\tau} > 0$ large enough). Choose $T > 0$ and $\lambda \in C^\infty([0,T],[0,t+T])$ such that $\lambda(0) = 0$, $\dot{\lambda}(0) = 1$, $\lambda(T) = T+t$, and $\dot{\lambda}(T) = 1$ (see Figure 1).

Notice that, for every $\delta > 0$, we can assume that $|\dot{\lambda}(\tau) - 1|, |\ddot{\lambda}(\tau)| \leq \delta$ for every $\tau \in [0,T]$ (fixing T in dependence on δ).

Define $\xi(\cdot)$ and $\nu(\cdot)$ through the relation

$$(\xi(\tau), \nu(\tau)) = e^{\tau f_0}(q_0), \qquad \tau \in [0, t+T].$$

We claim that $q(\cdot)$ can be defined as

$$q(\tau) = (\xi(\lambda(\tau)), \dot{\lambda}(\tau)\nu(\lambda(\tau))). \tag{7}$$

By definition, $q(\cdot)$ satisfies the required boundary conditions. The time-derivative of $q(\cdot)$ is given by

$$\dot{q}(\tau) = (\dot{\lambda}(\tau)\nu(\lambda(\tau)), \ddot{\lambda}(\tau)\nu(\lambda(\tau)) + \dot{\lambda}(\tau)^2\psi(\xi(\lambda(\tau)))).$$

In order to prove that $q(\cdot)$ is admissible, we have to check that

$$\|\psi(\xi(\lambda(\tau)))(1 - \dot{\lambda}(\tau)^2) - \ddot{\lambda}(\tau)\nu(\lambda(\tau))\| \leq \varepsilon.$$

Let $M = \sup_{\tau \in \mathbf{R}} \|\nu(\tau)\|$. Then

$$\begin{aligned} &\|\psi(\xi(\lambda(\tau)))(1 - \dot{\lambda}(\tau)^2) - \ddot{\lambda}(\tau)\nu(\lambda(\tau))\| \leq \\ &\qquad c|1 + \dot{\lambda}(\tau)|\,|1 - \dot{\lambda}(\tau)| + M|\ddot{\lambda}(\tau)| \leq c(2+\delta)\delta + M\delta, \end{aligned}$$

which can be made smaller than ε if δ is small.

An important remark from the point of view of applications, is that the energy required to the rendez-vous operation (i.e., the total energy of the trajectory $q(\cdot)$ defined by (7)) can be made as small as desired. To prove it, first notice that $\lambda^{(3)}$ can be assumed to have constant sign, in such a way that

$$\int_0^T |\ddot{\lambda}(\tau)| d\tau \leq 2 \max_{\tau \in [0,T]} |\lambda(\tau) - 1| \leq 2\delta.$$

Therefore, taking $\delta < 1$,

$$\begin{aligned} &\int_0^T \|\psi(\xi(\lambda(\tau)))(1 - \dot{\lambda}(\tau)^2) - \ddot{\lambda}(\tau)\nu(\lambda(\tau))\|^2 d\tau \leq \\ &\qquad (2+\delta)c\int_0^T (1 - \dot{\lambda}(\tau))d\tau + 2M\delta \leq 3ct + 2M\delta, \end{aligned}$$

which can be made as small as desired, since, as we already noticed, we can assume c and δ to be arbitrarily small.

It must be said that, as long as the applications we have in mind are related to spacecraft subject to the gravitational field of the Earth, the model stops to be accurate when the distance from the Earth becomes too large, since the gravitational attraction of the Earth stops to be predominant. Therefore, the previous remark on the smallness of the energy required to change the longitude along a non-elliptic trajectory should be taken as purely qualitative.

IV. Some generalizations

Let us go back to the system (Σ_ε) introduced in Section II. Many of the arguments introduced in Section III can be extended to the general case, under suitable assumptions on ψ and Ω.

Denote

$$f_0(x,v) = (v, \psi(x,v)).$$

A first extension which we are able to prove is the following.

Proposition 4.1: Let Ω_1 be the subset of Ω given by all points q such that $\sup_{t\geq 0} \|f_0(e^{tf_0}(q))\|$ is bounded. Assume that, for every $(x,v) \in \Omega_1$ and for every $\lambda \in \mathbf{R} \setminus \{0\}$, $(x, \lambda v)$ belongs to Ω. Let f_0 be recurrent at every point of $\Omega_0 = \Omega \setminus \Omega_1$. If $\varepsilon \mapsto (\Sigma_\varepsilon)$ has the unrestricted controllability property, then it is also URVC.

Proof. What has to be checked is that the argument above proving the feasibility of the "longitude" variation along an orbit can still be applied. That is, we want to show that, given $\varepsilon > 0$, $q_0 = (x_0, v_0) \in \Omega$ and $t \in \mathbf{R}$, there exists an admissible trajectory $q : [0,T] \to \Omega$ such that $q(0) = q_0$ and $q(T) = e^{(T+t)f_0}(q_0)$.

We already showed that this can be done by proving that there exists $\lambda \in \mathcal{C}^\infty([0,T],[0,t+T])$ such that $\lambda(0) = 0$, $\dot{\lambda}(0) = 1$, $\lambda(T) = T+t$, $\dot{\lambda}(T) = 1$, and $\tau \mapsto (\xi(\lambda(\tau)), \dot{\lambda}(\tau)\nu(\lambda(\tau)))$ is admissible for (Σ_ε), where $(\xi(\tau), \nu(\tau)) = e^{\tau f_0}(q_0)$. In order to prove the admissibility of $\tau \mapsto (\xi(\lambda(\tau)), \dot{\lambda}(\tau)\nu(\lambda(\tau)))$ we have to check that its support is contained in Ω and that

$$\|\psi(\xi(\lambda(\tau)))(1 - \dot{\lambda}(\tau)^2) - \ddot{\lambda}(\tau)\nu(\lambda(\tau))\| \leq \varepsilon$$

for every $\tau \in [0,T]$

If q_0 belongs to Ω_1, then the proof follows the pattern described above.

Fix q_0 in Ω_0. The idea is to take $\dot\lambda(\tau)=1$ when $(\xi(\lambda(\tau)),\nu(\lambda(\tau)))$ is far from q_0, and to make small variations in $\lambda(\tau)-\tau$ at every passage near f_0. For every $\delta>0$, let

$$W_\delta=\{(x,v)\in\mathbf{R}^{2n}\mid \|x-x_0\|,\|v-v_0\|\le\delta\}\,.$$

Fix $\delta>0$ such that $W_{2\delta}\subset\Omega$. Let t_n be an increasing sequence of time instants such that $e^{t_nf_0}(q_0)$ belongs to W_δ for every $n\ge1$. Let $M=\max_{q\in W_{2\delta}}\|f_0(q)\|$. Thus, $e^{tf_0}(q_0)\in W_{2\delta}$ for every t such that $|t-t_n|\ge\delta/M=:\delta'$. Without loss of generalization, $t_{n+1}-t_n\le2\delta'$ for every $n\ge1$.

Then $\lambda(\cdot)$ can be taken in the form

$$\lambda(\tau)=\tau+\sum_{n=1}^{N}\phi\left(\tau-t_n-(n-1)\frac{t}{N}\right)$$

where $\phi\in\mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ is such that $\dot\phi(\tau)\in[-1,1]$ for every τ and

$$\begin{array}{rcll}\phi(\tau)&=&0&\text{for every }\tau\le-\delta'\,,\\ 2M(1-\dot\phi(\tau)+|\ddot\phi(\tau)|)&\le&\varepsilon&\text{for every }\tau\in[-\delta',0]\,,\\ \phi(\tau)&=&\frac{t}{N}&\text{for every }\tau\ge0\,.\end{array}$$

The fact that, for every τ, $(\xi(\lambda(\tau)),\dot\lambda(\tau)\nu(\lambda(\tau)))$ belongs to Ω follows from the hypotheses of the proposition. □

The corollary below is an example of how the hypotheses of Proposition 4.1 can be strengthened in such a way to imply directly the unrestricted complete controllability of $\varepsilon\mapsto(\Sigma_\varepsilon)$.

Throughout the rest of this section, let us assume that ψ is even with respect to v, that is, for every $(x,v)\in\Omega$,

$$(x,-v)\in\Omega\quad\text{ and }\quad\psi(x,-v)=\psi(x,v)\,.\tag{8}$$

Notice that, under assumption (8), if $t\mapsto(x(t),v(t))$ is an admissible trajectory for (Σ_ε), then the same is true for $t\mapsto(x(-t),-v(-t))$.

We find useful to define, for every $(x,v)\in\Omega$, $\pi_x(x,v)=x$ and $\pi_v(x,v)=v$.

Corollary 4.2: Let $R>0$ be such that $(\mathbf{R}^n\setminus B^n_R)\times\mathbf{R}^n\subset\Omega$, and assume that $\|\psi(x,v)\|$ tends to zero as $\|x\|$ goes to infinity, uniformly in v. Let Ω_1 be the subset of Ω given by all $q\in\Omega$ such that $\|\pi_x(e^{tf_0}(q))\|$ tends to infinity as t goes to infinity. Assume that f_0 is recurrent at every point of $\Omega_0=\Omega\setminus\Omega_1$. Then $\varepsilon\mapsto(\Sigma_\varepsilon)$ is URVC.

Proof. Fix $\varepsilon>0$. It is easy to check that every control system in the form (Σ_ε) is bracket generating. Hence, the complete controllability of (Σ_ε) is proved if we show that $-f_0$ is in the Lie saturate of the family of admissible vector fields for (Σ_ε) (see [7]).

Since at points of Ω_0 the arguments of [7, Theorem 5, Chapter 4] still hold, the proof is complete if we show that $e^{-Tf_0}(q_0)$ is reachable from q_0 for every $q_0\in\Omega_1$ and every $T>0$. In order to do so, it is enough to show that, for every $q_0=(x_0,v_0)\in\Omega_1$, the point $(x_0,-v_0)$ is reachable from q_0. Indeed, if we write $e^{-Tf_0}(q_0)=(x_1,v_1)$, then $(x_1,-v_1)$ is reachable from $(x_0,-v_0)$, as a consequence of (8). Therefore, if $(x_0,-v_0)$ is reachable from q_0 and (x_1,v_1) is reachable from $(x_1,-v_1)$ (which is in Ω_1), then $e^{-Tf_0}(q_0)$ is reachable from q_0.

Fix $q_0=(x_0,v_0)\in\Omega_1$.

Let $\gamma:[0,L]\to[0,+\infty)\times\mathbf{R}^{n-1}$ be a smooth trajectory such that $\gamma(0)=\gamma(L)=0$, $\dot\gamma(0)=(1,0,\ldots,0)=-\dot\gamma(L)$. The idea is to look for an admissible trajectory steering q_0 to $(x_0,-v_0)$ of the following type: Take $u=0$ for a time $\bar\tau$, follow a rotated-dilated copy of $(\gamma(\cdot),\dot\gamma(\cdot))$ starting from $e^{\bar\tau f_0}(q_0)$, and finally let u be equal to zero until the trajectory reaches $(x_0,-v_0)$.

For every $r>0$, let

$$c_r=\max\{\|\psi(x,v)\|\mid(x,v)\in\Omega,\|x\|\ge r\}\,.$$

By hypothesis, $c_r\to0$ as r goes to infinity. Fix $r>R$ such that $c_r<\varepsilon/2$ and take $\bar\tau$ such that $\|\pi_x(e^{\bar\tau f_0}(q_0))\|\ge r$.

Since $q_0\in\Omega_1$, we can assume that

$$\pi_x(e^{\bar\tau f_0}(q_0))\cdot\pi_v(e^{\bar\tau f_0}(q_0))>0\,.$$

Therefore, for every $\lambda>0$, the trajectory

$$t\mapsto\pi_x(e^{\bar\tau f_0}(q_0))+A(\gamma(\lambda t))\tag{9}$$

is contained in $\mathbf{R}^n\setminus B^n_r$, where $A\in SO(n)$ is a unitary matrix which sends $(1,0,\ldots,0)$ to $\pi_v(e^{\bar\tau f_0}(q_0))/\|\pi_v(e^{\bar\tau f_0}(q_0))\|$.

Let $M=\max_{t\in[0,L]}\|\ddot\gamma(t)\|$. Then, taking $\lambda<\sqrt{\frac{\varepsilon}{2M}}$, (9) defines an admissible trajectory of (Σ_ε). □

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