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Limit Cycle Analysis Using a System Right Inverse

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***Abstract*— The variational system obtained by linearizing a dynamical system along a limit cycle is always non-invertible. This follows because the limit cycle is only unique modulo time translation. It is shown that questions such as uniqueness, robustness, and computation of limit cycles can be addressed using a right inverse of the variational system. Small gain arguments are used in the analysis.**

I. Introduction

Limit cycles are essential in feedback systems such as electronic oscillators, auto tuners, and in synchronization mechanisms in robotics and biology. It is in these applications important to predict the existence, uniqueness, location, and robustness of the limit cycle. The renewed interest in these fundamental questions is motivated by new and emerging applications where oscillators appear in feedback loops. Examples ranges from various synchronization problems [6], [9], [10], [7], control of rhythmic motion [11], [2], and problems in systems biology [2]. System theoretic concepts and methodologies are now being introduced to address robust stability problems for limit cycles and oscillators, see e.g. [1], [8]. In this paper we use small gain conditions on a variational system to estimate robustness in regions around a nominal limit cycle trajectory.

In [3] we developed results that allow estimation of a robustness margin of limit cycles in a class of systems on Luré form. The small gain conditions in [3] provide conditions for existence as well as stability of the limit cycle. In this paper we focus on questions relating to the existence and uniqueness of a limit cycle to the same class of systems. We provide a new uniqueness result and we discuss the computation and the robustness of limit cycles. Emphasis is put on the use of $\mathbf{L}_2$-methods and frequency domain representation of the operators.

Notation

We let $\mathbf{L}_2(1)$ denote the space of square integrable 1-periodic functions and $C(1)$ is the set of continuous 1-periodic functions equipped with the norm $\|v\|_{C(1)} = \sup_{t\in[0,1]}|v(t)|$, where $|\cdot|$ always denotes the Euclidean norm. We make extensive use of the topological inclusion $C(1) \hookrightarrow L_2(1)$ which follows since $\|v\|_{\mathbf{L}_2(1)} \le \|v\|_{C(1)}$. We will almost always denote norms and induced norms on $\mathbf{L}_2(1)$ without the suffix, i.e. $\|\cdot\| := \|\cdot\|_{\mathbf{L}_2(1)}$ and $\|\cdot\| := \|\cdot\|_{\mathbf{L}_2(1)\to\mathbf{L}_2(1)}$. At several places we consider the space $C(1)\times\mathbf{R}$ with the norm $\|(y,T)\|_{C(1)\times\mathbf{R}} = (\|y\|^2_{C(1)}+|T|^2)^{1/2}$ and similarly for $\mathbf{L}_2(1)\times\mathbf{R}$ but normally with the suffix suppressed.

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II. Limit Cycle Oscillations

We consider on $\mathbf{L}_\infty(-\infty,\infty)$ the system defined by

$$y = H\Phi(y)$$

where H is a bounded, causal, linear time-invariant operator and Φ is a memoryless nonlinear operator defined by the time-domain relation

$$(\Phi(y))(t) = \varphi(y(t))$$

where $\varphi : \mathbf{R}^m \to \mathbf{R}^p$ is C^1. We will assume Lipschitz continuity of the nonlinearity and its derivative. This is used to estimate various norms and Lipschitz constants. The system can be viewed as a feedback interconnection of a stable LTI plant with a memoryless nonlinearity. We assume that the linear part of the dynamics is defined by a strictly proper and stable transfer function. With this we mean that the impulse response function of the transfer functions $H(s)$ and $sH(s)$ satisfy

$$h(t) = \mathcal{L}^{-1}\{H(s)\} \in \mathbf{L}_1[0,\infty)$$
$$\dot h(t) = \dot h_c(t) + \sum_{k=0}^{\infty} \dot h_k\delta(t-T_k)$$

where $\dot h_c(\cdot)\in\mathbf{L}_1[0,\infty)$ and $\sum_{k=0}^\infty|\dot h_k|<\infty$, $T_k\ge 0$. We have the following time domain representation on functions in $C(1)$ (and $\mathbf{L}_2(1)$)

$$y(t) = \int_{-\infty}^t h(t-\tau)v(\tau)\tau = \int_0^1 \tilde h(t,\tau)v(\tau)d\tau$$

where $\tilde h : [0,1]\times[0,1]\to\mathbf{R}^{p\times m}$ is defined by

$$\tilde h(t,\tau) = \begin{cases} h(t-\tau)+\sum_{k=1}^\infty h(t+k-\tau), & 0\le\tau\le t\le 1\\ \sum_{k=1}^\infty h(t+k-\tau), & 0\le t<\tau\le 1\end{cases}$$

and it satisfies the periodicity property $\tilde h(1,\tau) = \tilde h(0,\tau)$, for $\tau\in[0,1]$.

We are interested in the existence, uniqueness, robustness, and computation of limit cycles for this class of feedback systems. A *limit cycle* is a nontrivial *isolated* periodic solution of the feedback equation. To simplify our development we always normalize the period time to be one. This results in the following system equation where the true period time appear as a parameter in the dynamic equation

$$y = H(s/T)\Phi(y) \tag{1}$$

Let $z = (y, T) \in C(1) \times \mathbf{R}$ and define $F : C(1) \times \mathbf{R} \to C(1)$ by

$$F(z) = y - H(s/T)\Phi(y). \tag{2}$$

Any solution to $F(z) = 0$ with $T > 0$ corresponds to a nontrivial limit cycle if it is isolated. The meaning of isolated periodic solution will be defined in the next section.

Uniqueness of the Limit Cycle

We first notice that a limit cycle at best is unique modulo time translations. Indeed, let $z_0 = (y_0, T_0)$ be a nontrivial 1-periodic solution of (1) and let $S_d : C(1) \times \mathbf{R} \to C(1) \times \mathbf{R}$ be a time translation operator defined by $S_d(y(t), T) = (y(t-d), T)$. Then in time domain

$$\begin{aligned}
&(F(S_d z_0))(t) \\
&\quad = y_0(t-d) - \int_{-\infty}^{t} T_0 h(T_0(t-\tau))\varphi(y_0(\tau - d))d\tau \\
&\quad = y_0(t-d) - \int_{-\infty}^{t-d} T_0 h(T_0(t-d-\tau))\varphi(y_0(\tau))d\tau \\
&\quad = (F(z_0))(t-d) = 0
\end{aligned} \tag{3}$$

which holds for any $d \in \mathbf{R}$. This shows that the time translated 1-periodic solution is still a valid periodic solution of (1). A 1-periodic solution of (1) is called a *limit cycle* if it is *isolated* in the sense that no periodic solution exists in a neighborhood of the nominal manifold $\mathcal{Z}_0 = \{S_d z_0 : d \in [0, 1)\}$.

In the next result we use the Fréchet derivative $F'(z_0) \in \mathcal{L}(C(1) \times \mathbf{R}, C(1))$ with the block structure

$$F'(z_0) = \begin{bmatrix} F'_y(z_0) & F'_T(z_0) \end{bmatrix} = \begin{bmatrix} I - L(z_0) & K(z_0) \end{bmatrix}$$

where $L(z_0) : C(1) \to C(1)$, $K(z_0) : \mathbf{R} \to C(1)$ are defined as

$$\begin{aligned}
L(z_0) &= H(s/T_0)\Phi'(y_0) \\
K(z_0) &= \frac{s}{T_0^2} H'(s/T_0)\Phi(y_0).
\end{aligned} \tag{4}$$

where $(\Phi'(y_0))(t) = \varphi'(y_0(t))$. Differentiation of (3) with respect to d at $d = 0$ gives

$$\begin{aligned}
\dot{y}_0(t) &= \int_{-\infty}^{t} T_0 h(T_0(t-\tau))\varphi'(y_0(\tau))\dot{y}_0(\tau)d\tau \\
&= (L(z_0)\dot{y}_0)(t)
\end{aligned}$$

This shows that $(\dot{y}_0, 0) \in \operatorname{Ker} F'(z_0)$. A right inverse corresponding to $F'(z_0)$ is a bounded linear operator $F^\dagger : \mathcal{L}(C(1), C(1) \times \mathbf{R})$ satisfying $F'(z_0)F'(z_0)^\dagger = 1$. One can show that a periodic solution of (1) is a limit cycle, i.e. it is isolated, if there exists a right inverse. The next lemma shows that $\operatorname{Ker} F'(z_0) = \operatorname{span}\,\{(\dot{y}_0, 0)\}$ is a necessary and sufficient condition for the existence of a right inverse. The lemma also provides one particular right inverse [4].

Proposition 1: There exists a right inverse $F'(z_0)^\dagger$ if and only if $\operatorname{Ker} F'(z_0) = \operatorname{span}\,\{(\dot{y}_0, 0)\}$. It is easy to see that a family of right inverses can be constructed as

$$F^\dagger = \begin{bmatrix} I \\ g \end{bmatrix} (F'_y + F'_T g)^{-1} \tag{5}$$

where $g \in C(1)^* \times \mathbf{R}$ is chosen so that the inverse exists and is bounded. This is true if $\langle e, g \rangle \neq 0$, where $e = (\dot{y}_0, 0)$.

We will sometimes use that the right inverse is robust to perturbations as stated in the next lemma.

Lemma 1 (Small Gain Lemma): Let X, Y be Banach spaces and suppose $F'_0 \in \mathcal{L}(X, Y)$ has a right inverse $F_0^\dagger \in \mathcal{L}(Y, X)$. Then there exists $F^\dagger \in \mathcal{L}(Y, X)$ such that $(F'_0 + \Delta)F^\dagger = I$, for all $\Delta \in \mathcal{L}(X, Y)$ with $\|\Delta\|_{X \to Y} < 1/\|F_0^\dagger\|_{Y \to X}$. One possible choice for this right inverse is

$$\begin{aligned}
F^\dagger &= {F'}_0^\dagger (I + \Delta {F'}_0^\dagger)^{-1} \\
&= {F'}_0^\dagger (F' {F'}_0^\dagger)^{-1}, \quad \text{where} \quad F' = F'_0 + \Delta
\end{aligned}$$

Our analysis will be local and we frequently use extension of various operators to $\mathbf{L}_2$ space to simplify computations. For this reason we introduce the following notation

Definition 1: will use the notation

$$\begin{aligned}
B_r(z_0) &= \{z \in C(1) : \|z - z_0\|_{\mathbf{L}_2(1) \times \mathbf{R}} \leq r\} \\
B_r(\mathcal{Z}_0) &= \cup_{d \in [0,1]} B_r(S_d z_0) \\
&= \{z \in C(1) \times \mathbf{R} : \|z - S_d z_0\| \leq r;\ d \in [0, 1]\}
\end{aligned} \tag{6}$$

Note that the functions in $B_r(z_0)$ are assumed continuous to allow us to do calculus in the $C(1)$-topology while norms will be evaluated using $\mathbf{L}_2(1)$-topology to simplify the analysis and some computation. Note also that any limit cycle solution of (1) must be absolutely continuous so it belongs to $C(1)$.

The following properties will be useful

1) The operator F is naturally defined on the $C(1)$ space but for computational reasons we will use the extensions of the operators to $\mathbf{L}_2(1)$, i.e., $F : B_r(\mathcal{Z}) \to \mathbf{L}_2(1)$, $F' : B_r(\mathcal{Z}) \to \mathcal{L}(\mathbf{L}_2(1) \times \mathbf{R}, \mathbf{L}_2(1))$ and $F^\dagger : B_r(\mathcal{Z}) \to \mathcal{L}(\mathbf{L}_2(1), \mathbf{L}_2(1) \times \mathbf{R})$.
2) In several applications we need $F'(z_0)^\dagger F'(z_0) z_0 = z_0$ or more generally that $F'(z_0)^\dagger F'(z_0) : \mathbf{L}_2(1) \times \mathbf{R} \to \mathbf{L}_2(1) \times \mathbf{R}$ is an orthogonal projection onto $\operatorname{Ker}(F'(z_0))^\perp$. For the right inverse in (5) the first condition holds if $\langle y_0, g \rangle = T_0$ while the second cannot be true. We will in the next section see how a right inverse with the desired properties can be derived in frequency domain.

Notice that $\sup_{z \in B_r(\mathcal{Z}_0)} \|F(z)\| = \sup_{z \in B_r(z_0)} \|F(z)\|$ and $\sup_{z \in B_r(\mathcal{Z}_0)} \|F'(z)\| = \sup_{z \in B_r(z_0)} \|F'(z)\|$ due to the time invariance of F and the periodicity of F'.

We will also make the assumption that F and F' are locally Lipschitz continuous on the set $B_r(\mathcal{Z})$, in the sense that there exists a constant $Lip_r[F]$ such that

$$\|F(z_2) - F(z_1)\| \leq Lip_r[F]\|z_2 - z_1\|, \quad \forall z_2, z_1 \in B_r(z_0)$$

and similarly for F'. We briefly discuss the computation of Lipschitz constants in Section VI.

We have the following result

Proposition 2: Consider a solution $z_0 = (y_0, T_0) \in C(1) \times \mathbf{R}$ of (1) and let

$$B_r^0(z_0) = \{z \in C(1) \times \mathbf{R} : z \in (\operatorname{Ker} F'(z_0))^\perp; \|z - z_0\| \le r\}.$$

where $\operatorname{Ker}(F'(z_0))^\perp = \{z \in \mathbf{L}_2(1) \times \mathbf{R} : \langle z, (\dot{y}_0, 0)\rangle = 0\}$.

Suppose

(i) $z_0 = F'(z_0)^\dagger F'(z_0) z_0$

(ii) $\|(I - F'(z_0)^\dagger F'(z_0))z\| \le \gamma_1 \|z\|$, $\forall z \in B_r^0(z_0)$

(iii) $\|F'(z_0)^\dagger\| Lip_r[F'(z_0) - F] = \gamma_2$

(iv) $\gamma_1 + \gamma_2 < 1$

Then z_0 is a unique (modulo time translation) limit cycle in $B_r(\mathcal{Z}_0)$. In other words, if $\bar{z} = (\bar{y}, \bar{T}) \in B_r(\mathcal{Z}_0)$ is such that $F(\bar{z}) = 0$ then $\bar{y}(t) = y_0(t + d)$ for some $d \in [0, 1]$ and $\bar{T} = T_0$.

Remark 1: Note that $\gamma_1 = 0$ if $F'(z_0)^\dagger F'(z_0)$ is an orthogonal projection onto $(\operatorname{Ker} F'(z_0))^\perp$.

III. The Right Inverse

We will next consider various realizations of the of right inverse using time-domain as well as frequency domain representations. We will only consider the extensions of the operators to $\mathbf{L}_2(1)$ where norm computation is simplified. Particularly simple representations will be obtained in the frequency domain where we easily can exploit that the strictly proper part of $F'(z_0)$, namely $G = \begin{bmatrix} -L & K \end{bmatrix}$, is a compact operator. This can be used to derive a perfect right inverse.

Proposition 3: Suppose $\dim \operatorname{Ker}(F'(z)) = 1$. Then there exists a singular value decomposition[1] $F'(z) = \sum_{k=1}^{\infty} \sigma_k u_k v_k^*$, where $0 < \sigma_1 \le \sigma_2 \le \dots$ and $\{u_k\}_{k=1}^{\infty}$, $\{v_k\}_{k=1}^{\infty}$ are complete orthonormal sequences in $\mathbf{L}_2(1)$ and $(\operatorname{Ker}(F'(z)))^\perp \subset \mathbf{L}_2(1) \times \mathbf{R}$, respectively. The right inverse

$$F'(z)^\dagger = \sum_{k=1}^{\infty} \sigma_k^{-1} v_k u_k^* \tag{7}$$

is the right inverse with the minimum induced norm $\|F'(z)^\dagger\| = \sigma_1^{-1}$. It has the property that $F'(z)^\dagger F'(z)$ is an orthogonal projection onto $(\operatorname{Ker}(F'(z)))^\perp$.

A. Frequency Domain Representation

Let

$$l_2(\mathbf{C}^m) = \{\{\hat{y}[k]\}_{k=-\infty}^{\infty} : \hat{y}[k] \in \mathbf{C}^m; \sum_{k=-\infty}^{\infty} |\hat{y}[k]|^2 < \infty\}.$$

and let $\mathcal{F} : \mathbf{L}_2(1) \to l_2(\mathbf{C}^m)$ denote the Fourier transform defined by $\hat{y}[k] = \int_0^1 y(t) e^{-j2\pi kt} dt$. The Fourier transform is an isometric isomorphism between these two spaces and it will here be used to construct perfect right inverses along the lines of Proposition 3.

[1]Should be interpreted as $F'(z)w = \sum_{k=1}^{\infty} \sigma_k u_k \langle w, v_k \rangle$.

We will use the notation $\omega = 2\pi/T$. Let $\hat{z} = (\hat{y}, T) \in l_2(\mathbf{C}^m) \times \mathbf{R}$, where $\hat{y} = \mathcal{F}y$ is the Fourier series representation. In frequency domain

$$\hat{F}(\hat{z}) = \hat{y} - \hat{H}(j\omega\mathcal{K})\hat{\Phi}(\hat{y})$$

where $\hat{\Phi}(\hat{y}) : l_2(\mathbf{C}^m) \to l_2(\mathbf{C}^m)$ is defined by $\hat{\Phi}(\hat{y}) = \mathcal{F}\varphi(\mathcal{F}^{-1}\hat{y})$ and $\hat{H}(j\omega\mathcal{K})$ is the infinite dimensional diagonal matrix with elements $H(j\omega k)$. The derivative $F'(\hat{z}) : l_2(\mathbf{C}^m) \times \mathbf{R} \to l_2(\mathbf{C}^m)$ is defined as

$$\begin{aligned}\hat{F}'(\hat{z}) &= \begin{bmatrix} \hat{F}'_y(\hat{z}) & \hat{F}'_T(\hat{z}) \end{bmatrix} = \begin{bmatrix} I - \hat{L}(\hat{z}) & \hat{K}(\hat{z}) \end{bmatrix} \\ &:= \begin{bmatrix} I - \hat{H}(j\omega\mathcal{K})\hat{\Phi}'(\hat{y}) & \frac{j\omega^2}{2\pi}\mathcal{K}\hat{H}'(j\omega\mathcal{K})\hat{\Phi}(\hat{y}) \end{bmatrix}\end{aligned}$$

where $\hat{\Phi}'(\hat{y}) = \mathcal{F}\varphi'(\mathcal{F}^{-1}\hat{y})\mathcal{F}^{-1}$. More explicitly we have

$$\hat{K}(\hat{z})[k] = \frac{j\omega^2 k}{2\pi} H'(j\omega k)\hat{\Phi}(\hat{y})[k]$$

and the (k, l) coefficient of the infinite dimensional matrix $L(\hat{z})$ is

$$\hat{L}(\hat{z})[k, l] = H(j\omega k)\hat{\Phi}'(\hat{y})[k - l]$$

Since H is strictly proper it will be possible to truncate the Fourier representation at some sufficiently high index to obtain a matrix representation of required accuracy. If P_N denotes orthogonal projection onto the $2N+1$ dimensional space spanned by the Fourier bases $\{e^{j\omega kt}\}_{-N}^{N}$ then we let

$$\hat{F}'_N(\hat{z}) := P_N \begin{bmatrix} I - \hat{L}(\hat{z})P_N & \hat{K}(\hat{z}) \end{bmatrix} \tag{8}$$

For given $\hat{z}$ this matrix belongs to $C^{(2N+1)\times(2N+2)}$ and the matrix SVD can be used to compute the right inverse as the standard pseudo inverse. We next derive conditions under which the truncated right inverse implies right invertibility of $\hat{F}'(\hat{z})$. For this purpose let $Q_N := I - P_N$ and

$$\begin{aligned}\hat{L}_{\bar{N},\bar{N}}(\hat{z}) &:= Q_N \hat{L}(\hat{z}) Q_N, \\ \hat{L}_{N,\bar{N}}(\hat{z}) &:= P_N \hat{L}(\hat{z}) Q_N, \\ \hat{L}_{\bar{N},N}(\hat{z}) &:= Q_N \hat{L}(\hat{z}) P_N, \\ \hat{K}_{\bar{N}}(\hat{z}) &:= Q_N \hat{K}(\hat{z}).\end{aligned} \tag{9}$$

We have the following proposition

Proposition 4: Suppose

$$\|\hat{F}_N^\dagger\| \cdot \|\hat{L}_{N\bar{N}}\| \cdot \| \begin{bmatrix} \hat{L}_{\bar{N},N} & \hat{K}_{\bar{N}} \end{bmatrix} \| < 1 - \|\hat{L}_{\bar{N},\bar{N}}\| \tag{10}$$

Then $\hat{F}'(\hat{z})$ has a bounded right inverse.

B. Time-Domain Representation

First consider $F : C(1) \times \mathbf{R} \to C(1)$ defined by $F((y, T)) = y - H(s/T)\varphi(y)$. If $H(s) = C(sI - A)^{-1}B$ then $F : (y, T) \mapsto y_f$ has state space realization

$$\begin{aligned}\dot{x}_1 &= T(Ax_1 + B\varphi(y)), \quad x_1(1) = x_1(0) \\ y_f &= y - Cx_1\end{aligned}$$

The operator $F'(z) : (v, \delta T) \mapsto w$ ($z = (y, T)$) is in the finite dimensional case defined by the state space realization

$$\begin{aligned} \dot{x}_2 &= T(Ax_2 + B\varphi'(y)v) + \dot{x}_1 \delta T, \quad x_2(1) = x_2(0) \\ w &= v - Cx_2 \end{aligned} \tag{11}$$

We can use the right inverse $F'(z)^\dagger : w \mapsto (v, \delta T)$ defined by (with the choice $x_3(0) = x_2(0)$)

$$\begin{aligned} \dot{x}_3 &= T((A + B\varphi'(y)C)x_3 + B\varphi'(y)w) + \dot{x}_1 c^T x_3(0) \\ (v, \delta T) &= (w + Cx_3, c^T x_3(0)), \qquad x_3(1) = x_3(0) \end{aligned} \tag{12}$$

where the vector c must chosen such that $I - \Phi_{cl}(1,0) - \int_0^1 \Phi_{cl}(1,\tau)\dot{x}_1(\tau)d\tau c^T$ is invertible, where Φ_{cl} is the transition matrix corresponding to $A_{cl} = T(A + B\varphi'(y)C)$. This condition ensures that the periodicity condition $x_3(1) = x_3(0)$ can be satisfied, which in turn implies boundedness of the right inverse. If we in addition want $z = F'(z)^\dagger F'(z)z$ then we need $c^T x_3(0) = T$. If $z = z_0 = (y_0, T_0)$, a limit cycle, then invertibility follows if $1 \notin \text{eig}(\Phi_{cl}(1,0) + \dot{x}_0(0)c^T)$, where $\dot{x}_0(t) = T_0(Ax_0(t) + B\varphi(y_0(t)))$. This is the case if $c^T \dot{x}_0(0) \neq 0$.

Norm Computation: For given choice of c we compute

$$\gamma_0 = \| F_z'(z_0)^\dagger \|_{\mathbf{L}_2(1) \to \mathbf{L}_2(1) \times \mathbf{R}}$$

by solving the optimization problem

$$\gamma_0^2 = \inf \gamma^2 \quad \text{subject to} \quad J(x, w, \gamma) \leq 0, \ \forall (x, w) \in \mathcal{L} \tag{13}$$

where

$$\begin{aligned} J(x, w, \gamma) &= |c^T x(0)|^2 + \int_0^1 (|w + Cx|^2 - \gamma^2 |w|^2) dt \\ \mathcal{L} &= \{(x, w) \in \mathbf{L}_2(1) : \dot{x}(t) = A_{cl}(t)x(t) + B_{cl}(t)w(t) \\ &\qquad + \dot{x}_0(t)c^T x(0); \ x(1) = x(0)\} \end{aligned}$$

and $A_{cl}(t) = A + B\varphi'(y_0(t))C$ and $B_{cl}(t) = B\varphi'(y_0(t))$. We obtain an upper bound by using LQ optimal control techniques. If we let $Q_0 = cc^T$, $Q = C^T C$, $S = C^T$, $R = (1 - \gamma^2)I$, then γ is an upper bound on the optimization problem (13), i.e. $\gamma > \|F_z'(z_0)^\dagger\|_{\mathbf{L}_2(1) \to \mathbf{L}_2(1) \times \mathbf{R}}$, if and only if there exists $\epsilon > 0$ such that

$$\sup_{(x,w) \in \mathcal{L}} x(0)^T Q_0 x(0) + \int_0^1 (x^T Q x + 2x^T S w + w^T R w) dt \leq -\epsilon |x(0)|^2. \tag{14}$$

The next proposition gives necessary and sufficient conditions for this to hold.

Proposition 5: Let $\dot{M}(t, t_0) = H(t)M(t, t_0)$, with initial condition $M(t_0, t_0) = I$ and

$$H = \begin{bmatrix} A_{cl} - B_{cl}R^{-1}S^T & -B_{cl}R^{-1}B_{cl}^T \\ -(Q - SR^{-1}S^T) & -(A_{cl} - B_{cl}R^{-1}S^T)^T \end{bmatrix}$$

have block partition consistent with the dimensions of H

$$M(t, \tau) = \begin{bmatrix} M_{11}(t, \tau) & M_{12}(t, \tau) \\ M_{21}(t, \tau) & M_{22}(t, \tau) \end{bmatrix}.$$

There exists $\epsilon > 0$ such that (14) holds if and only if

(i) $R < 0$, i.e., $\gamma > 1$,
(ii) $\det(M_{12}(t, 0)) \neq 0$ for $t \in (0, 1]$,
(iii) $\bar{J} < 0$, where the matrix $\bar{J}$ is defined by

$$\bar{J} = Q_0 + \frac{1}{2} \int_0^1 (X^T Q X + 2X^T S W + W^T R W) dt$$

where

$$\begin{aligned} \begin{bmatrix} X(t) \\ \Psi(t) \end{bmatrix} &= \begin{bmatrix} M_{11}(t,0) + M_{12}(t,0)L_0 \\ M_{21}(t,0) + M_{22}(t,0)L_0 \end{bmatrix} + \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \\ W(t) &= -R^{-1}(S^T X(t) + B_{cl}(t)^T \Psi(t)) \\ V_1(t) &= \int_0^t M_{11}(t, \tau)\dot{x}_0(\tau) k d\tau \\ V_2(t) &= \int_0^t M_{21}(t, \tau)\dot{x}_0(\tau) k d\tau \end{aligned}$$

and where $L_0 = M_{12}(1,0)^{-1}(I - M_{11}(1,0) - V_1(1))$.

IV. Computation of Limit Cycle

In this section we consider the Newton iteration

$$z_{k+1} = z_k - F'(z_k)^\dagger F(z_k), \quad k = 1, 2, \ldots$$

for the computation of a limit cycle. We first provide two convergence properties for this Newton iteration. To do this we define the distance to the limit cycle manifold as

$$d(z, \mathcal{Z}_0) = \min_{d \in [0,1]} \|z - S_d z_0\| \tag{15}$$

We let $d_k^* = \text{argmin}_{d \in [0,1]} \|z_k - S_d z_0\|$ and $z_k^* = S_{d_k^*} z_0$.

Proposition 6: Let z_k be generated by (15) with $z_1 \in B_r(\mathcal{Z}_0)$. Suppose $r > 0$ is chosen such that $cr < 1$, where

$$c = Lip_r[F'] \sup_{z \in B_r(\mathcal{Z}_0)} \|F'(z)^\dagger\|^2 \sup_{z \in B_r(\mathcal{Z}_0)} \|F'(z)\|^2 \times \frac{\|F_O'(z_0)^\dagger\|}{1 - r Lip_r[F'] \|F_O'(z_0)^\dagger\|}$$

is assumed positive. Here $F_O'(z_0)^\dagger$ is the right inverse with[2] $\text{Im}\, F_O'(z_0)^\dagger = (\text{Ker}\, F'(z_0))^\perp \subset \mathbf{L}_2(1) \times \mathbf{R}$.

Under these conditions we have $z_k \in B_r(\mathcal{Z}_0)$ $\forall k$ and

$$d(z_{k+1}, \mathcal{Z}_0) \leq c d(z_k, \mathcal{Z}_0)^2$$

If in addition $\eta \sup_{z \in B_r(\mathcal{Z}_0)} \|F(z)\| < 1$, where

$$\eta = Lip_r[F'] \sup_{z \in B_r(\mathcal{Z}_0)} \|F'(z)^\dagger\|^2$$

then $\|F(z_{k+1})\| \leq \eta \|F(z_k)\|^2$ for all k.

[2]It was shown in Proposition 3 how such right inverses can be constructed.

Remark 2: If $F'(z_k)^\dagger = F'_O(z_k)^\dagger$ then the bound improves to

$$c = Lip_r[F'] \sup_{z\in B_r(\mathcal{Z}_0)} \|F'(z)^\dagger\|^3 \sup_{z\in B_r(\mathcal{Z}_0)} \|F'(z)\|^2$$

Remark 3: The first part of the conclusion of this proposition shows that the distance to the limit cycle manifold decreases quadratically. The second part show that near the true limit cycle manifold $\mathcal{Z}_0$ the function value converges quadratically to zero.

A. Implementation of the Newton iteration

By exploiting the frequency domain representation and the pseudo inverse we obtain a Newton iteration that can be implemented using least squares computation. Consider the Newton iteration

$$z_{k+1} = z_k - F'_O(z_k)^\dagger F(z_k) = z_k - \sum_{k=1}^{\infty} \sigma_k^{-1} v_k u_k^* F(z_k)$$

obtained by using (7) in Proposition 3. It is easy to see that it is the least squares solution to the problem

$$\min_{z_{k+1}} \|F'(z_k)(z_{k+1} - z_k) + F(z_k)\|$$

and its solution satisfies $z_{k+1} - z_k \perp \text{Ker } F'(z_k)$. After truncation to the $2N+1$ dimensional space spanned by the Fourier bases $\{e^{j\omega kt}\}_{-N}^{N}$ we approximate the Newton iteration by the least squares problem

$$\min_{\hat{z}_{N+1}} \|\hat{F}'_N(\hat{z}_N)(\hat{z}_{k+1} - \hat{z}_k) - \hat{F}_N(\hat{z}_N)\| \qquad (16)$$

where $\hat{z}_N = (P_N\hat{y}, T)$ and $\hat{F}_N(\hat{z}_N) = P_N\hat{F}(\hat{z}_N)$. To improve efficiency we can exploit that the problem has more parameters and constraints than necessary. Indeed, since the time-domain signal is real valued the Fourier coefficients satisfies $\hat{y}[-k] = \overline{\hat{y}[k]}$. Likewise $H(-j\omega k) = \overline{H(j\omega k)}$.

An alternative to using the right inverse would be to add one constraint to the equation system and apply the standard Newton algorithm. The standard way to do this in truncated Frequency domain implementations is to add a constraint that fixes the phase [5]. This is a good strategy if the iteration is initialized appropriately.

Example 1: We have $\mathcal{F}\{y_*(t+d)\}[k] = e^{j2\pi dk}\hat{y}_*$ so the limit cycle is unique modulo an undetermined phase in the frequency domain. It can be made unique if we fix the phase, e.g. as $\langle v, z_*\rangle = \text{im}\,\hat{y}_*[1] = 0$, i.e. the phase is fixed such that the first harmonic is real valued.

V. A Robustness Result

In many applications we do not know the model exactly and the Newton iteration normally terminates before an exact solution has been found. We will next provide conditions for the limit cycle solution to sustain perturbations of the dynamics. Consider the system

$$y = H_\Delta(s/T)\Phi(y) \qquad (17)$$

where H_Δ is a set of uncertain transfer functions. One possible uncertainty parameterization is the linear fractional transformation $H_\Delta = \mathcal{F}_l(H,\Delta) = H_{11} + H_{12}\Delta(I - H_{22}\Delta)^{-1}H_{21}$, where $\Delta \in \boldsymbol{\Delta}$ (some structured set of transfer functions). We use a truncated version of the nominal (approximate) dynamics for computation of an approximate limit cycle using the Newton iteration. The next theorem state conditions under which there exist a limit cycle in some neighborhood $B_r(\bar{z})$ of $\bar{z}$ for every $\Delta \in \boldsymbol{\Delta}$. Below we use the notation

$$\begin{aligned}
F(z,\Delta) &= y - H_\Delta(s/T)\Phi(y) \\
F_N(z,\Delta) &= P_N F(P_N z,\Delta), \qquad F_N(z) = F_N(z,0) \\
F'_N(z,\Delta) &= P_N F'_z(P_N z,\Delta), \qquad F'_N(z) = F'_N(z,0)
\end{aligned}$$

where P_N is the projection to the $2N+1$ dimensional subspace defined by the Fourier basis $\{e^{j2\pi kt}\}_{k=-N}^{N}$. We also use the notation $L_N, L_{N,\bar{N}}, L_{\bar{N},N}, L_{\bar{N},\bar{N}}, K_N$ and $K_{\bar{N}}$ introduced in (8) and (9) (we do not specify if we consider time domain or frequency domain).

Proposition 7: Consider a Newton iteration with a nominal and truncated right inverse

$$z_{k+1} = z_k - F'_N(z_k)^\dagger F_N(z_k).$$

Assume we terminate the Newton iteration at $\bar{z}$ satisfying
(A) $\bar{z} = F'_N(\bar{z})^\dagger F'_N(\bar{z})\bar{z}$.
(B) $F_N(\bar{z}) = \epsilon$
(C) $\bar{z} = (\bar{y}, T)$, where $\bar{y}(t) = c_0 + 2\text{Re}\sum_{k=1}^{N} c_k e^{i2\pi kt}$.
Suppose under these conditions that there exists $r > 0$ such that

$$\begin{aligned}
\gamma_1 &= \sup_{z\in B_r(\bar{z}),\Delta\in\boldsymbol{\Delta}} \|F'_N(\bar{z})^\dagger(F'_N(\bar{z}) - F'_N(z,\Delta))\| \\
\gamma_2 &= \sup_{z\in B_r(\bar{z}),\Delta\in\boldsymbol{\Delta}} \|F'_N(\bar{z})^\dagger L_{N\bar{N}}(z,\Delta) \\
&\qquad\qquad + Q_N(F'(z,\Delta) - 1)\| \\
\epsilon_1 &= \sup_{\Delta\in\boldsymbol{\Delta}} \|F'_N(\bar{z})^\dagger(F_N(\bar{z},\Delta) - F_N(\bar{z}))\| \\
\epsilon_2 &= \sup_{\Delta\in\boldsymbol{\Delta}} \|Q_N(F(\bar{z},\Delta) - I)\| \\
\epsilon_3 &= \epsilon\|F'_N(\bar{z})^\dagger\|
\end{aligned}$$

satisfies $\gamma = \gamma_1 + \gamma_2 < 1$ and $\epsilon_1 + \epsilon_2 + \epsilon_3 < (1-\gamma)r$. Then for every $\Delta \in \boldsymbol{\Delta}$ there exists a unique (modulo time translation) $z_\Delta \in B_r(\bar{z})$ such that $F(z_\Delta) = 0$.

Remark 4: We have that γ_2, ϵ_2 and ϵ_3 are due to the truncation. If N is sufficiently large then these bounds are negligible and the robustness test simplifies to $\gamma_1 < 1$ and $\epsilon_1 < (1-\gamma_1)r$.

VI. Estimation of Lipschitz Constants

We will assume that the nonlinearity satisfies the Lipschitz constants

$$\begin{aligned}
|\varphi(y_1) - \varphi(y_2)| &\le k_1|y_1 - y_2| \\
|\varphi'(y_1) - \varphi'(y_2)| &\le k_2|y_1 - y_2|
\end{aligned} \qquad (18)$$

The next lemma shows how to estimate $C(1)$ bounds for the system output using the Lipschitz constants in (18).

Lemma 2: Suppose $(y, T) \in B_r(z_0)$ and that (18) holds. Then

$$\begin{aligned}\|y - y_0\|_{C(1)} &= \|H(s/T)\Phi(y) - H(s)\Phi(y_0)\|_{C(1)} \\ &\leq \|H(s/T) - H(s)\|_{\mathbf{L}_2(1)\to C(1)}(\|\Phi(y_0)\| + k_1 r) \\ &\quad + \|H(s)\|_{\mathbf{L}_2(1)\to C(1)} k_1 r =: \eta \quad (19)\end{aligned}$$

where

$$\|H(s)\|^2_{\mathbf{L}_2(1)\to C(1)} = \max_{t\in[0,1]} \int_0^1 |\tilde{h}(t,\tau)|^2 d\tau$$

By using this lemma we do not need to require that the Lipschitz constants in (18) are global. It is enough that (18) holds for all $y_1, y_2 \in \cup_{t\in[0,1]}\{y : |y - y_0(t)| \leq \eta\}$, where η is defined in (19) and y_0 is the trajectory of the 1-periodic nominal limit cycle. The reason is that the system maps all signals in $B_r(z_0)$ to this set and therefore we can modify the nonlinearity so that it is bounded or linear outside this range. This will not affect the conclusion of our analysis in $B_r(z^0)$.

To estimate the Lipschitz constant of $F'(s)$ we let $\delta F'_z = F'_z(z_2) - F'_z(z_1)$ and use $\|\delta F'_z\| = \sqrt{\|\delta F'_y\|^2 + \|\delta F'_T\|^2}$, where

$$\begin{aligned}\|\delta F'_T\| &= \sup_{z_2, z_1 \in B_r(z_0)} \|K(z_2) - K(z_1)\| \\ &\leq \sup_{T_2} \|\frac{s}{T_2^2} H(s/T_2)\| 2k_1 r \\ &+ \sup_{T_2, T_1} \|\frac{s}{T_2^2} H(s/T_2) - \frac{s}{T_1^2} H(s/T_1)\|(\|\Phi(y_0)\| + k_1 r)\end{aligned}$$

and

$$\begin{aligned}\|\delta F'_y\| &= \sup_{z_1, z_2 \in B_r(z_0)} \|L(z_2) - L(z_1)\| \\ &\leq \sup_{T_2} \|H(s/T_2)\| 2k_2 r \\ &+ \sup_{T_2, T_1} \|H(s/T_2) - H(s/T_1)\|(\|\Phi(y_0)\| + k_2 r)\end{aligned}$$

VII. EXAMPLES

Consider Van der Pol's equation

$$\ddot{y}(t) + m(y(t)^2 - 1)\dot{y}(t) + y(t) = 0$$

To represent this system on the form (1) we introduce the new coordinates

$$\begin{aligned}x_1 &= -\dot{y} - m(y^3/3 - y) \\ x_2 &= y\end{aligned}$$

The transformed system has the form (1) with $\varphi(y) = -my^3/3 + (2 + m)y$ and

$$H(s) = \frac{s}{s^2 + 2s + 1}$$

With let $m = 2$ and use Newton iteration using the right inverse implemented as a least squares problem. The algorithm converges in four steps as illustrated in Figure 1.

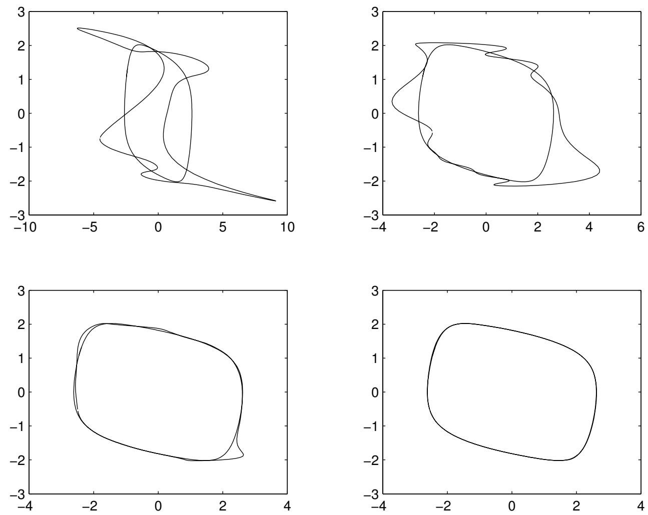

Fig. 1. Convergence of the Newton iteration applied to the Van der Pol oscillator.

VIII. CONCLUDING REMARKS

We have discussed a general framework for limit cycle analysis. The analysis can take place in either time domain or frequency domain. The frequency domain implementation have some advantages in that the best possible right inverse can readily be obtained using the SVD.

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