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Open-loop Unstable Feedback Systems with Double-Sided Inputs: an Explicit Demonstration of Self-Consistency

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Abstract—**The standard formulation of linear shift-invariant feedback systems in the doubly infinite time axis setting lacks self-consistency with seemingly irreconcilable difficulties having been identified when the system is open-loop unstable. The available options for circumventing these difficulties for discrete-time SISO systems are highlighted and the manner in which they are exploited to obtain a self-consistent framework by reformulating the feedback systems in the space of distributions is clarified. In addition, it is explicitly demonstrated that causality and stability of a standard example are implied by the causality and stability of the equivalent system when reformulated in the self-consistent framework.**

I. INTRODUCTION

THE self-consistency of linear time-invariant feedback systems or, more precisely, of the mathematical models/representations of the input signals, output signals and sub-systems, is not obvious. Recently, there has been some considerable interest in the doubly infinite time-axis setting (the signals having support $(-\infty,\infty)$), [3]-[12]. Unfortunately, instead of delivering a self-consistent framework, this work has identified some seemingly irresolvable difficulties when the system is open loop unstable, [3]-[11]. Of course, the analysis of feedback systems can be restricted to the singly infinite time axis setting (the signals having support $[0,\infty)$) the self-consistency of which is well established, [1],[2]. Nevertheless, the former setting is not uncommon in control theory, e.g. Wiener-Hopf optimal control and LQG control. Furthermore, control systems are becoming ever more complex, varied and disparate in nature requiring extensions to the standard framework. In this paper, in Section II, the recent work, [3]-[11], is examined to highlight the underlying premises and, thereby, identify the available options for circumventing the above difficulties for discrete-time SISO systems. In Section III, the self-consistent framework, proposed in [12], is discussed, the manner in which the available options are exploited being clarified. In addition, further to [12], it is explicitly demonstrated that causality and stability of a standard example are implied by the causality and stability of the equivalent system when reformulated in the self-consistent framework.

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II. INTRINSIC DIFFICULTIES

Consider the system of figure 1 for the operator $\mathcal{T}: D_{\mathcal{T}} \to R_{\mathcal{T}}$. The elements in a coherent analysis framework for feedback systems are the classes of functions representing the inputs and outputs and the class of operators representing the systems. Since the feedback system is equivalent to the relationship

$$[I+\mathcal{T}]y = x \quad (x-y) \in R_{\mathcal{T}}, \quad y \in D_{\mathcal{T}} \qquad (1)$$

the feedback loop is well defined provided $[I+\mathcal{T}]^{-1}$ exists. To illustrate the role of these elements consider the following example.

Example 1: Let the class of inputs and outputs be the integers and the operator be identity element, *I*. Since $[I+\mathcal{T}]^{-1} = \frac{1}{2} I$, the feedback system is not self consistent. This difficulty can be resolved by enlargement of the class of inputs and outputs to the rational numbers.

Since linear systems are being represented, the operators must be linear and the classes of inputs and outputs linear spaces. Ideally, $D_{\mathcal{T}}$ and $R_{\mathcal{T}}$ should be the same linear space to ensure compatibility when embedded in a larger system.

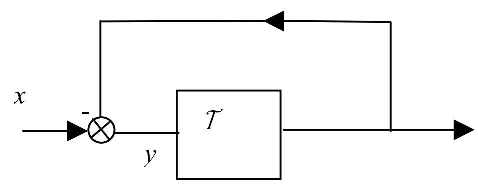

Fig. 1. Feedback system.

For the singly infinite time-axis, an explicit analysis in the input-output setting, establishing the self-consistency of feedback systems, is undertaken in [1] (continuous-time) and [2] (discrete-time). In the former, a dynamical system is considered to be a linear operator $P: \mathfrak{D}_P \subseteq L_2[0,+\infty) \to L_2[0,+\infty)$ with the graph of *P*, $\mathcal{G}_P$, defined by

$$\mathcal{G}_P = \begin{pmatrix} I \\ P \end{pmatrix} \mathfrak{D}_P \subset L_2[0,\infty) \times L_2[0,\infty). \qquad (2)$$

The relation of the properties of the dynamical system, such as stability and causality, to the mathematical properties of its graph is then investigated. One such property is stabilisability. Consider the feedback system in figure 2 denoted by [P,C]. The feedback system, [P,C], is stabilisable if there exists a C such that the operator

$$F = \begin{pmatrix} I & C \\ P & I \end{pmatrix} : D_P \times D_P \to L_2[0,\infty)\times L_2[0,\infty) : \begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \quad (3)$$

has a bounded inverse. Stabilisability requires, firstly, the feedback loop to be self-consistent then, secondly, the linear operator corresponding to the closed loop system to be bounded. The term, graph theory, is employed in this paper when referring to the above approach to the analysis of feedback systems. The analyses in continuous-time and in discrete-time are wholly equivalent with the results in one readily transferred to the other with minimal adjustment.

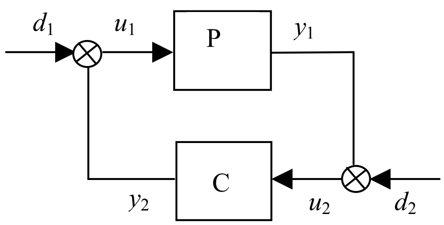

Fig. 2. Linear feedback system

In the development of [1] (translated to discrete-time) and [2], the class of functions is $\ell_2(Z_+)$. In graph theory, the definition of the operator is implicit rather than explicit. However, any linear shift-invariant operator on $\ell_2(Z_+)$ can be represented by a convolution, [5]. Hence, although not explicitly stated, the class of systems in [1] is the convolutions on $\ell_2(Z_+)$. The conditions for stabilisability are sufficient to ensure the feedback system is self-consistent.

To facilitate the usual analysis techniques for feedback systems, one final requirement for a framework is an equivalent representation in the transfer domain. In [1] (translated to discrete-time) and [2], the z-transform is integral to the derivation of results, it being an isometric isomorphism from $\ell_2(Z_+)$ to the Hardy space $\mathcal{H}_2(D)$ where D is the interior of the unit disc. (Note, each element of $\widetilde{X}(q) \in \mathcal{H}_2(D)$ is defined by $\widetilde{X}(q) = X(z)|_{z=q^{-1}}$ where $X(z)$ is the z-transform of some $x \in \ell_2(Z_+)$). Hence, implicit to [1] is the equivalent transfer domain representation through the standard z-transform.

For the doubly infinite time axis when P is unstable, the graph of the operator $P : \mathcal{D}_P \subseteq L_2(-\infty,+\infty) \to L_2(-\infty,+\infty)$ need not be closed, a necessary condition [3] for stabilisability of the feedback system in figure 2. This observation has serious consequences for any attempt to establish a self-consistent framework. Graph theory for the doubly infinite time axis is investigated in [3]-[7].

A. Simple Example

In [7], a simple example is investigated to illustrate the need to establish a self-consistent framework to prevent inconsistencies. The discrete-time, first-order convolution system

$$y[i] = b\sum_{n\ge 0} a^n (u[i-n-1] + v[i-n+1]) + d[i]\,;\; u[i] = -ky[i] \quad (4)$$

is discussed. The output, y, the input, u, and the disturbance or noise terms, v and d, are all possibly double-sided. The following Theorem (Theorem 3, [8]) is proved by a wholly time-domain argument.

Theorem 1: *Consider the feedback configuration (4). Let a-kb=0. There exists a (unique) solution y[i] for any i if and only if*

$$\lim_{i\to-\infty} a^{-i}(-kd[i] + v[i]) = 0\,. \quad (5)$$

Let the open-loop system be unstable, i.e. $|a|>1$. It follows from Theorem 1 that, when (kb-a)=0, the closed-loop system does not have a solution for every square summable v and d.

The above system, (4), is also investigated in the transform domain. Let $b\neq 0$ and v and d be double-sided square summable real sequences. The feedback system (4) becomes

$$Y(z) = G(z)[U(z) + V(z)] + D(z) \quad ; \quad U(z) = -kY(z) \quad (6)$$

where $Y(z)$, $U(z)$, $V(z)$ and $D(z)$ are the bilateral z-transforms of y, u, v and d, respectively, and $G(z) = bz^{-1}/(1-az^{-1})$ is the usual transfer function for the open-loop system. On eliminating $U(z)$,

$$[1+kG(z)]Y(z) = G(z)V(z) + D(z)\,. \quad (7)$$

Hence,

$$Y(z) = [1+kG(z)]^{-1}G(z)V(z) + [1+kG(z)]^{-1}D(z)\,. \quad (8)$$

The conclusion from the above analysis, which is not qualified by any restriction on $V(z)$ and $D(z)$, is that, when $|a|>1$, the closed-loop system is stable provided $|kb-a|<1$, including $|kb-a|=0$ in contradiction to the time-domain analysis.

It is worthwhile reconsidering the above z-domain analysis. In the conventional treatment, [13], the domain of $X(z)$, the z-transform of $x[i]$, is the region, $0 \le \underline{R}_X < |z| < \overline{R}_X$, in which $\sum_{i=-\infty}^{\infty} x[i]\,|z|^{-i}$ converges; that is, X(z) is analytic for $\underline{R}_X < |z| < \overline{R}_X$ and the sequence, {x[i]}, is the coefficients of the Laurent Series for $\widetilde{X}(q) = X(z)|_{z=q^{-1}}$. When {$x$[i]} is a square-summable doubly-infinite sequence, $\underline{R}_X < 1 < \overline{R}_X$. Hence, it is more precise to define the z-transform of x[i] by the doublet {$X(z)$,D_X}, where D_X is the region $\underline{R}_X \le |z| \le \overline{R}_X$, rather than by $X(z)$ alone. Similar considerations apply to the transfer function, $G(z) = \sum_{i=0}^{\infty} a^i z^{-i}$. It is more precise to define the transfer function by the doublet {$G(z)$,D_G} where

D_G is the region in which the summation converges. For the open-loop system (4), D_G is the region $|z|>|a|$. (Changing D_G is tantamount to changing the system since the coefficients of the Laurent Series and so the convolution in (4) changes.) Addition is now the doublet manipulation

$$\{A(z), D_A\} + \{B(z), D_B\} = \{A(z) + B(z), D_A \cap D_B\} \quad (9)$$

and multiplication

$$\{A(z), D_A\}.\{B(z), D_B\} = \{A(z)B(z), D_A \cap D_B\}. \quad (10)$$

Hence, the right-hand side in (7) is more precisely restated as $\{G(z)V(z) + D(z), (D_G \cap D_V) \cap D_D = D_G \cap D_W\}$, where $W(z) = -kD(z) + V(z)$. When $|a| > \overline{R}_W > 1$, $D_G \cap D_W = \phi$ and the right hand side of (7) does not exist in a meaningful sense. Consequently, the closed-loop system only has a solution when $|a| < \overline{R}_W$, specifically, $a^{-i} w[i] \to 0$ as $i \to -\infty$ is necessary, $b^{-i} w[i] \to 0$ for some $b>|a|$ sufficient.

The z-domain analysis is now in close agreement to the time-domain analysis. The source of the problem is not the step from (7) to (8) as stated in [8] but is already present in (7). Nevertheless, contradictory to the implications of Theorem 1, it remains a fact that the closed-loop system is stable and the output exists and is square summable when $|kb - a| < 1$, including $|kb - a| = 0$.

B. Graph Theory Analysis

In [3], a continuous-time system, $P_1 : D_1 \subset L_2(-\infty,\infty) \to L_2(-\infty,\infty)$, is discussed within the context of graph theory. The system, P_1, is defined by the convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \quad (11)$$

with $u \in L_2(-\infty,\infty)$ and $h(t) = e^t (t \geq 0), 0 (t < 0)$. It is established that the graph for P_1 is not closed and, hence, does not satisfy the conditions for stabilisability. The argument is outlined below. On the restricted domain, $D_1 \cap L_2[0,\infty)$, P_1 coincides with the system, $P_2 : D_2 \subset L^2(-\infty,\infty) \to L^2(-\infty,\infty)$, defined by (11) but with $h(t) = 0 (t > 0), -e^t (t \leq 0)$. The graph for P_2 in the Fourier domain is the closure of $\bigcup_{T \geq 0} e^{sT} G \mathcal{H}_2(C_+)$, where $G = [(s-1)/(s+1) \quad 1/(s+1)]^T$ and $\mathcal{H}_2(C_+)$ is the Hardy space on the right-half plane. Furthermore, the graph for P_1 in the Fourier domain contains $\bigcup_{T \geq 0} e^{sT} G \mathcal{H}_2(C_+)$. Hence, if the graph for P_1 is closed it must contain the graph for P_2. But there are elements in the graph for P_2 (for example, the non-causal input-output pair $u(t) = e^{-t} (t \geq 0), 0\ (t < 0)$ and $-e^{-|t|}/2$) that are not contained in the graph for P_1 (causal). Consequently, the graph for P_1 cannot be a closed graph. Moreover, the closure of the graph for P_1 must contain the graph for P_2 and be non-causal, [5]. An equivalent discrete-time system exhibiting the same difficulties, [7], is $P : D_P \subset \ell_2(Z) \to \ell_2(Z)$,

$$Pu[i] = \sum_{n=-\infty}^{i} 2^{i-n} u[n] \quad , \quad u \in D_P \quad (12)$$

The observation, that the closure of the graph for a system may not be causal, motivated the need to determine the necessary and sufficient conditions for the graph of a system to be causally closable. The discrete-time system, $P : D_P \subset \ell_2(Z) \to \ell_2(Z)$, defined by the convolution

$$Pu[i] = \sum_{n=-\infty}^{i} g[i-n]u[n] \quad , \quad u \in D_P \quad (13)$$

is investigated in [5]. (MIMO systems are investigated in [7]). The following Theorem (Theorem 15, [5]) is established.

Theorem 2 *Let P be an LTI system on $\ell_2(Z)$, given by (13). Further, let $D_P \cap \ell_2(Z_+) \neq \{0\}$, then the following statements are equivalent*
1) P causally closable;
2) $\widetilde{G}(q) = \sum_{i \in Z_+} g[i] q^i$ belongs to the Smirnoff class.

where the Smirnoff class [2] is defined by

Definition 1 *An analytic function $f : D_f \subset C \to C$ is said to belong to the Smirnoff class if it can be written as $f = f_1 / f_2$ with $f_1, f_2 \in H_\infty$ and f_2 outer.*

An outer function can have no zeroes strictly inside the unit disc [2] and, so, a function belonging to the Smirnoff class cannot have any poles in the unit disc. However, $G(z) = \widetilde{G}(q)|_{q=z^{-1}}$ is the usual system transfer function. Hence, the graph for an exponentially unstable system cannot be causally closable. Apparently, graph theory is unable to resolve the difficulties encountered with unstable systems and double-side inputs.

C. Attempts at Resolution

To resolve the difficulties with unstable systems and double-side inputs, one possibility is to replace its operator by some extension. In [4] and [6], it is proposed to replace the operator for a convolution system by its closure. (Closing the graph of the operator is not the same as closing the operator). Closure of the operator does not necessarily preserve causality; for example, in [3], the closure of the operator includes the non-causal solution. Furthermore, it is shown that the convolution system, $P : \ell_p(Z) \to \ell_p(Z)$, defined by the impulse response $\{ba^{k-1}\}|_{k \geq 1}$, $|a|>1$, has no $\ell_p(Z_-)$ closure. A consequence of this observation is that, although finite $\ell_p(Z)$ stabilisability requires finite $\ell_p(Z_-)$ stabilisability, no finite $\ell_p(Z_-)$ gain stabilisable $\ell_p(Z_-)$

extension of the system is possible. A subspace, $\tilde{\ell}_p(Z) \subset \ell_p(Z)$, must be specified such that the operator for this system is closable in $\tilde{\ell}_p(Z_-) \subset \tilde{\ell}_p(Z) \cap \ell_p(Z_-)$. The subspace, $\tilde{\ell}_p(Z)$, needs to be chosen such that $\lim_{i\to-\infty} a^{-i}x[i] = 0, \forall x \in \tilde{\ell}_p(Z_-)$. The conclusion reached is that it is very difficult to perform a complete analysis of this situation and the difficulties remain unresolved.

A further possibility is proposed in [9]. The standard model for a system

$$y = G(u+v) + d \quad ; \quad u = C(r-y) \tag{14}$$

where G and C are possibly unstable convolution operators in $\ell_1(Z)$, is replaced by

$$U_W \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} Nv + Dd \\ Xr \end{bmatrix}, U_W = \begin{bmatrix} D & -N \\ X & Y \end{bmatrix} \tag{15}$$

where $N, D, X, Y : \ell_\infty(Z) \to \ell_\infty(Z)$ are stable, causal, discrete-time, convolution operators in ℓ_1; more precisely, the coefficients of their impulse response are in $\ell_1(Z_+)$. There is a causal unique solution in $\ell_\infty(Z) \times \ell_\infty(Z)$ provided U_W^{-1} is also a causal convolution operator in ℓ_1. However, on the space of all double-sided sequences, the null-space of U_W is not empty, thereby, rendering the solutions non-unique. Consequently, when the system inputs are zero, the system output may not be zero. The system, (15), is, therefore, not linear. The system, (15), is further modified to

$$\begin{bmatrix} DH & -NH \\ XV & YV \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} NHv + DHd \\ XVr \end{bmatrix} \tag{16}$$

with N and V any causal convolution operators. To regain uniqueness, a solution is required to satisfy (16) for all N and V. Unfortunately, the way to directly apply transform methods to these modified models remains unclear.

III. Difficulties Resolved

In the graph theory analysis, examined in Section II, the class of inputs and outputs is chosen to be $\ell_2(Z)$ and the class of operators the convolutions on $\ell_2(Z)$. Since it is integral to the derivation of results, the transition to the transfer domain description is through the z-transform. In the attempts at resolution, the class of inputs and outputs is chosen to be $\ell_p(Z)$, for some $p \geq 0$, and the class of operators again the convolutions. As in [11], the analysis is wholly in the time-domain and the transition to the transfer domain description is not discussed.

With the above choice of classes for the inputs, outputs and operators, there appears to be no prospect of a self-consistent framework for feedback systems. However, in example 1, the lack of self-consistency is resolved by enlarging the class of inputs and outputs from the integers to the rational numbers. By analogy, one option for feedback systems is, thus, enlargement of the class of inputs and outputs. Furthermore, such simple, if unjustified, arguments as (6) to (8) must be supported by the transfer domain representation equivalent to that enlargement. The option, in this case, is to make the transition to the transfer domain less direct. These two options are exploited in the self-consistent framework discussed below, [12].

A. Equivalent Reformulation

In [12], a causal but possibly unstable open-loop system is first recast as an equivalent stable but possibly non-causal system. Establishing that the closed-loop system for the latter is causal establishes that the closed-loop system for the former is stable. For the simple example of [8], (4), the equivalent system is

$$\begin{aligned} &y[i] = -b\sum_{n<0} a^n (u[i-n-1] + v[i-n+1]) + d[i] \,, |a| > 1 \\ &u[i] = -ky[i]. \end{aligned} \tag{17}$$

The solution to (17), $\forall v[i], d[i] \in \ell_p, p = 1,2$, is

$$y[i] = b\sum_{n>0} (kb-a)^{n-1} (v[i-n] + kd[i-n]) + d[i] \ . \tag{18}$$

That stability of system, (4), can be inferred from the causality of system, (18), follows from Theorem 3 below.

Let $T_S : \mathbb{S}_S \subset \ell_p \to \mathfrak{R}_S$, p=1,2, be a linear shift-invariant operator such that

$$T_S x[i] = x[i] - \Phi[i] * x[i], \forall x[i] \in \mathbb{S}_S$$

$$\Phi[i] = kb(a-kb)^{(i-1)}\theta[i] \,, |a| > 1 \,, |a-kb| < 1; \theta[i] = \begin{cases} 1 \,, i > 0 \\ 0 \,, i \leq 0 \end{cases}$$

and, with the analogous definitions, let

$$T_P x[i] = \Pi[i] * x[i], \forall x[i] \in \mathbb{S}_P \quad ; \quad \Pi[i] = -kba^{(i-1)}(1-\theta[i])$$

$$T_G x[i] = \Gamma[i] * x[i], \forall x[i] \in \mathbb{S}_G \quad ; \quad \Gamma[i] = kba^{(i-1)}\theta[i]$$

$$T_R x[i] = \Psi[i] * x[i], \forall x[i] \in \mathbb{S}_R \quad ; \quad \Psi[i] = kba^{(i-1)} .$$

Theorem 3: (i) $\forall x[i] \in \ell_p$, $(I + T_P)(T_S x[i]) = x[i]$ and equations

$$y[i] = T_p u[i] \,, u[i] = r[i] - y[i] \quad ; \quad r[i], u[i], y[i] \in \ell_p \tag{19}$$

have, $\forall r[i] \in \ell_p$, the solution

$$y[i] = T_P T_S r[i] = \Phi[i] * r[i] \quad , \quad u[i] = T_S r[i] .$$

(ii) $\forall x[i] \in a^{-i}\ell_p \cap \ell_p$, $T_R(T_S x[i]) = 0$ and equations

$$y[i] = T_G u[i] \,, u[i] = r[i] - y[i] \quad ; \quad r[i], u[i], y[i] \in \ell_p \tag{20}$$

have, $\forall r[i] \in a^{-i}\ell_p \cap \ell_p \subset \ell_p$, the solution

$$y[i] = T_G T_S r[i] = T_P T_S r[i] = \Phi[i] * r[i] \quad , \quad u[i] = T_S r[i] .$$

Proof: Clearly, $\mathbb{S}_S = \mathbb{S}_P = \ell_p$ and $\mathfrak{R}_S, \mathfrak{R}_P \subset \ell_p$ but $\mathbb{S}_G = \mathbb{S}_R = \{x[i] : a^{-i}x[i] \in \ell_1\} \cap \ell_p$ and $\mathfrak{R}_G, \mathfrak{R}_R \subset \{x[i] : x[i] \in a^i \ell_\infty\}$. Hence, $\forall x[i] \in \mathbb{S}_S$,

$$\begin{aligned} &(I + T_P)(T_S x[i]) = \\ &\quad x[i] + \Pi[i] * x[i] - \Phi[i] * x[i] - (\Pi[i] * \Phi[i]) * x[i] = x[i] \end{aligned}$$

since
$\Pi[i]*\Phi[i]=$

$$-\sum_{n=-\infty}^{\infty} kba^{(i-n-1)}(1-\theta(i-n))kb(a-kb)^{(n-1)}\theta[n]=\Pi[i]-\Phi[i].$$

Part (i) follows immediately. Furthermore, $(a^{-i}(a-kb)^{i}\theta[i])*(a^{-i}x[i])\in\ell_1$ when $a^{-i}x[i]\in\ell_1$ and

$$\Psi[i]*\Phi[i]=\sum_{n=-\infty}^{\infty} kba^{(i-n-1)}\theta[n]kb(a-kb)^{(n-1)}=\Psi[i].$$

Hence, $\forall x[i]\in\mathfrak{S}_R$, $\Psi[i]*(\Phi[i]*x[i])$ exists and equals $(\Psi[i]*\Phi[i])*x[i]$. Consequently,

$$T_R(T_S x[i])=\Psi[i]*x[i]-(\Psi[i]*\Phi[i])*x[i]=0$$

and part (ii) follows immediately since $T_G=T_P+T_R$.

The solutions to equations (19) and (20) are the same, for all $r[i]\in a^{-i}\ell_p\cap\ell_p$. Furthermore, the open-loop systems for (4) and (17) are equivalent to the operators T_G and T_P, respectively, and the solution (18) to T_PT_S. Hence, system, (17), being stable (and causal) implies system, (4), is causal (and stable) but only for the inputs $r[i]\in a^{-i}\ell_p\cap\ell_p$.

In [11], to recast a causal but possibly unstable open-loop system as an equivalent stable but possibly non-causal system, the transfer function for the former is analytically continued into an analytic region containing the unit circle. The transfer function for system, (4), is $\{b/(z-a),|z|>a\}$ and for system, (17), is $\{b/(z-a),|z|<a\}$. For the latter, restricting the domain to $|z|=1$, $b/(z-a)|_{z=e^{j\omega T}}$ is a periodic function the Fourier coefficients of which are the time series for the impulse response of system, (17). The transition to the transfer domain is now less direct being the analytic continuation of the transfer function. However, it enables a straightforward enlargement of the class of inputs and outputs to the distributions, [14].

B. Distributions

Let $\mathfrak{S}$ be the linear space of functions defined almost everywhere on the real line. A sub-space, $\hat{\mathfrak{S}}\subset\mathfrak{S}$, is shift-invariant when, $\forall f(t)\in\hat{\mathfrak{S}}$, $f(t-a)\in\hat{\mathfrak{S}}$ for all a. Clearly, $\mathfrak{S}$ itself is shift-invariant. Let $\mathfrak{F}$ be the linear space of linear functionals with domain a shift-invariant sub-space of $\mathfrak{S}$. In addition, let $\mathfrak{F}_D\subset\mathfrak{F}$ be the sub-space of functionals with domain containing D and continuous in D where D is the class of good functions with finite support. Furthermore, let $\mathfrak{F}_T\subset\mathfrak{F}_D\subset\mathfrak{F}$ be the sub-space of functionals with domain containing S and continuous in S where S is the class of good functions of infinite support. The restriction of the functionals in $\mathfrak{F}_D$ to D is the linear space of distributions, $\mathfrak{D}$, and the restriction of the functionals in $\mathfrak{F}_T$ to S is the linear space of tempered distributions, $\mathfrak{D}_S$. The value assigned to each $f(t)$ in its domain, by the functional, $x\in\mathfrak{F}$, is denoted by $x[f(t)]$. A shifted functional is indicated by a subscript; that is, x_a is defined such that $x_a[f(t)]=x[f(t+a)]$ for all $f(t)$ in the domain of x. An operator, T, with domain in $\mathfrak{F}$ is shift-invariant if $y_a=Tx_a$, $\forall a$, whenever $y=Tx$.

The following linear sub-spaces of $\mathfrak{D}$ are required

$$\mathfrak{D}^T=\left\{x\in\mathfrak{D}:x=\sum_{k=-\infty}^{\infty}a_k\delta_{kT}\right\}$$

$$\mathfrak{D}_E^T=\left\{x\in\mathfrak{D}^T:a_k/(1+|k|)^N\text{ square summable for some }N\geq 0\right\}$$

$$\mathfrak{D}_{EN}^T=\left\{x\in\mathfrak{D}^T:a_k/(1+|k|)^N\text{ square summable}\right\}\quad;\quad N\geq 0$$

for some $T>0$, where the functional $\delta_T\in\mathfrak{D}$ is defined by

$$\delta_\tau[\phi(t)]=\phi(\tau),\ \forall\phi\in\mathrm{D}\tag{21}$$

The definitions of $\mathfrak{D}^T$, $\mathfrak{D}_E^T$ and $\mathfrak{D}_{EN}^T$ are specific to some value of the parameter, T, and $\mathfrak{D}_E^T$ and $\mathfrak{D}_{EN}^T$ are sub-classes of $\mathfrak{D}_S$, the class of tempered distributions.

Each functional $x\in\mathfrak{D}$ is related by a linear bijection to a functional $X\in\mathfrak{U}$, the class of ultra-distributions, such that $x[\phi^*(t)]=2\pi X[\Phi^*(\omega)]$ for all $\phi(t)\in\mathrm{D}$ with $\Phi(\omega)$ the Fourier transform of $\phi(t)$. The functionals, x and X, constitute a Fourier transform pair with $X=\mathfrak{F}\{x\}$ and $x=\mathfrak{F}^{-1}\{X\}$. The sub-classes, $\mathfrak{U}_S$, $\mathfrak{U}^T$, $\mathfrak{U}_E^T$ and $\mathfrak{U}_{EN}^T$ of $\mathfrak{U}$ are defined as those for which the members are the Fourier transforms of the members of the corresponding sub-class of $\mathfrak{D}$. $\mathfrak{U}^T$ and $\mathfrak{U}_E^T$ are the sub-classes of $\mathfrak{U}$ consisting, respectively, of all periodic functionals in $\mathfrak{U}$ and $\mathfrak{U}_S$ of period $2\pi/T$.

The most general extension to Fourier series is provided by $\mathfrak{U}^T$. For any sequence $\{x[i]\}$, the functional $x\in\mathfrak{D}^T$ with $a_i=x[i]$ is related by the Fourier transform to a periodic functional $X\in\mathfrak{U}^T$ such that $X=\mathfrak{F}\{x\}$ and $x=\mathfrak{F}^{-1}\{X\}=\sum_{i=-\infty}^{\infty}x[i]\delta_{iT}$. There thus exists a linear bijection between the class of all sequences and $\mathfrak{U}^T$. Furthermore,

$$\mathfrak{F}\{\textstyle\sum_{i=-\infty}^{\infty}x[i]\delta_{iT}\}=\sum_{i=-\infty}^{\infty}x[i]e_{iT}$$

where e_{kT} is the regular functional defined by the function $e^{-jk\omega T}$. The functional $\sum_{i=-\infty}^{\infty}x[i]e_{iT}$ is the Fourier series and the sequence $\{x[i]\}$ the Fourier coefficients for X.

C. Enlargement of Signal Class

In [12], the classes of inputs and outputs are both chosen to be $\mathfrak{D}_E^T$, i.e. the discrete-time signal, $x[i]$, is represented by the distribution, $\sum_{i=-\infty}^{\infty}x[i]\delta_{iT}$. The class of systems is chosen to be those convolutes with Fourier transforms, i.e. system functions, the periodic multipliers in $\mathfrak{U}_S$ of period $2\pi/T$. In this framework, the representation for system (17) is

$$y=\mathrm{P}*(u-v)+d\quad;\quad u=-ky\tag{22}$$

where $y,u,v,d \in \mathfrak{D}_E^T$ and $\mathrm{P} = -b\sum_{n\le 0} a^{n-1}\delta_{nT}$. The solution to (22), $\forall v,d \in \mathfrak{D}_{EN}^T$, is

$$y = -\mathrm{H} * (v + kd) + d \qquad (23)$$

where $\mathrm{H} = -b\sum_{n>0}(kb-a)^{n-1}\delta_{nT}$. $\mathfrak{F}\{\mathrm{P}\}$ and $\mathfrak{F}\{\mathrm{H}\}$ are the multipliers in $\mathcal{U}_\mathrm{S}$ defined by the functions, $b/(e^{j\omega T} - a)$ and $b/(e^{j\omega T} + kb - a)$, respectively. Both P and H are stable, [12], in the sense of mapping $\mathfrak{D}_{EN}^T$ into $\mathfrak{D}_{EN}^T$, $\forall N$, but only H is causal. The representation for system (4) is

$$y = \mathrm{G} * (u - v) + d \quad ; \quad u = -ky \qquad (24)$$

where $\mathrm{G} = b\sum_{n>0} a^{n-1}\delta_{nT}$. Note, that G is not a convolute and so the equations, (24), do not necessarily have a solution. That existence of stable solutions for (22) can be inferred from the causality of (23) follows from Theorem 4 below.

Let $\widetilde{S} : \mathfrak{S}_\mathrm{S} \subset \mathfrak{S} \to \mathfrak{R}_\mathrm{S} \subset \mathfrak{S}$ be a linear operator such that

$$\widetilde{S}f(t) = f(t) - \sum_{i=-\infty}^{\infty} kb(a-kb)^{-(i+1)}\theta[-i]f(t-iT), \forall f(t) \in \mathfrak{S}_\mathrm{S}.$$

Note that both $\mathfrak{S}_\mathrm{S}$ and $\mathfrak{R}_\mathrm{S}$ are shift-invariant sub-spaces of $\mathfrak{S}$ and that $\mathrm{D} \subset \mathfrak{S}_\mathrm{S}$. Let $\mathfrak{I}_\mathrm{S} \subset \mathfrak{I}$ be the restriction of the functionals in $\mathfrak{I}$ to $\mathfrak{R}_\mathrm{S}$. The shift-invariant linear operator, T_S, is defined by

$$y = \mathrm{T}_\mathrm{S}x : y[f(t)] = x[\widetilde{S}f(t)], \forall x \in \mathfrak{I}_\mathrm{S}.$$

The domain of y is the $f(t)$ such that $\widetilde{S}f(t)$ is in the domain of x. Similarly, the shift-invariant linear operators, T_P, T_G and T_R, are defined by

$$y = \mathrm{T}_\mathrm{P}x : y[f(t)] = x[\widetilde{P}f(t)], \forall x \in \mathfrak{I}_\mathrm{P}$$

$$y = \mathrm{T}_\mathrm{G}x : y[f(t)] = x[\widetilde{G}f(t)], \forall x \in \mathfrak{I}_\mathrm{G}$$

$$y = \mathrm{T}_\mathrm{R}x : y[f(t)] = x[\widetilde{R}f(t)], \forall x \in \mathfrak{I}_\mathrm{R}.$$

with the analogous definitions, where

$$\widetilde{P}f(t) = -\sum_{i=-\infty}^{\infty} kba^{-(i+1)}(1-\theta[-i])f(t-iT), \forall f(t) \in \mathfrak{S}_\mathrm{P}$$

$$\widetilde{G}f(t) = \sum_{i=-\infty}^{\infty} kba^{-(i+1)}\theta[-i]f(t-iT), \forall f(t) \in \mathfrak{S}_\mathrm{G}$$

$$\widetilde{R}f(t) = \sum_{i=-\infty}^{\infty} kba^{-(i+1)}f(t-iT), \forall f(t) \in \mathfrak{S}_\mathrm{R}.$$

Theorem 4: (i) $\forall x \in \mathfrak{I}_\mathrm{D}, \phi(t) \in \mathrm{D}, (\mathrm{I} + \mathrm{T}_\mathrm{P})\mathrm{T}_\mathrm{S}x[\phi(t)] = x[\phi(t)]$ and equations

$$y = \mathrm{T}_\mathrm{p}u, \; u = r - y \quad ; \quad r,u,y \subset \mathfrak{I}_\mathrm{D} \qquad (25)$$

have, $\forall r \in \mathfrak{D}_\mathrm{S} \subset \mathfrak{I}_\mathrm{D}$, the solution

$$y = \mathrm{T}_\mathrm{P}\mathrm{T}_\mathrm{S}r = (\mathrm{I} - \mathrm{T}_\mathrm{S})r \quad , \quad u = \mathrm{T}_\mathrm{S}r.$$

(ii) $\forall x \in \mathfrak{I}_\mathrm{D}, \phi(t) \in \mathrm{D}$, $\mathrm{T}_\mathrm{R}\mathrm{T}_\mathrm{S}x[\phi(t)] = 0$

and equations

$$y = \mathrm{T}_\mathrm{G}u, \; u = r - y \quad ; \quad r,u,y \subset \mathfrak{I}_\mathrm{D} \qquad (26)$$

have, $\forall r \in \mathfrak{D}_\mathrm{S} \subset \mathfrak{I}_\mathrm{D}$, the solution.

Proof: Clearly, $\mathrm{T}_\mathrm{S}x, \mathrm{T}_\mathrm{P}x \in \mathfrak{I}_\mathrm{D}, \forall x \in \mathfrak{I}_\mathrm{T}$ but $\mathrm{T}_\mathrm{G}x$ and $\mathrm{T}_\mathrm{R}x$ need not. $\forall x \in \mathfrak{I}_\mathrm{D}, \phi(t) \in \mathrm{D}$, $(\mathrm{I} + \mathrm{T}_\mathrm{P})\mathrm{T}_\mathrm{S}x[\phi(t)] = x[\phi(t)]$ since $(\mathrm{I} + \mathrm{T}_\mathrm{P})\mathrm{T}_\mathrm{S}x[\phi(t)] = x[\widetilde{S}\phi(t) + \widetilde{S}\widetilde{P}\phi(t)]$ and, $\forall \phi(t) \in \mathrm{D}$, $\widetilde{S}\widetilde{P}\phi(t) = (I - \widetilde{S})\phi(t)$. Part (i) follows immediately. Also, for all $x \in \mathfrak{I}_\mathrm{D}, \phi(t) \in \mathrm{D}$, $\mathrm{T}_\mathrm{R}\mathrm{T}_\mathrm{S}x[\phi(t)] = 0$ since $\mathrm{T}_\mathrm{R}\mathrm{T}_\mathrm{S}x[\phi(t)] = x[\widetilde{S}\widetilde{R}\phi(t)]$ and, $\forall \phi(t) \in \mathrm{D}, \widetilde{S}\widetilde{R}\phi(t) = 0$. Part (ii) follows immediately.

The solutions to equations (25) and (26) are the same, for all $r \in \mathfrak{D}_\mathrm{S}$. Furthermore, the open-loop systems for (22) and (24) are equivalent to the operators T_G and T_P, respectively, and the solution (23) to $\mathrm{T}_\mathrm{P}\mathrm{T}_\mathrm{S}$. Hence, the solutions to (24) exist $\forall v,d \in \mathfrak{D}_\mathrm{E}^\mathrm{T}$ and are stable.

IV. Conclusion

Clearly by reformulating in the framework of [12], self-consistency of discrete-time unstable systems with doubly infinite time axis is achieved. The class of inputs and outputs is greatly enlarged to any polynomially bounded signal. In the equivalent transform domain, the standard transfer function analysis applies.

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