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Discrete-Time Adaptive Stabilization and Disturbance Rejection for Minimum Phase Plants

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Abstract Discrete-time adaptive disturbance rejection is relevant to active noise and vibration control. In this paper we develop an adaptive feedback disturbance rejection algorithm, for minimum phase plants, that does not require a measurement of the disturbance and requires only minimal information about the plant, namely the first nonzero moving average coefficient from control to performance. We also prove Lyapunov stability and asymptotically perfect disturbance rejection for disturbances generated by LTI Plants.

1. Introduction

A large portion of the feedback adaptive control literature is devoted to proving stability of the closed loop system and boundedness of solutions [1–6]. In many applications however, the plant is open-loop stable or stability can be achieved with a controller based on a nominal model and the overriding concern is performance with respect to disturbance rejection [7, 8].

In this paper we develop a method for achieving simultaneous adaptive stabilization and disturbance rejection. Conventional one step ahead adaptive control methods [3] and predictive control methods [9], minimize the predicted error between the desired and actual output to determine the control signal. In the present work we minimize a cost function based on a retrospective output. Since the retrospective cost can be computed without using a measurement of the disturbance (See Section 2), the parameter update law is also independent of the disturbance.

A disturbance rejection algorithm based on the retrospective cost as well as non-minimal plant and controller realizations was introduced in [10, **?**, 11]. The approach proved successful in applications and exhibits significant robustness [12]. However a proof of closed loop stability and disturbance rejection performance have not been given.

In the present paper we utilize minimal time series models to describe both the plant and controller, we do not require the plant to be asymptotically stable, and we prove closed-loop Lyapunov stability and asymptotically perfect disturbance rejection. Furthermore we do not require knowledge of all moving average coefficients from control to performance as in [10, 11], but rather we require only the first nonzero moving average coefficient and the relative degree from control to performance.

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The contents of the paper are as follows. In section 2 we describe the model the plant and controller models and also state all relevant assumptions. The adaptive algorithm is described in section 3. In section 4 we formulate a state space representation of the closed-loop system. Lyapunov stability of the closed-loop system and asymptotically perfect disturbance rejection for bounded disturbances is demonstrated in section 5. Finally in section 6 we use the proposed algorithm to reject tonal and broadband disturbances in an acoustic duct.

2. Plant and Controller Models

Consider the linear time invariant discrete-time system $G_{\rm p}$ described by the time series model

$$z(k) = \sum_{j=1}^{n} -\alpha_j z(k-j) + \sum_{j=0}^{n} \Omega_j w(k-j) + \sum_{j=d}^{n} \Pi_j u(k-j), \tag{2.1}$$

where the control vector $u(k) \in \mathbb{R}^{m_u}$, the disturbance vector $w(k) \in \mathbb{R}^{m_w}$ and the performance vector $z(k) \in \mathbb{R}^{l_z}$, $\alpha_j \in \mathbb{R}$, $\Omega_j \in \mathbb{R}^{l_z \times m_w}$ and $\Pi_j \in \mathbb{R}^{l_z \times m_u}$. Notice that the delay (relative degree) d in the path from u to z appears explicitly. It will play a role in the later development. We make the following assumptions about the plant (2.1).

Assumption 2.1: The delay d is known.

Assumption 2.2: The number of inputs is greater than or equal to the number of performance variables, i.e., $m_u \geq l_z$.

Assumption 2.3: The matrix $\Pi_d \in \mathbb{R}^{l_z \times m_u}$ is known and has full row rank.

Assumption 2.4: The transfer function matrix G_{zu} from u to z is stably invertible.

Assumption 2.5: The performance $z(k)$ is measured.

Assumption 2.6: The disturbance $w(k)$ is generated by the free response of an LTI system.

Let $G_{\rm c}$ be a controller of order n_c given by the time series model

$$u(k) = -\sum_{j=1}^{n_c} \hat{\Gamma}_j(k) u(k-j) + \sum_{j=1}^{n_c} \hat{\Upsilon}_j(k) z(k-j) \tag{2.2}$$

where $\hat{\Gamma}_j \in \mathbb{R}^{m_u \times m_u}$ and $\hat{\Upsilon}_j \in \mathbb{R}^{m_u \times l_z}$. Next, define $q_1 \triangleq n_c m_u$, $q_2 \triangleq n_c l_z$, $q_3 \triangleq q_1 + q_2$,

$$U(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_c) \end{bmatrix} \in \mathbb{R}^{q_1}, \tag{2.3}$$

$$Z(k) \triangleq \begin{bmatrix} z(k-1) \\ \vdots \\ z(k-n_c) \end{bmatrix} \in \mathbb{R}^{q_2}, \qquad (2.4)$$

$$\phi(k) \quad \triangleq \quad \begin{bmatrix} U(k) \\ Z(k) \end{bmatrix} \in \mathbb{R}^{q_3}, \qquad (2.5)$$

and

$$\hat{\Theta}(k) \triangleq \left[-\hat{\Gamma}_1(k) \cdots -\hat{\Gamma}_{n_c}(k)\ \hat{\Upsilon}_1(k) \cdots \hat{\Upsilon}_{n_c}(k)\right] \in \mathbb{R}^{m_u \times q_3}.$$

Then

$$u(k) = \hat{\Theta}(k)\phi(k), \qquad (2.6)$$

and the closed-loop performance (2.1) , (2.6) is given by

$$z(k) = \sum_{j=1}^{n} -\alpha_j z(k-j) + \sum_{j=0}^{n} \Omega_j w(k-j) + \sum_{j=d+1}^{n} \Pi_j u(k-j) + \Pi_d \hat{\Theta}(k-d)\phi(k-d). \qquad (2.7)$$

We define the *retrospective performance* signal

$$z_r(k) \triangleq \sum_{j=1}^{n} -\alpha_j z(k-j) + \sum_{j=0}^{n} \Omega_j w(k-j) + \sum_{j=d+1}^{n} \Pi_j u(k-j) + \Pi_d \hat{\Theta}(k) R \phi(k-d), \quad (2.8)$$

and use the following assumption concerning deadbeat disturbance rejection.

Assumption 2.7: There exists $\Theta^* \in \mathbb{R}^{m_u \times q_5}$ and $k_0 \geq d$ such that, for all $k \geq k_0$

$$0 = -\sum_{j=1}^{n} \alpha_j z(k-j) + \sum_{j=d_1}^{n} \Omega_j w(k-j) + \sum_{j=d+1}^{n} \Pi_j u(k-j) + \Pi_d \Theta^* \phi(k-d). \qquad (2.9)$$

We emphasize that Assumption 2.7 will be only used to formulate the closed-loop error dynamics, that is the adaptive controller is not claimed to achieve deadbeat disturbance rejection.

Now let $\otimes$ denote the Kronecker product and define the measurement regressor

$$\Psi(k) \triangleq \phi(k) \otimes \Pi_d^{\mathrm{T}} \in \mathbb{R}^{q_4 \times l_z}, \qquad (2.10)$$

the control regressor

$$\psi(k) \triangleq \phi(k) \otimes I_{mu} \in \mathbb{R}^{q_4 \times m_u}, \qquad (2.11)$$

the equivalent measurement

$$\xi(k) \triangleq z(k) - \Pi_d u(k-d) \in \mathbb{R}^{l_z}, \qquad (2.12)$$

the estimated parameter vector

$$\hat{\theta}(k) \triangleq \mathrm{vec}\left[\hat{\Theta}(k)\right] \in \mathbb{R}^{q_4},$$

the deadbeat parameter vector

$$\theta^* \triangleq \mathrm{vec}\left[\Theta^*\right] \in \mathbb{R}^{q_4}, \qquad (2.13)$$

and the parameter error vector

$$\tilde{\theta}(k) \triangleq \hat{\theta}(k) - \theta^*, \qquad (2.14)$$

where $q_4 \triangleq m_u q_3$. Then the performance signal is given by

$$z(k) = \xi(k) + \Psi^{\mathrm{T}}(k-d)\hat{\theta}(k-d) \qquad (2.15)$$

$$= \Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k-d), \qquad (2.16)$$

the retrospective performance signal is given by

$$z_r(k) = \xi(k) + \Psi^{\mathrm{T}}(k-d)\hat{\theta}(k) \qquad (2.17)$$

$$= \Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k), \qquad (2.18)$$

and the control is given by

$$u(k) = \psi^{\mathrm{T}}(k)\hat{\theta}(k). \qquad (2.19)$$

It follows from (2.12) and (2.17) that the retrospective performance signal z_r can be computed without a measurement of the disturbance $w(k)$.

3. Adaptive Algorithm

The adaptive feedback mechanism shown in Figure 1 consists of an instantaneously linear controller G_{c} given by (2.6) and a parameter up date law that modifies the controller parameters at every time step k. To obtain the parameter update law we first define a cost function that is quadratic in the *a posteriori* retrospective performance

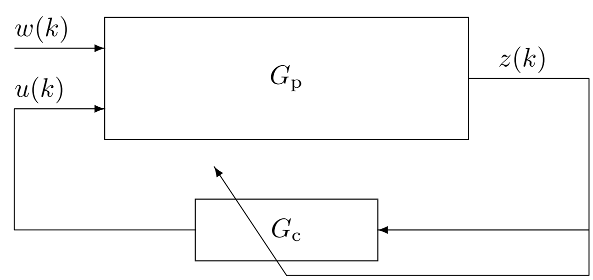

Fig. 1. The Adaptation Mechanism

$$J(k,\hat{\theta}) \triangleq \sum_{j=1}^{k} \left[\xi(k) + \Psi^{\mathrm{T}}(k-d)\hat{\theta}(k+1)\right]^{\mathrm{T}} \left[\xi(k) + \Psi^{\mathrm{T}}(k-d)\hat{\theta}(k+1)\right]. \qquad (3.1)$$

Then we use a recursive least squares estimate of $\hat{\theta}(k+1)$ to minimize $J(k,\hat{\theta})$, for details see, for example [13]. The RLS estimate for $\hat{\theta}(k)$ is given by

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \mathcal{P}(k+1)\Psi(k-d)\left[\xi(k) + \Psi^{\mathrm{T}}(k-d)\hat{\theta}(k)\right], \qquad (3.2)$$

$$\mathcal{P}(k+1) = \mathcal{P}(k) - \mathcal{P}(k)\Psi(k-d) \left(I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k)\Psi(k-d)\right)^{-1} \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k), \qquad (3.3)$$

where $\mathcal{P}(0) > 0$.

4. Closed-Loop State Space Representation

Define

$$\tilde{\vartheta}(k) \triangleq \begin{bmatrix} \tilde{\theta}(k) \\ \vdots \\ \tilde{\theta}(k-d) \end{bmatrix}. \quad (4.1)$$

Then a state vector for the closed loop system (2.1), (2.6), (3.2) and (3.3) is defined by

$$X(k) \triangleq \begin{bmatrix} Z(k) \\ \tilde{\vartheta}(k) \\ \mathrm{vec}(\mathcal{P}(k)) \\ \vdots \\ \mathrm{vec}(\mathcal{P}(k-d)) \end{bmatrix}. \quad (4.2)$$

For $k \geq k_0$ state equations of the closed-loop system can be represented as

$$Z(k+1) = AZ(k) + B\Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k-d), \quad (4.3)$$

$$\tilde{\theta}(j+1) = \tilde{\theta}(j) - \mathcal{P}(j+1)\Psi(j-d)\Psi^{\mathrm{T}}(j-d)\tilde{\theta}(j), \quad (4.4)$$

$$\mathrm{vec}\left[\mathcal{P}(j+1)\right] = \mathrm{vec}\left[\mathcal{P}(j) - \mathcal{P}(j)\Psi(j-d) \cdot \left[1 + \Psi^{\mathrm{T}}(j-d)\mathcal{P}(j)\Psi(j-d)\right]^{-1} \cdot \Psi^{\mathrm{T}}(j-d)\mathcal{P}(j)\right], \quad (4.5)$$

where $j = k-d, \ldots k$,

$$A \triangleq \left[\begin{array}{cc} \multicolumn{2}{c}{0_{l_z \times q_2}} \\ \hline I_{(q_2-l_z)\times(q_2-l_z)} & 0_{(q_2-l_z)\times l_z} \end{array} \right], \quad (4.6)$$

is nilpotent and thus asymptotically stable, and

$$B \triangleq \begin{bmatrix} I_{l_z \times l_z} \\ 0_{(q_2-l_z)\times l_z} \end{bmatrix}. \quad (4.7)$$

Note that every equilibrium of the error system (4.3)-(4.5) is of the form $(0, \tilde{\theta}_q, \mathcal{P}_q)$, where $\mathcal{P}_q \geq 0$.

5. Stability of the Adaptive System

To demonstrate that the origin of the system (4.3)-(4.5) is Lyapunov stable we require the following lemmas.

Lemma 5.1: Define

$$V_{\mathcal{P}}(\mathcal{P}) \triangleq \mathrm{tr}\ \mathcal{P}^2, \quad (5.1)$$

$$\triangle V_{\mathcal{P}}(k) \triangleq \mathrm{tr}\left[\mathcal{P}^2(k+1) - \mathcal{P}^2(k)\right], \quad (5.2)$$

$$V_{\tilde{\theta}}(\tilde{\theta}, \mathcal{P}) \triangleq \tilde{\theta}^{\mathrm{T}}\mathcal{P}^{-1}\tilde{\theta}, \quad (5.3)$$

and

$$\triangle V_{\tilde{\theta}}(k) \triangleq \tilde{\theta}^{\mathrm{T}}(k+1)\mathcal{P}^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^{\mathrm{T}}(k)\mathcal{P}^{-1}(k)\tilde{\theta}(k). \quad (5.4)$$

Then

$$\triangle V_{\mathcal{P}}(k) \leq 0, \quad (5.5)$$

and

$$\triangle V_{\tilde{\theta}}(k) = -\tilde{\theta}^{\mathrm{T}}(k)\Psi(k-d)\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k)\Psi(k-d)\right]^{-1} \cdot \Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k) \leq 0. \quad (5.6)$$

Proof The result follows from the standard properties of RLS. see [3, p. 60], [14, p. 58] and [9, p. 202]. □

Lemma 5.2: For all $k > d$,

$$z^{\mathrm{T}}(k)\left[1 + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1} z(k) \leq \sum_{k-d}^{k} \tilde{\theta}^{\mathrm{T}}(i)\Psi(i-d)\left[1 + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1} \cdot \Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i).$$

Proof. Successive self substitutions of (4.4) yield

$$\tilde{\theta}(k) = \tilde{\theta}(k-d) - \sum_{i=1}^{d} \mathcal{P}(k-i+1)\Psi(k-d-i)\Psi^{\mathrm{T}}(k-d-i)\tilde{\theta}(k-i) = \tilde{\theta}(k-d) - \sum_{i=k-d}^{k-1} \mathcal{P}(i+1)\Psi(i-d)\Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i). \quad (5.7)$$

Multiplying both sides of (5.7) by $\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2}\Psi^{\mathrm{T}}(k-d)$ and using (2.16) we have

$$\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} z(k) = \left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2}\Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k) + \sum_{i=k-d}^{k-1}\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} \cdot \Psi^{\mathrm{T}}(k-d)\mathcal{P}(i+1)\Psi(i-d)\Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i).$$

Now use of (4.5) yields

$$\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} z(k) = \left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2}\Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k) + \sum_{i=k-d}^{k-1}\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} \cdot \Psi^{\mathrm{T}}(k-d)\mathcal{P}(i)\Psi(i-d)\left[I + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1} \cdot \Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i).$$

Using the triangle inequality and the fact that $\mathcal{P}(i) \leq \mathcal{P}(k-d)$ it follows that

$$\left\|\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} z(k)\right\|_2 \leq \left\|\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2}\Psi^{\mathrm{T}}(k-d)\tilde{\theta}(k)\right\|_2 + \sum_{i=k-d}^{k-1}\left\|\left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(i)\Psi(k-d)\right]^{-1/2} \cdot \Psi^{\mathrm{T}}(k-d)\mathcal{P}^{1/2}(i)\mathcal{P}^{1/2}(i)\Psi(i-d) \cdot \left[I + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1/2}\right\|_2 \cdot \left\|\left[I + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1/2}\Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i)\right\|_2.$$

Now using the Cauchy-Schwarz inequality we have

$$\left\| \left[I + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1/2} z(k) \right\|_2$$
$$\leq \sum_{i=k-d}^{k} \left\| \left[I + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1/2} \Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i) \right\|_2$$

□

Lemma 5.3: Define

$$\Pi(k) \triangleq \mathrm{diag} \left[\begin{array}{ccc} \mathcal{P}^{-1}(k) & \cdots & \mathcal{P}^{-1}(k+d) \end{array} \right], \quad (5.8)$$

$$V_{\tilde{\vartheta}}(\tilde{\vartheta}, \Pi) \triangleq \tilde{\vartheta}^{\mathrm{T}} \Pi^{-1} \tilde{\vartheta}, \quad (5.9)$$

$$\begin{aligned} \triangle V_{\tilde{\vartheta}}(k) \triangleq\ & \tilde{\vartheta}^{\mathrm{T}}(k+1)\Pi^{-1}(k+1)\tilde{\vartheta}(k+1) \\ & - \tilde{\vartheta}^{\mathrm{T}}(k)\Pi^{-1}(k)\tilde{\vartheta}(k), \end{aligned} \quad (5.10)$$

$\varrho \triangleq \lambda_{\min}\left[\Pi_d \Pi_d^{\mathrm{T}}\right]$, and $\gamma \triangleq \lambda_{\min}\left[\mathcal{P}(0)\right]$. Then

$$\triangle V_{\tilde{\vartheta}}(k) \quad \leq \quad \frac{-z^{\mathrm{T}}(k)z(k)}{1 + \varrho\gamma\left(\|U(k)\|_2^2 + \|Z(k)\|_2^2\right)}. \quad (5.11)$$

Proof. From (4.1), (5.8) and (5.10) it follows that

$$\begin{aligned} \triangle V_{\tilde{\vartheta}}(k) = & \sum_{i=k-d+1}^{k+1} \tilde{\theta}^{\mathrm{T}}(i)\mathcal{P}^{-1}(i)\tilde{\theta}(i) \\ & - \sum_{i=k-d}^{k} \tilde{\theta}^{\mathrm{T}}(i)\mathcal{P}^{-1}(i)\tilde{\theta}(i). \end{aligned} \quad (5.12)$$

Now using Lemma 5.2 we have

$$\begin{aligned} & \triangle V_{\tilde{\vartheta}} = -\sum_{k-d}^{k} \tilde{\theta}^{\mathrm{T}}(i)\Psi(i-d) \\ & \cdot \left[1 + \Psi^{\mathrm{T}}(i-d)\mathcal{P}(i)\Psi(i-d)\right]^{-1} \Psi^{\mathrm{T}}(i-d)\tilde{\theta}(i) \\ & \leq -z^{\mathrm{T}}(k)\left[1 + \Psi^{\mathrm{T}}(k-d)\mathcal{P}(k-d)\Psi(k-d)\right]^{-1} z(k). \end{aligned} \quad (5.13)$$

Noting that $\mathcal{P}(k-d) \leq \mathcal{P}(0)$ for $k \geq d$, and

$$\begin{aligned} \Psi^{\mathrm{T}}(k-d)\Psi(k-d) &= \phi^{\mathrm{T}}(k-d)\phi(k-d) \otimes \Pi_d \Pi_d^{\mathrm{T}} \\ &= \phi^{\mathrm{T}}(k-d)\phi(k-d)\Pi_d \Pi_d^{\mathrm{T}} \end{aligned} \quad (5.14)$$

we can write

$$\begin{aligned} \triangle V_{\tilde{\theta}}(k) &\leq \frac{-z^{\mathrm{T}}(k)z(k)}{1 + \varrho\gamma\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\ &= \frac{-z^{\mathrm{T}}(k)z(k)}{1 + \varrho\gamma\left(\|U(k)\|_2^2 + \|Z(k)\|_2^2\right)}. \end{aligned}$$

□

Lemma 5.4: Define

$$V_\Pi(\Pi) \triangleq \mathrm{tr}\left[\Pi^{\mathrm{T}}\Pi\right] \quad (5.15)$$

and

$$\triangle V_\Pi(k) \triangleq \mathrm{tr}\left[\Pi^{\mathrm{T}}(k+1)\Pi(k+1)\right] - \mathrm{tr}\left[\Pi^{\mathrm{T}}(k)\Pi(k)\right]. \quad (5.16)$$

Then

$$\triangle V_\Pi(k) \leq 0, \qquad k \geq 0.$$

Proof. From (5.5) and (5.8) it follows that

$$\begin{aligned} \triangle V_\Pi(k) &= \sum_{i=k-d+1}^{k+1} \mathrm{tr}\ \mathcal{P}^2(i) - \sum_{i=k-d}^{k} \mathrm{tr}\ \mathcal{P}^2(i) \\ &= -\sum_{i=k-d}^{k} \triangle V_{\mathcal{P}}(k) \leq 0. \end{aligned}$$

□

Lemma 5.5: Let $P, R \in \mathbb{R}^{n\times n}$ be positive-definite matrices that satisfy

$$P = A^{\mathrm{T}}PA + R + I, \quad (5.17)$$

and let

$$\sigma \triangleq \sqrt{\lambda_{\max}(A^{\mathrm{T}}PA)}. \quad (5.18)$$

Furthermore, let $\mu > 0$ and define

$$V_Z(Z) \triangleq \ln(1 + \mu Z^{\mathrm{T}}PZ), \quad (5.19)$$

$$\triangle V_Z(k) \triangleq V_Z(Z(k+1)) - V_Z(Z(k)),$$

$\rho \triangleq \lambda_{\min}\left[P\right]$, and $\beta \triangleq \lambda_{\max}\left[B^{\mathrm{T}}PB\right]$. Then

$$\begin{aligned} \triangle V_Z(k) \leq & -\mu \frac{Z^{\mathrm{T}}(k)RZ(k)}{1 + \rho\mu\|Z(k)\|_2^2} \\ & + \mu \frac{(\sigma^2+1)\beta\left[z^{\mathrm{T}}(k)z(k)\right]}{1 + \rho\mu\|Z(k)\|_2^2}. \end{aligned}$$

Proof. Define

$$F \triangleq \frac{1}{\sigma}P^{1/2}A, \quad G \triangleq \sigma P^{1/2}B, \quad \mathcal{J}(Z) \triangleq Z^{\mathrm{T}}PZ.$$

Then,

$$\begin{aligned} \triangle \mathcal{J}_Z(k) &\triangleq Z^{\mathrm{T}}(k+1)PZ(k+1) - Z^{\mathrm{T}}(k)PZ(k) \\ &= Z^{\mathrm{T}}(k)A^{\mathrm{T}}PAZ(k) + Z^{\mathrm{T}}(k)A^{\mathrm{T}}PBz(k) \\ &\quad + z^{\mathrm{T}}(k)B^{\mathrm{T}}PAZ(k) + z^{\mathrm{T}}(k)(k)B^{\mathrm{T}}PBz(k) \\ &\quad - Z^{\mathrm{T}}(k)PZ(k). \end{aligned}$$

Adding and subtracting $Z^{\mathrm{T}}F^{\mathrm{T}}FZ$ and $z^{\mathrm{T}}G^{\mathrm{T}}Gz$, and omitting the explicit dependence on k we have

$$\begin{aligned} \triangle \mathcal{J}_Z(k) &= Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z \\ &\quad - \left[\begin{array}{cc} Z^{\mathrm{T}} & -z^{\mathrm{T}} \end{array}\right] \left[\begin{array}{cc} F^{\mathrm{T}}F & F^{\mathrm{T}}G \\ G^{\mathrm{T}}F & G^{\mathrm{T}}G \end{array}\right] \left[\begin{array}{c} Z \\ -z \end{array}\right] \\ &\quad + z^{\mathrm{T}}\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z \\ &\leq Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z \\ &\quad + z^{\mathrm{T}}\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z. \end{aligned}$$

Noting that

$$F^{\mathrm{T}}F = \frac{A^{\mathrm{T}}PA}{\sigma^2} \qquad \leq \frac{\lambda_{\max}\left(A^{\mathrm{T}}PA\right)I_n}{\lambda_{\max}(A^{\mathrm{T}}PA)} = I_n,$$

it follows from (5.17) that

$$A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F \leq A^{\mathrm{T}}PA - P + I = -R.$$

Therefore,

$$Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z \leq -Z^{\mathrm{T}}RZ,$$

which implies that

$$\triangle \mathcal{J}_Z(k) \le -Z^{\mathrm{T}}(k)RZ(k) + z^{\mathrm{T}}(k)\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z(k). \quad (5.20)$$

Since $G^{\mathrm{T}}G = \sigma^2 B^{\mathrm{T}}PB$, it follows from (5.20) that

$$\triangle \mathcal{J}_Z(k) \le -Z^{\mathrm{T}}(k)RZ(k) + (\sigma^2+1)\beta\left[z^{\mathrm{T}}(k)z(k)\right].$$

Now, since $\ln x \le x-1$ for all $x>0$,

$$\triangle V_Z(k) = \ln\left(1+\mu\frac{\triangle \mathcal{J}_Z(k)}{1+\mu Z^{\mathrm{T}}(k)PZ(k)}\right) \le -\mu\frac{Z^{\mathrm{T}}(k)RZ(k)}{1+\rho\mu\|Z(k)\|_2^2} + \mu\frac{(\sigma^2+1)\beta\left[z^{\mathrm{T}}(k)z(k)\right]}{1+\rho\mu\|Z(k)\|_2^2}. \quad \square$$

We now present the main stability result.

Theorem 5.1: Assume that Assumptions 2.1-2.6 are satisfied. Then every equilibrium of the system (4.3)-(4.5) is Lyapunov stable, and $z(k)\to 0$ as $k\to\infty$.

Proof. Consider the Lyapunov function candidate

$$V(X) \stackrel{\triangle}{=} aV_\phi(\phi) + V_{\tilde{\vartheta}}(\tilde{\vartheta},\Pi) + V_\Pi(\Pi). \quad (5.21)$$

Let $P, R \in \mathbb{R}^{n\times n}$ be positive definite and satisfy (5.17), and let $a>0$. Then using lemmas 5.5, 5.3 and 5.4 it follows that

$$\triangle V(k) \stackrel{\triangle}{=} V(X(k+1)) - V(X(k)) \le a\mu\frac{-Z^{\mathrm{T}}(k)RZ(k) + (\sigma^2+1)\beta\|Z(k)\|_2^2}{1+\rho\mu\|Z(k)\|_2^2} - \frac{\|Z(k)\|_2^2}{1+\varrho\gamma\left(\|U(k)\|_2^2+\|Z(k)\|_2^2\right)}. \quad (5.22)$$

Since G_{zu} has a stable inverse by Assumption 2.4, there exist $c_1 \ge 0$ and $c_2>0$ such that

$$\|U(k)\|_2^2 \le c_1 + c_2\|Z(k)\|_2^2. \quad (5.23)$$

See [3, p. 487]. Using (5.23) in (5.22) we have

$$\triangle V(k) \le a\mu\frac{-Z^{\mathrm{T}}(k)RZ(k) + (\sigma^2+1)\beta\|Z(k)\|_2^2}{1+\rho\mu\|Z(k)\|_2^2} - \frac{\nu\|Z(k)\|_2^2}{1+\delta\|Z(k)\|_2^2}, \quad (5.24)$$

where $\nu = 1/\varrho\gamma c_1$ and $\delta = (1+c_2)/c_1$. Next define $\mu \stackrel{\triangle}{=} \delta/\rho$ and $a = \nu\rho/\delta\beta\left(\sigma^2+1\right)$. Then

$$\triangle V(k) \le -a\mu\frac{Z^{\mathrm{T}}(k)RZ(k)}{1+\rho\mu\|Z(k)\|_2^2} \le 0. \quad (5.25)$$

Since $V(X)$ is positive definite and radially unbounded it follows from (5.25) that every equilibrium of the error system (4.3)-(4.5) is Lyapunov stable. Furthermore, using Theorem A1 of [6] it follows that $z(k)\to 0$ as $k\to\infty$. $\square$

6. Example

Example 6.1: Consider the rectangular cross-section acoustic duct shown in Figure 2. We treat the duct as a

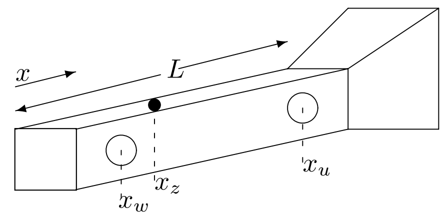

Fig. 2. Acoustic Duct

one dimensional waveguide with spatial coordinate x, where $0 \le x \le L$. We use the mathematical model for the acoustic duct derived in [15], where the speed of acoustic waves is 343 m/s, the density of air is 1.21 kg/m^3, and the duct model includes five modes. Let the disturbance speaker be placed at x_w, the control speaker at x_u, and the performance microphone at x_z. For $L=6$ m, $x_w = 0.1$ m, $x_z = 0.3$ m, and $x_u = 5.95$ m the state equations were developed in [16].

The modal frequencies of the duct are 85.4167 Hz, 170.8333 Hz, 256.25 Hz, 341.6667 Hz and 427.0833 Hz. The following simulations were performed at a sample rate of 2000 Hz.

1) The disturbance speaker is excited at the modal frequency 427.0833 Hz. The controller order $n_c = 10$. Figure 3 shows the closed-loop performance versus time and figure 4 shows the open and closed loop frequency response of the transfer function from disturbance to performance G_{zw}.
2) The disturbance speaker is excited at the modal frequencies 427.0833 Hz and 85.4167 Hz. The controller order $n_c = 20$. Figure 5 shows the closed-loop performance versus time and figure 6 shows the open and closed loop frequency response of G_{zw}.
3) The disturbance is white noise band limited to 1000 Hz. The closed-loop frequency response with $n_c = 28$ is shown in Figure 7. Although the class of disturbances considered in Theorem 5.1 does not include white noise, the simulation results demonstrate disturbance rejection.

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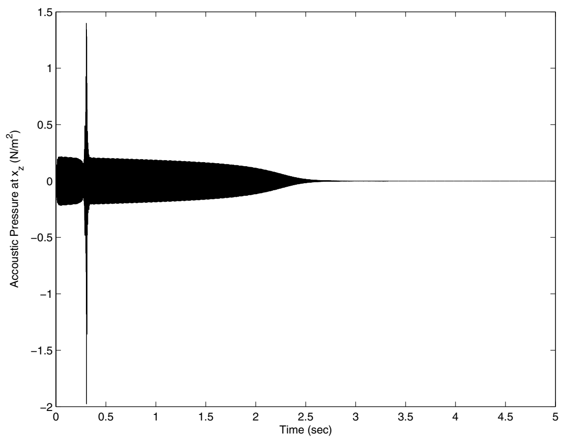

Fig. 3. Time History of z with Single Tone Disturbance and $n_c = 10$

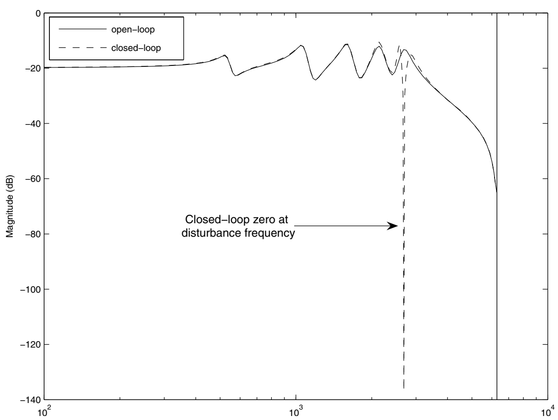

Fig. 4. Bode Plot of G_{zw} with Single Tone Disturbance and $n_c = 10$

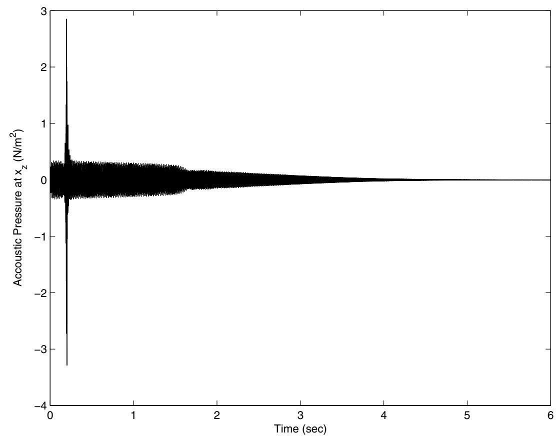

Fig. 5. Time History of z with Dual Tone Disturbance and $n_c = 20$

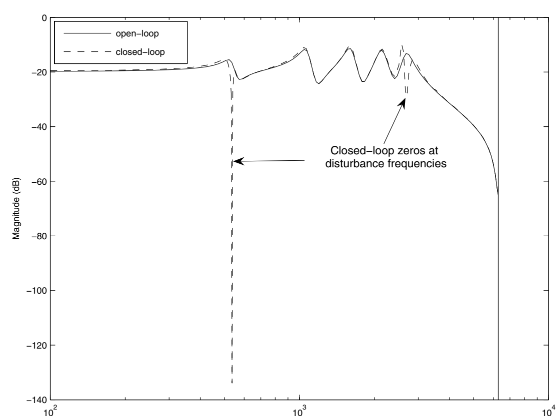

Fig. 6. Bode Plot of G_{zw} with Dual Tone Disturbance and $n_c = 20$

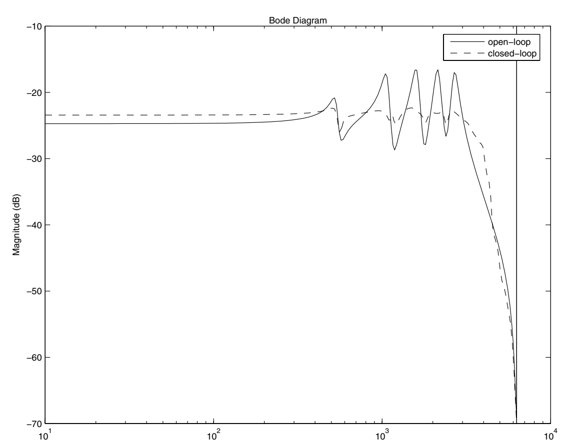

Fig. 7. Broadband Disturbance with $n_c = 28$

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