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# Energy-Based Nonlinear Control of Underactuated Euler-Lagrange Systems Subject to Impacts

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*Abstract*— **In this paper, energy-based nonlinear controllers are designed to globally asymptotically stabilize an underactuated mechanical system. An interesting aspect of the problem is that the equilibrium points of some states are defined by contact with a surface while the equilibrium points of remaining states are defined by noncontact positions. To stabilize the states of the system an energy coupling strategy is employed. The energy coupling approach is motivated by the desire to improve the transient response of the system. A Lyapunov stability analysis and numerical simulations are provided to demonstrate the stability and performance of the developed controllers.**

## I. Introduction

The control of mechanical systems subject to impact is a theoretically interesting problem with practical importance. Large stresses arise as a consequence of impact, demanding that the impact forces be properly recognized and controlled to prevent system failure. As described in [12], some useful short-duration effects such as high stresses, rapid dissipation of energy, and fast acceleration and deceleration may be achieved from low-energy sources by controlling the impact of robots operating at low force levels. Some robotic examples in which controlled contacts are required include: the impact between a walking robot and the ground, the interaction of a robot manipulator with an object, and the cooperation and contact of multi-robots. One of the difficulties in controlling impact is that the equations of motion are quite different when the system status changes quickly from a noncontact condition to a contact condition. Thus, it is challenging to develop a uniform controller that behaves well in both free motion and contact conditions.

For the past decade, many researchers have addressed the modeling and control of impact [1]-[3], [7], [9]-[14]. In [14], a switching control strategy is designed to guarantee the stability of the impact controller. In [10], a stable discontinuous transition controller is proposed to deal with the contact transition problem. In [9], Lee et al. use a hybrid impedance/time-delay controller that establishes a stable contact and achieves the desired dynamics for contact or noncontact conditions. In [11], a discontinuous Lyapunov-based control scheme is introduced to regulate the impact of a hydraulic actuator coming in contact with a nonmoving environment. In [12], a continuous PD controller is proposed to control the impact of an underactuated system where the actuators are used to regulate the contact coordinates, and the noncontact coordinates are indirectly regulated. Particularly, two kinds of models are proposed in [12] to describe the impact phenomenon: smooth impact and nonsmooth impact. In [7], static and dynamic PD controllers are proposed to address global asymptotic stabilization problem of the underactuated mechanical system subject to smooth impact.

The result in this paper is motivated by the idea that energy-based controllers (e.g., [4] [5]) can be used to couple the states of the underactuated system as a means to improve the transient response over an uncoupled controller (e.g., the PD controllers in [7] and [12]). The energy coupling controllers designed in this paper globally asymptotically stabilize the generalized free motion and contact coordinates of an underactuated mechanical system subject to impact conditions. An interesting aspect of the problem is that the equilibrium points of some states are defined by contact with a surface while the equilibrium points of the remaining states are defined by noncontact positions. A Lyapunov stability analysis and numerical simulations are provided to demonstrate the stability and performance of the developed controllers.

## II. Motivating Example

An example of the class of systems considered in this paper can be described by the mass spring system introduced in [7] (several other academic examples are also provided in [7]) that is depicted in Fig. 1. As depicted in Fig. 1, the system consists of two masses $M_1$ and $M_2$ that are coupled to each other and to a fixed surface through springs. The generalized coordinates, denoted by $q(t) \in \mathbb{R}^n$, that denote the positions of the masses are defined as

$$q \triangleq \begin{bmatrix} q_u & q_c \end{bmatrix}^T \tag{1}$$

where $q_{ui}(t) \in \mathbb{R}, \forall i = 1, ..., n-m$ and $q_{cj}(t) \in \mathbb{R}, \forall j = 1, ..., m$ defined as

$$q_u \triangleq \begin{bmatrix} q_{u1} & \cdots & q_{u(n-m)} \end{bmatrix}^T \tag{2}$$

$$q_c \triangleq \begin{bmatrix} q_{c1} & \cdots & q_{cm} \end{bmatrix}^T \tag{3}$$

denote the generalized coordinates of the states associated with masses that are not in contact and that are in contact

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with a surface, respectively. For the system in Fig. 1, $n = 2$ and $m = 1$. Based on the definitions given in (1) and (3) the following relationship can be developed

$$q_c = S^T q, \tag{4}$$

where $S \in \mathbb{R}^{n \times m}$ denotes a constant transformation matrix.

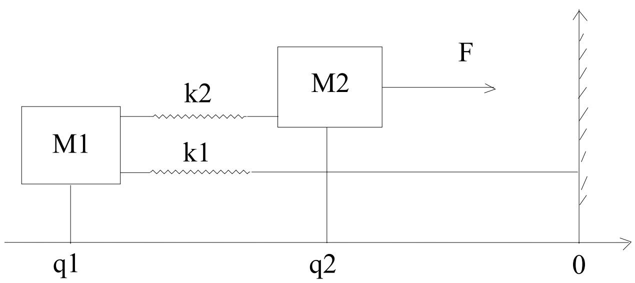


Fig. 1. The mass spring system represents an academic example of a general underactuated Euler-Lagrange system with contact and noncontact states.

The objective for this example is to design an input force $F(t)$ so that $M_2$ is regulated to be in contact with the surface of a fixed object while $M_1$ is regulated to a stable noncontact equilibrium point. Intuitively, if the controller does not actively compensate for oscillation by $q_1(t) = q_u(t)$ the state will exhibit a long settling time, especially for systems with low stiffness. The control development in this paper is motivated by the desire to couple the states through an energy-based method as a means to improve the transient response. Specifically, in the subsequent sections two energy based controllers are developed for general Euler-Lagrange systems in free motion and in contact conditions. The stability of the controllers is analyzed through a Lyapunov-based analysis. A simulation is also provided for the example of mass spring system to illustrate the performance of the developed energy coupling controllers.

## III. Dynamic Model

The dynamic model for an $n$-degrees-of-freedom (DOF) Euler-Lagrange system in free motion and in contact conditions is assumed to have the following form [7]:

$$M\ddot{q} + C(q,\dot{q})\dot{q} + h(q) + \sum_{i=1}^{m} K_i S \Lambda_i S^T q = Su. \tag{5}$$

In (5), $M(q) \in \mathbb{R}^{n \times n}$ denotes the inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ denotes the centripetal-Coriolis effects, $h(q) \in \mathbb{R}^n$ denotes conservative forces (e.g., spring forces, gravity), $u(t) \in \mathbb{R}^m$ denotes the control input, $K_i \in \mathbb{R}\ \forall i = 1, ..., m$ denote positive constants determined by the free motion system and by the contact surface, and $\Lambda_i(q_{ci}) \in \mathbb{R}$ denotes a function defined as

$$\Lambda_i \triangleq \left\{ \begin{array}{ll} 0, & q_{ci} < q_{ci}^* \\ 1, & q_{ci} \geq q_{ci}^* \end{array} \right\}$$

where $q^* \triangleq \left[ \begin{array}{cc} q_u^* & q_c^* \end{array} \right]^T \in \mathbb{R}^n$ denotes a constant vector of the equilibrium points defined as

$$q^* = \left[ \begin{array}{cccccc} q_{u1}^* & \cdots & q_{u(n-m)}^* & q_{c1}^* & \cdots & q_{cm}^* \end{array} \right]^T$$

where $q_u^* \in \mathbb{R}^{n-m}$ and $q_c^* \in \mathbb{R}^m$ denote the equilibrium points for the noncontact states and the contact states (at the contact position), respectively.

The energy, denoted by $E(q,\dot{q})$, for systems described by (5) can be written as

$$E = \frac{1}{2}\dot{q}^T M \dot{q} + \left[ E_s(q) - E_s(q^*) + \sum_{i=1}^{m} \frac{1}{2} K_i \Lambda_i (q_{ci} - q_{ci}^*)^2 \right] \tag{6}$$

where the first two elements in the brackets represent the potential energy of the free motion system, and the last term in the brackets denotes the potential energy caused by the contact.

**Assumption 1:** The inertia matrix $M(q)$ is assumed to be symmetric, positive definite, and can be upper and lower bounded by the following inequalities

$$a_1 ||\xi||^2 \leq \xi^T M \xi \leq a_2 ||\xi||^2, \qquad \forall \xi \in \mathbb{R}^n \tag{7}$$

where $a_1, a_2 \in \mathbb{R}$ are positive constants. The following skew-symmetric relationship is also assumed to be satisfied

$$\xi^T (\frac{1}{2}\dot{M}(q) - C(q,\dot{q}))\xi = 0, \qquad \forall \xi \in \mathbb{R}^n. \tag{8}$$

**Assumption 2:** Since $h(q)$ in (5) is assumed to be a conservative force, the associated work is equal to the change in potential energy as

$$\dot{E}_s(q) - \dot{q}^T h(q) = 0. \tag{9}$$

The conservative spring force for the example mass spring problem is given by

$$h(q) = \left[ \begin{array}{c} (K_1 + K_2)(q_1 - q_1^*) - K_2(q_2 - q_2^*) \\ -K_2(q_1 - q_1^*) + K_2(q_2 - q_2^*) \end{array} \right] \tag{10}$$

and the potential energy of the system is

$$E_s = \frac{1}{2}K_1(q_1 - q_1^*)^2 + \frac{1}{2}K_2(q_1 - q_2 - q_1^* + q_2^*)^2. \tag{11}$$

The differential expression in (9) is satisfied for (10) and (11).

**Assumption 3:** The system energy $E(q,\dot{q})$ is a continuously differentiable, positive definite and radially unbounded function where it is assumed that if $E(q,\dot{q}) \in L_\infty$ then $q_u(t) \in L_\infty$.

**Assumption 4:** The system (5) with output $q_c(t)$ and $\dot{q}_c(t)$ is assumed to be zero-state observable [8] in the sense that no solution of (5) with $u(t) = 0$ can stay identically in the set $\Phi = \{(q,\dot{q}) \in R^{2n} | \, \|\dot{q}_c\| = 0, q_c = q_c^*\}$ other than the trivial solution given by $q(t) = q^*$ and $\|\dot{q}(t)\| = 0$. That is, since the system is assumed to be zero-state observable

$$\begin{aligned} \text{if } \|\dot{q}_c\| &= 0, \; q_c = q_c^*, \text{ and } \|u\| = 0 \\ \text{then } q &= q^* \text{ and } \|\dot{q}\| = 0. \end{aligned} \tag{12}$$

*Remark 1:* Assumptions 6 and 7 of [12] are comparable to Assumption 4 in this paper, and are required in [12] to guarantee the steady state solution is unique, and $q(t) = q^*$ and $\|\dot{q}(t)\| = 0$.

## IV. Control Development

### A. Control Objective

The motivation of this research is to globally asymptotically regulate the states of an underactuated system to equilibrium points that are defined by impact and nonimpact conditions (i.e., $(q, \dot{q}) \to (q^*, 0)$). The control objective is based on the assumption that $q(t)$ and $\dot{q}(t)$ are measurable, and the states $q(t)$ and $\dot{q}(t)$ are zero-state observable with respect to the output $q_c(t)$ and $\dot{q}_c(t)$. The following regulation error system, denoted by $e(t) \in \mathbb{R}^n$, is introduced to quantify the control objective

$$e = \begin{bmatrix} e_u & e_c \end{bmatrix}^T \tag{13}$$

where $e_c(t) \in \mathbb{R}^m$ and $e_u(t) \in \mathbb{R}^{n-m}$ denote the states of the regulation error for the contact and noncontact coordinates, respectively, and are defined as

$$e_c = q_c - q_c^* \qquad e_u = q_u - q_u^*. \tag{14}$$

Based on (12)-(14) the subsequent development will focus on the objective to prove that

$$\lim_{t\to\infty} \|\dot{e}_c(t)\| = 0 \qquad \lim_{t\to\infty} \|e_c(t)\| = 0 \qquad \lim_{t\to\infty} \|u(t)\| = 0.$$

### B. Open-Loop Error System

The following expression can be obtained after taking the second time derivative of (13) and utilizing (5) and (7)

$$\ddot{e} = M^{-1}(Su - \bar{C}(e,\dot{e})\dot{e} - \bar{h}(e) - \sum_{i=1}^{m} K_i S \Lambda_i S^T e), \tag{15}$$

where

$$\bar{C}(e,\dot{e}) = C(e + q^*, \dot{e}), \qquad \bar{h}(e) = h(e + q^*). \tag{16}$$

After premultiplying (15) by $S^T$ the dynamics of the contact coordinates can be written as

$$\ddot{e}_c = \frac{Pu + W}{\det(M)} \tag{17}$$

where the auxiliary signals $P(q) \in \mathbb{R}^{m\times m}$ and $W(e,\dot{e}) \in \mathbb{R}^m$ are defined as

$$P \triangleq S^T adj(M) S \tag{18}$$

$$W \triangleq -S^T adj(M)(\bar{C}(e,\dot{e})\dot{e} + \bar{h}(e) + \sum_{i=1}^{m} K_i S \Lambda_i S^T e). \tag{19}$$

Based on (6) and (13), the system energy can be rewritten as

$$E = \frac{1}{2}\dot{e}^T M \dot{e} + E_s(e + q^*) - E_s(q^*) + \sum_{i=1}^{m} \frac{1}{2} K_i \Lambda_i e_{ci}^2. \tag{20}$$

Taking the derivative of (20) and substituting (15) into the resulting expression yields

$$\dot{E} = \dot{e}^T (\frac{1}{2}\dot{M} - \bar{C}(e,\dot{e}))\dot{e} + \dot{e}^T \left(Su - \bar{h}(e)\right) + \dot{E}_s(e + q^*) \tag{21}$$

where (7) has been utilized. The expression in (21) can be reduced as

$$\dot{E} = \dot{e}^T S u = \dot{e}_c^T u \tag{22}$$

where (8) and (9) were utilized.

### C. Nonlinear Energy Coupling Controller

Based on (17), (22), and the subsequent stability analysis, a nonlinear energy coupling controller is designed as

$$u = [\Omega]^{-1}(-K_d \dot{e}_c - K_p e_c - \frac{K_v W}{\det(M)}) \tag{23}$$

where $\Omega(e,\dot{e}) \in \mathbb{R}^{m\times m}$ is defined as

$$\Omega \triangleq K_E E I_m + \frac{K_v P}{\det(M)}, \tag{24}$$

$K_d, K_p, K_v, K_E \in \mathbb{R}$ are positive constant feedback gains, $I_m$ denotes the $m \times m$ identity matrix, and $P(q)$ and $W(e,\dot{e})$ were defined in (18) and (19), respectively. Since $E(e,\dot{e})$ and $M(q)$ are assumed to be positive definite, Theorem 4.2.1 of [6] can be invoked to ensure that $\Omega(e,\dot{e})$ is positive definite; hence, $\Omega(e,\dot{e})$ is invertible. After substituting (23) into (17), the closed-loop error system for $\ddot{e}_c(t)$ can be obtained as

$$\begin{aligned} \ddot{e}_c &= \frac{P[\Omega]^{-1}(-K_d \dot{e}_c - K_p e_c - \frac{K_v W}{\det(M)})}{\det(M)} \\ &\quad + \frac{W}{\det(M)}. \end{aligned} \tag{25}$$

After substituting (23) into (22), a closed-loop expression for the system energy can be obtained as

$$\dot{E} = \dot{e}_c^T [\Omega]^{-1}(-K_d \dot{e}_c - K_p e_c - \frac{K_v W}{\det(M)}). \tag{26}$$

*Theorem 1:* The equilibrium points of the open-loop system in (5) with the controller defined in (23) are globally asymptotically stable in the sense that

$$\lim_{t\to\infty} q(t) = q^* \text{ and } \lim_{t\to\infty} \|\dot{q}(t)\| = 0. \tag{27}$$

**Proof:** Let $V_1(e,\dot{e}) \in \mathbb{R}$ denote the following continuously differentiable, positive definite, radially unbounded function (i.e., a Lyapunov function candidate)

$$V_1 = \frac{1}{2} K_E E^2 + \frac{1}{2} K_p e_c^T e_c + \frac{1}{2} K_v \dot{e}_c^T \dot{e}_c. \tag{28}$$

Based on the closed-loop error systems in (25) and (26), the time derivative of (28) can be expressed as

$$\begin{aligned}\dot{V}_1 &= K_E E\dot{e}_c^T[\Omega]^{-1}(-K_d\dot{e}_c - K_p e_c - \frac{K_v}{\det(M)}W) \quad (29)\\ &+K_p\dot{e}_c^T e_c + K_v\dot{e}_c^T\frac{W}{\det(M)}\\ &-\frac{K_v\dot{e}_c^T}{\det(M)}(P[\Omega]^{-1}(K_d\dot{e}_c + K_p e_c + \frac{K_v}{\det(M)}W)).\end{aligned}$$

The expression in (29) can be rewritten as

$$\begin{aligned}\dot{V}_1 &= \dot{e}_c^T(K_E E I_m + K_v\frac{P}{\det(M)})[\Omega]^{-1}(-K_d\dot{e}_c \quad (30)\\ &-K_p e_c - \frac{K_v}{\det(M)}W) + K_p\dot{e}_c^T e_c + K_v\dot{e}_c^T\frac{W}{\det(M)}.\end{aligned}$$

The expression in (30) can be simplified as

$$\dot{V}_1 = -K_d\dot{e}_c^T\dot{e}_c \leq 0 \quad (31)$$

where (24) was utilized (i.e., $\dot{V}_1(e,\dot{e})$ is negative semi-definite). From (28) and (31), the origin of the closed-loop system is stable in the sense of Lyapunov and $V_1(e,\dot{e}) \in \mathcal{L}_\infty$; hence, $E(e,\dot{e})$, $e_c(t)$, $\dot{e}_c(t) \in \mathcal{L}_\infty$. Since $E(e,\dot{e}) \in \mathcal{L}_\infty$, (20) can be used to prove that $\dot{e}(t) \in \mathcal{L}_\infty$, and Assumption 3 can be used to conclude that $e_u(t) \in \mathcal{L}_\infty$; hence, $e(t) \in \mathcal{L}_\infty$. Since $e(t)$, $\dot{e}(t) \in \mathcal{L}_\infty$, (1), (13), and (14) can be used to prove that $q_u(t)$, $q_c(t)$, $q(t)$, $\dot{q}_u(t)$, $\dot{q}_c(t)$, $\dot{q}(t) \in \mathcal{L}_\infty$. The definitions in (18), (19), and (24) can now be used to prove that $P(q)$, $W(e,\dot{e})$, $\Omega(e,\dot{e}) \in \mathcal{L}_\infty$.

The proceeding arguments can be used along with (23) to prove that $u(t) \in \mathcal{L}_\infty$. Based on the fact that all of the closed-loop signals remain bounded, LaSalle's Invariance Theorem can now be utilized to prove Theorem 1. To this end, let $\bar{\Phi}$ denote the following set

$$\bar{\Phi} = \{(e,\dot{e}) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_1 = 0\}. \quad (32)$$

In the set $\bar{\Phi}$, it is clear from (31) that

$$\dot{e}_c(t) = 0 \qquad \ddot{e}_c(t) = 0, \quad (33)$$

and hence, from (26) we can conclude that

$$\dot{E}(e,\dot{e}) = 0. \quad (34)$$

The expressions in (26), (28), and (32)-(34) can be used to prove that $e_c(t)$, $E(e,\dot{e})$, and $V_1(E, e_c, \dot{e}_c)$ are constant. To prove that $u(t)$ is constant in $\bar{\Phi}$, we rewrite (17) as

$$\frac{1}{\det(M)}W = \ddot{e}_c - \frac{1}{\det(M)}Pu \quad (35)$$

and substitute (35) into (23) to obtain the following expression

$$u = [\Omega]^{-1}(-K_d\dot{e}_c - K_p e_c - K_v(\ddot{e}_c - \frac{1}{\det(M)}Pu)). \quad (36)$$

After multiplying $\Omega(e,\dot{e})$ on both sides of (36), the following simplified relationship can be developed

$$u = \frac{-K_d\dot{e}_c - K_p e_c - K_v\ddot{e}_c}{K_E E}. \quad (37)$$

Since $e_c(t)$ and $E(e,\dot{e})$ have been proven to be constant in $\bar{\Phi}$, (37) can be used to conclude that $u(t)$ is equal to the following constant

$$u = \frac{-K_p e_c}{K_E E}. \quad (38)$$

To continue the analysis, we consider the following cases: $e_{ci}(t) < 0$, $e_{ci}(t) > 0$, or $e_{ci}(t) = 0$ for all $i = 1, ..., m$. Based on (38), if $e_{ci}(t) < 0$ then $u(t)$ will be the only force acting on the system, and it will equal some positive force that will cause $e_{ci}(t)$ to change and violate the results in (33). Based on (38), if $e_{ci}(t) > 0$ then $u(t)$ will equal some negative force that acts on the system in the same direction as the reactive forces from the contact. These additive forces will cause $e_{ci}(t)$ to change and violate the results in (33). Based on (38), if $\|e_c(t)\| = 0$ then $\|u(t)\| = 0$. Hence, the only possible value for $e_c(t)$ and $u(t)$ in $\bar{\Phi}$ is $\|u(t)\| = \|e_c(t)\| = 0$. The assumption that the system is zero-state observable [8] (i.e., Assumption 4) can be used along with the facts that $\|u(t)\| = \|e_c(t)\| = \|\dot{e}_c(t)\| = 0$, to conclude the result in (27).

### *D. Control Extension*

To illustrate how additional controllers can also be derived, an alternative energy coupling controller is designed as

$$\begin{aligned}u &= \frac{-K_d\dot{e}_c - K_p e_c - K_v P^{-1}W}{K_E + K_V} \quad (39)\\ &-\frac{K_v(\frac{d}{dt}(\det(M)P^{-1}))\dot{e}_c}{2\,(K_E + K_V)}\end{aligned}$$

where $K_d, K_p, K_E, K_V \in \mathbb{R}$ are positive constant control parameters, $P(q)$ is introduced in (18), and $W(e,\dot{e})$ is defined in (19). Since $M(q)$ is assumed to be positive definite, (18) can be used to conclude that $P(q)$ is also positive definite and invertible. After substituting (39) into (15) and then premultiplying by $S^T$, the following closed-loop error system is obtained

$$\begin{aligned}\ddot{e}_c &= \frac{-P\,(K_d\dot{e}_c + K_p e_c) - K_v W}{\det(M)\,(K_E + K_V)} \quad (40)\\ &-\frac{1}{2}\frac{PK_v(\frac{d}{dt}(\det(M)P^{-1}))}{\det(M)\,(K_E + K_V)}\dot{e}_c + \frac{W}{\det(M)}.\end{aligned}$$

After substituting (39) into (22), a closed-loop expression for the system energy can be obtained as

$$\begin{aligned}\dot{E} &= \frac{-K_d\dot{e}_c^T\dot{e}_c - K_p\dot{e}_c^T e_c - K_v\dot{e}_c^T P^{-1}W}{K_E + K_V} \quad (41)\\ &-\frac{\dot{e}_c^T K_v(\frac{d}{dt}(\det(M)P^{-1}))\dot{e}_c}{2\,(K_E + K_V)}.\end{aligned}$$

*Theorem 2:* The equilibrium points of the open-loop system in (5) with the controller defined in (39) are globally asymptotically stable in the sense that

$$\lim_{t\to\infty} q(t) = q^* \text{ and } \lim_{t\to\infty}\|\dot{q}(t)\| = 0. \quad (42)$$

**Proof:** Let $V_2(e,\dot{e}) \in \mathbb{R}$ denote the following continuously differentiable, positive definite, radially unbounded function (i.e., a Lyapunov function candidate)

$$V_2 = K_E E + \frac{1}{2}K_p e_c^T e_c + \frac{1}{2}K_v\dot{e}_c^T(\det(M)P^{-1})\dot{e}_c. \quad (43)$$

Based on (40) and (41), the time derivative of (43) can be simplified as

$$\dot{V}_2 = -K_d \dot{e}_c^T \dot{e}_c \tag{44}$$

and hence, the origin of the closed-loop system is stable in the sense of Lyapunov, and $V_2(e, \dot{e}) \in \mathcal{L}_\infty$. Since $V_2(t) \in \mathcal{L}_\infty$, (43) can be used to prove that $E(e, \dot{e}), e_c(t), \dot{e}_c(t) \in \mathcal{L}_\infty$. Based on the fact that $E(e, \dot{e}) \in \mathcal{L}_\infty$, (20) can be used to prove that $\dot{e}(t) \in \mathcal{L}_\infty$, and Assumption 3 can be used to conclude that $e_u(t) \in \mathcal{L}_\infty$; hence, $e(t) \in \mathcal{L}_\infty$. Given that $e(t)$, $\dot{e}(t) \in \mathcal{L}_\infty$, (1), (13), and (14) can be used to prove that $q_u(t)$, $q_c(t)$, $q(t)$, $\dot{q}_u(t)$, $\dot{q}_c(t)$, $\dot{q}(t) \in \mathcal{L}_\infty$. The definitions in (18) and (19) can now be used to prove that $P(q)$, $P^{-1}(q)$, $\frac{dP^{-1}(q)}{dt}$, $\frac{d\det(M(q))}{dt}$, $W(e, \dot{e}) \in \mathcal{L}_\infty$.

The proceeding arguments can be used along with (39) to prove that $u(t) \in \mathcal{L}_\infty$. Based on the fact that all of the closed-loop signals remain bounded, LaSalle's Invariance Theorem can now be utilized to prove Theorem 2. To this end, let $\bar{\Phi}$ denote the following set

$$\bar{\Phi} = \{(e, \dot{e}) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_2 = 0\}. \tag{45}$$

In the set $\bar{\Phi}$, it is clear from (44) that

$$\dot{e}_c(t) = 0 \qquad \ddot{e}_c(t) = 0, \tag{46}$$

and hence, from (26) we can conclude that

$$\dot{E}(e, \dot{e}) = 0. \tag{47}$$

The expressions in (26), (43), and (45)-(47) can be used to prove that $e_c(t)$, $E(e, \dot{e})$, and $V_2(e, \dot{e})$ are constant. To prove that $u(t)$ is constant in $\bar{\Phi}$, we rewrite (17) as

$$P^{-1}W = \det(M)P^{-1}\ddot{e}_c - u. \tag{48}$$

and substitute (35) into (39) to obtain the following expression

$$\begin{aligned} u = {} & \frac{1}{K_E}(-K_d\dot{e}_c - K_p e_c - K_v \det(M)P^{-1}\ddot{e}_c \\ & -\frac{1}{2}K_v(\frac{d}{dt}(\det(M)P^{-1}))\dot{e}_c). \end{aligned} \tag{49}$$

Since $e_c(t)$ has been proven to be constant in $\bar{\Phi}$, (49) can be used to conclude that $u(t)$ is equal to the following constant

$$u = \frac{-K_p e_c}{K_E}. \tag{50}$$

The result in (42) can now be obtained by following the same arguments as that in the proof of Theorem 1.

## V. Numerical Simulation

To illustrate the performance of the energy-based controllers in (23) and (39), numerical simulations were performed for the example system depicted in Fig. 1. The equations of motion of the system are given by the following differential equations

$$\begin{aligned} & M_1\ddot{q}_1 + (K_1 + K_2)(q_1 - q_1^*) - K_2(q_2 - q_2^*) = 0 \\ & M_2\ddot{q}_2 - K_2(q_1 - q_1^*) + (K_2 + K_e\Lambda_2(q_2))(q_2 - q_2^*) = F \end{aligned} \tag{51}$$

where $q_1(t)$, $q_2(t)$ denote the positions of $M_1$ and $M_2$, respectively, and $q_1^*$, $q_2^*$ denote the equilibrium points of $M_1$ and $M_2$, respectively. After utilizing (13), the dynamics in (51) can be expressed as

$$\begin{aligned} M_1\ddot{e}_u + (K_1 + K_2)e_u - K_2 e_c &= 0 \\ M_2\ddot{e}_c - K_2 e_u + K_2 e_c + K_e\Lambda_2(e_c)e_c &= u \end{aligned} \tag{52}$$

where $u(t) = F(t)$. The energy of the system is given by

$$\begin{aligned} E = {} & \frac{1}{2}K_1 e_u^2 + \frac{1}{2}K_2(e_u - e_c)^2 + \frac{1}{2}K_e\Lambda_2(e_c)e_c^2 \\ & + \frac{1}{2}M_1\dot{e}_u^2 + \frac{1}{2}M_2\dot{e}_c^2 \end{aligned} \tag{53}$$

where the top line in (53) denotes the potential energy of the system and the bottom line represents the kinetic energy. Based on (51), the definitions in (4), (18), and (19) can be expressed as

$$S = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \tag{54}$$

$$P = M_1 \tag{55}$$

$$W = M_1(K_2 e_u - K_2 e_c - K_e\Lambda_2(e_c)e_c). \tag{56}$$

*Remark 2:* To verify that the system is zero-state observable (i.e., Assumption 4), we set $u(t) = 0$, $e_c(t) = 0$, and $\dot{e}_c(t) = 0$ (and hence $\ddot{e}_c(t) = 0$) to obtain the following dynamics

$$\begin{aligned} M_1\ddot{e}_u + (K_1 + K_2)e_u &= 0 \\ -K_2 e_u &= 0. \end{aligned} \tag{57}$$

The dynamics in (57) can be used to conclude that a necessary and sufficient condition for $e_u(t) = 0$ and $\dot{e}_u(t) = 0$, is for $e_c(t) = 0$, $\dot{e}_c(t) = 0$, and $u(t) = 0$.

Based on (51) and (54)-(56), the two energy-based controllers given in (23) and (39) can be expressed as follows.

- Energy coupling nonlinear controller in (23):

$$\begin{aligned} u = {} & \frac{-K_d\dot{e}_c - K_p e_c}{K_E E + \frac{K_v}{M_2}} \\ & - \frac{\frac{K_v}{M_2}(K_2(e_u - e_c) - K_e\Lambda_2(e_c)e_c)}{K_E E + \frac{K_v}{M_2}}. \end{aligned} \tag{58}$$

- Alternative energy coupling controller in (39):

$$\begin{aligned} u = {} & \frac{-K_d\dot{e}_c - K_p e_c}{K_E + K_V} \\ & - \frac{K_v(K_2(e_u - e_c) - K_e\Lambda_2(e_c)e_c)}{K_E + K_V}. \end{aligned} \tag{59}$$

*Remark 3:* In (58), the energy $E(e, \dot{e})$ appears directly in the controller giving the most coupling information. Whereas in (59), the energy does not appear directly, but some coupling information is still contained in the controller because of the use of energy in the controller synthesis and the stability analysis. As will be seen in the simulation results, the controller in (58) generates better transient response than the controller in (59) due to the more coupling information supplied by energy.

For the simulation, the physical parameters of the mass spring system were selected as

$$\begin{aligned} M_1 &= 1 \text{ [kg]}, \qquad M_2 = 1 \text{ [kg]}, \\ K_1 &= 10^3 \text{ [N/m]}, \qquad K_2 = 5 \times 10^3 \text{ [N/m]}, \\ K_e &= 10^6 \text{ [N/m]}. \end{aligned}$$

The equilibrium positions of $M_1$ and $M_2$ were set to the following values

$$\begin{bmatrix} q_1^* & q_2^* \end{bmatrix}^T = \begin{bmatrix} -1 & -0.5 \end{bmatrix}^T [m]$$

where the initial conditions for $q_1(t)$ and $q_2(t)$ were selected as

$$\begin{bmatrix} q_1(0) & q_2(0) \end{bmatrix}^T = \begin{bmatrix} -1.5 & -0.8 \end{bmatrix}^T [m].$$

The controllers in (58) and (59) were tuned to yield the best performance for approximately equal control efforts. The integral of the control effort squared $\int_0^1 u^2(\sigma)d\sigma$ for the controller in (58) is $2.4 \times 10^4 [N^2]$ and the counterpart for the controller in (59) is $2.1 \times 10^4 [N^2]$. From Fig. 2 and Fig. 3, the controller in (58) exhibits improved transient performance because the controller provides improved coupling of the underactuated states.

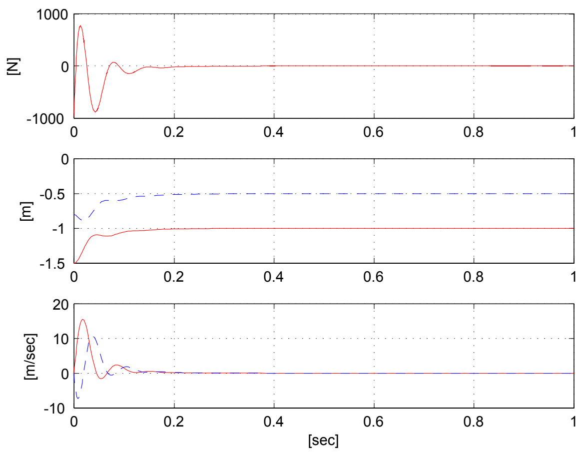


Fig. 2. Results for the energy-based coupling controller. The top figure depicts the control input. In the middle figure the solid curve is the position of $M_1$, and the dashed curve is the position of $M_2$. In the bottom figure the solid curve is the velocity of $M_1$, and the dashed curve is the velocity of $M_2$.

## VI. Conclusions

The efforts in this paper are inspired by the idea that improved transient response will result from using the system energy to couple the states in the controller. Based on this idea, two examples of energy-based controllers are proven to globally asymptotically stabilize a general class of underactuated Euler-Lagrange systems. A Lyapunov-based stability analysis and numerical simulations are provided to demonstrate the stability and performance of the developed controllers.

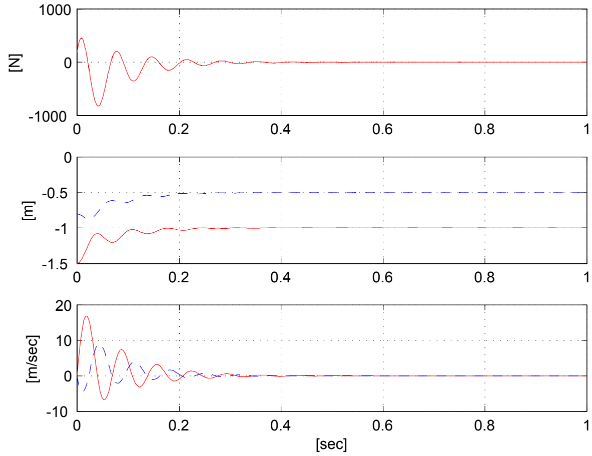


Fig. 3. Results for an alternative energy coupling controller. The top figure depicts the control input. In the middle figure the solid curve is the position of $M_1$, and the dashed curve is the position of $M_2$. In the bottom figure the solid curve is the velocity of $M_1$, and the dashed curve is the velocity of $M_2$.

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