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A viability approach to Hamilton-Jacobi equations: application to concave highway traffic flux functions

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Abstract— **This paper presents a new approach which links the solution to a particular Hamilton-Jacobi partial differential equation to the solution of an optimal control problem provided by viability theory. It constructs the solution to this partial differential equation through its hypograph, which is defined as the capture basin of a target under an auxiliary dynamics that we define. The target itself represents the hypograph of a desired function. It is applied to concave Hamiltonian functions and has implications for the control of conservation laws with concave flux functions. It is a building block towards controlling conservation laws with concave flux functions, though at this stage, the link with boundary control of hyperbolic conservation laws cannot be made explicitly.**

I. Introduction

We first introduce the Lighthill-Whitham-Richards partial differential equation which is a first order hyperbolic conservation law. We then show how this equation can be transformed into a *Hamilton-Jacobi* (HJ) *partial differential equation* (PDE), using a well known integration technique.

A commonly used first order model of highway traffic is the *Lighthill - Whitham - Richards* (LWR) *partial differential equation* (PDE) [6], [8]. This partial differential equation is derived from physical principles. Let us consider an infinite road. We denote by x the coordinate along the road. We call $\rho(t,x)$ the vehicle density on the highway, i.e. the number of vehicles per unit length. The flux of vehicles at a location x is defined as the number of vehicles crossing this location per time unit. The density varies with space (the density of vehicles on the highway is not necessarily homogeneous), and with time (it changes during the day). When an observer is standing at a particular location of the highway, he/she can observe a *phenomenological law*, which relates the *local density* of vehicles on the highway to the *flux* of vehicles at the location where he/she stands. We denote this flux function by $\psi(\cdot)$. In practice, observations show that for a uniform highway (same number of lanes all along) the flux function $\psi(\cdot)$ is a function of the density ρ which can be measured. For small densities or vehicles, the flux increases linearly with the density (the more vehicles, the more flux), with a slope ν. Beyond a *critical density*, called γ the flux stops to increase, because of the appearance of highway congestion. If the density increases further, the flux decreases because of congestion, until it eventually becomes zero for $\rho = \omega$. This density is called jam density. It corresponds to the situation in which vehicles are stuck on the highway. The LWR PDE is given by:

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial(\psi(\rho(t,x)))}{\partial x} = 0 \qquad (1)$$

Example — *Lighthill-Whitham-Richards partial differential equation with Greenshield flux function*. A very simple fit for the flux function is called the *Greenshield flux function* and is given by

$$\forall\ \rho\ \in\ [0,\omega],\ \ \psi(\rho)\ :=\ \frac{\nu}{\omega}\rho\,(\omega-\rho) \qquad (2)$$

where ω is the *jam density* and ν is the *free flow velocity*. With this expression of $\psi(\cdot)$, equation (1) becomes:

$$\frac{\partial \rho(t,x)}{\partial t} + \nu\left(1 - \frac{2\rho(t,x)}{\omega}\right)\frac{\partial \rho(t,x)}{\partial x} = 0 \qquad (3)$$

Other flux functions will be investigated in section III, in particular a well known trapezoidal flux function, available in the literature (see in particular Daganzo [4], [5]):

$$\psi_0(p)\ = \begin{cases} \nu^\flat p & \text{if} \quad p \le \gamma^\flat \\ \delta & \text{if} \quad p \in [\gamma^\flat, \gamma^\sharp] \\ \nu^\sharp(\omega - p) & \text{if} \quad p \ge \gamma^\sharp \end{cases}$$

where $\delta \le \frac{\omega\nu^\flat\nu^\sharp}{\nu^\flat+\nu^\sharp}$ is the maximal flux and

$$\gamma^\flat\ :=\ \frac{\delta}{\nu^\flat}\ \text{ and }\ \gamma^\sharp\ :=\ \frac{\nu^\sharp\omega - \delta}{\nu^\sharp}$$

the lower and upper critical densities. □

The LWR PDE exhibits discontinuous solutions, which are well known. These solutions have been mathematically defined in 1957 Oleinik [7], to the price of the adjunction of another requirement, and have been observed in physics since the 19th century. The discontinuity (or set valuedeness) of these solutions is not a problem. However, it creates serious difficulties when trying to control the solutions of such equations. Therefore, transforming these partial differential equation into a partial differential equation from which we know that the solution is continuous makes it much easier to control the corresponding solution.

A. *Using continuous solutions provided by Hamilton-Jacobi equations*

We use a well known transformation from scalar conservation laws into scalar Hamilton-Jacobi equations, motivated

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by the need to derive control policies for the LWR PDE of interest. We introduce the integral of ρ, defined by

$$\mathbf{N}(t,x) = \int_0^x \rho(t,u)du \quad (4)$$

Physically, the function $\mathbf{N}(t,x)$ represents the cumulated number of vehicles between positions 0 and x (because it is the integral of the density of cars between these two points, see Daganzo [4], [5] for more details). An integration of the partial differential equation with respect to the state variable provides us with:

$$\frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) = \psi\left(\frac{\partial \mathbf{N}(t,0)}{\partial x}\right) \quad (5)$$

The interpretation of the right hand side of (5) is a flux source term at the point $x = 0$. The term $\psi(\frac{\partial \mathbf{N}(t,0)}{\partial x})$ represents the flux q of the density at this point (since the density is the space derivative of $\mathbf{N}(t,x)$. Ideally, we would like to control the LWR PDE (1) using boundary conditions at one end of the domain, which would translate to controlling (5) using the boundary condition at 0. This problem was not solved at this stage. The contribution of this article is instead to construct a solution to (5) using tools of viability theory, as a preliminary step towards this ambitious goal. As will appear throughout this paper, the chosen method to construct the solution $\mathbf{N}(t,x)$ leaves us total freedom in choosing the boundary condition $\mathbf{N}(t,0)$. This gives us hope that this method will be able to be generalized so that the appropriate $\mathbf{N}(t,0)$ can be translated in terms of ρ to control the original LWR PDE directly.

We will therefore now focus our attention on the slightly more general Hamilton-Jacobi partial differential equation:

$$\begin{array}{r} \dfrac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\dfrac{\partial \mathbf{N}(\mathbf{t},\mathbf{x})}{\partial \mathbf{x}}\right) = \mathbf{l}(t,x) \\ \mathbf{N}(0,x) = \mathbf{N}_0(x) \end{array} \quad (6)$$

where ψ is a concave flux function and $\mathbf{l}(t,x)$ is an arbitrary function (given by the right hand side of (5) in the present case). The initial condition $\mathbf{N}_0$ represents the initial repartition of vehicles. We shall prove that such a solution exists and is unique, in a generalized sense (Frankowska and Barron/Jensen) because the solution is just upper semicontinuous. We shall provide an explicit formula of the solution:

$$\mathbf{N}(t,x) \;:=\; \sup_{u(\cdot)\in L^1(0,+\infty;\mathrm{Dom}(\varphi^*))} \left(\mathbf{N}_0\left(x+\int_0^t u(r)dr\right) + \int_0^t \left(\mathbf{l}\left(t-r, x+\int_0^r u(s)ds\right) - \varphi^*(u(r))\right)dr\right)$$

We deduce that that is is "sup-linear" (if the initial data is the supremum of initial data, then the solution is the supremum of the solutions) and depends continuously on the initial data for specific concepts of convergence.

B. Organization of this article

This article is organized as follows. In section II, we briefly survey the fundamental convex analysis tools used in the rest of this paper. In section III, we explain how to modify the flux functions so that the domain of the Fenchel transform is compact, a property needed in order to define the proper solution of the HJ PDE. Examples of these modifications are provided with the well-known Greenshield or trapezoidal flux function models commonly used in traffic engineering. Section IV presents a uniqueness and existence result for the corresponding HJ PDE. This fact is not new, but its characterization in terms of viability theory is new. This solution is constructed through its hypograph, which is the capture basin of a target (the hypograph of the initial conditions), under an auxiliary dynamics which we make explicit. This dynamics is set valued, i.e. we need to identify a feedback to solve for the capture basin. This feedback is explicited in Section V. Finally, an algorithm is developed in Section VI to compute the solution of the Hamilton-Jacobi equation numerically.

II. Fenchel Transform and sub and super differentials

Given a concave flux function ψ, we define the convex function φ by $\varphi(p) \;:=\; -\psi(p)$. We introduce the Fenchel transform φ^* of φ, defined by:

$$\varphi^*(u) := \sup_{p\in\mathrm{Dom}(\varphi)} [\langle p,u\rangle - \varphi(p)] = \sup_{p\in\mathrm{Dom}(\varphi)} [\langle p,u\rangle + \psi(p)] \quad (7)$$

Note that since $\varphi = \varphi^{**}$ if and only if φ is convex, lower semicontinuous, and non trivial (i.e. $\mathrm{Dom}(\varphi) := \{p \mid \varphi(p) < +\infty\} \neq \emptyset$), then we can recover the function ψ from φ^* by formula

$$\psi(p) \;:=\; \inf_{u\in\mathrm{Dom}(\varphi^*)} [\varphi^*(u) - \langle p,u\rangle] \quad (8)$$

Note the two following properties:

$$\left\{ \begin{array}{l} \sup_p \psi(p) \;=\; \varphi^*(0) \\ \forall\, u \in \mathrm{Dom}(\varphi^*), \;\; \varphi^*(u) \;\geq\; \sup_{\{p\in\psi^{-1}(0)\}} \langle p,u\rangle \end{array} \right.$$

Recall that if ψ is concave and finite on the finite dimensional vector space X, then it is locally Lipschitz and super differentiable (in the sense that $\forall\, p \in X,\;\; \partial_+\psi(p) \neq \emptyset$.)

Remark: Convex and Concave Analysis — Since the authors of most of books on convex analysis have chosen to study convex functions rather than concave ones, we have chosen to associate with the concave function ψ the Fenchel transform φ^* of φ rather than the "concave Fenchel" transform $\psi^{\boxtimes}$ defined by the concave function

$$\psi^{\boxtimes}(u) \;:=\; \inf_{p\in\mathrm{Dom}(\psi)} [\langle p,u\rangle - \psi(p)] = -\varphi^*(-u)$$

The basic theorem of convex analysis states that $\psi = \psi^{\boxtimes\boxtimes}$ if and only if ψ is concave, upper semicontinuous, and non trivial (i.e. $\mathrm{Dom}(\psi) \;:=\; \{p \mid \varphi(p) > -\infty\} \neq \emptyset$). Note that the hypograph of ψ is related to the epigraph of φ by the relation

$$(p,\lambda) \in \mathcal{H}yp(\psi) \text{ if and only if } (p,-\lambda) \in \mathcal{E}p(\varphi)$$

Definition 2.1: The hypoderivative $D_\downarrow\psi(p)$ and the epiderivative $D_\uparrow\varphi(p)$ are related to the tangent cones of the hypograph of ψ and epigraph of φ by the relations

$$\mathcal{H}yp(D_\downarrow\psi(p)) := T_{\mathcal{H}yp(\psi)}(p,\psi(p))$$

$$\mathcal{E}p(D_\uparrow\varphi(p)) := T_{\mathcal{E}p(\varphi)}(p,\varphi(p))$$

The superdifferential $\partial_+\psi(p)$ of the concave function ψ at p is defined by

$$u \in \partial_+\psi(p) \text{ if } \forall\, v \in X, \; \langle u,v\rangle \geq D_\downarrow\psi(p)(v)$$

and the subdifferential $\partial_-\varphi(p)$ is defined by

$$u \in \partial_-\varphi(p) \text{ if } \forall\, v \in X, \; \langle u,v\rangle \leq D_\uparrow\varphi(p)(v)$$

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and the subdifferential $\partial_-\varphi(p)$ is defined by

$$u \in \partial_-\varphi(p) \text{ if } \forall\, v \in X, \; \langle u,v\rangle \leq D_\uparrow\varphi(p)(v)$$

We infer that

$$\forall\, v \in X, \;\; D_\downarrow\psi(p)(v) = -D_\uparrow\varphi(p)(v)$$

and that

$$u \in \partial_+\psi(p) \text{ if and only if } u \in -\partial_-\varphi(p)$$

The *polar cone* P^- of a given set P is defined by:

$$P^- = \{p \in X^* \mid \forall x \in P, \; \langle p,x\rangle \leq 0\}$$

where X^* is the dual space of X and the normal cone $N_K(x) := T_K(x)^-$ to K at $x \in K$ we use in this paper is the polar cone to the contingent cone to K at $x \in K$. The superdifferential $\partial_+\psi(p)$ and the subdifferential $\partial_-\varphi(p)$ are related to the normal cones of the hypograph of ψ and epigraph of φ by the relations

$$u \in \partial_+\psi(p) \text{ if and only if } (-u,1) \in N_{\mathcal{H}yp(\psi)}(p,\psi(p))$$

and

$$u \in \partial_-\varphi(p) \text{ if and only if } (u,-1) \in N_{\mathcal{E}p(\varphi)}(p,\varphi(p))$$

Recall the Legendre inversion formula:

$$u \in -\partial_+\psi(p) \text{ if and only if } p \in \partial_-\varphi^*(u)$$

and the (decreasing) monotonicity property of superdifferential maps $p \rightsquigarrow \partial_+\psi(p)$ of a concave function:

$$\forall\, u_i \in \partial_+\psi(p_i), \; i=1,2, \; \langle u_1-u_2, p_1-p_2\rangle \leq 0$$

If K is a subset, we denote by $\sigma(K,p) := \sup_{u\in K}\langle p,u\rangle$ the support function of K. Its subdifferential $\partial_-\sigma(K,p)$ the support zone of p in K. See [1], [2] or [9] for more details. We shall need the following result on tangent and normal cones to hypographs:

Lemma 2.3: **A.** If $\psi : X \mapsto \mathbf{R}_+ \cup \{-\infty\}$ is an extended function and if $D_\downarrow\psi(p)(dp)$ is finite, then, for every $w < \psi(p)$ and every $\mu \in \mathbf{R}$, the pair (dp,μ) belongs to the contingent cone $T_{\mathcal{H}yp(\psi)}(p,w)$ to the epigraph of ψ at (p,w).

B. Consequently, a pair (u,λ) belongs to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w)$ to the epigraph of ψ at (p,w) if and only

1) if $w = \psi(p)$, $\lambda > 0$ and $u \in -\lambda\partial_+\psi(p)$,
2) if $w \leq \psi(p)$, $\lambda = 0$ and $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$.

C. In particular, if the domain of $D_\downarrow\psi(p)$ is dense in X, then (u,λ) belongs to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w)$ to the epigraph of ψ at (p,w) if and only if $\lambda > 0$ and $u \in -\lambda\partial_+\psi(p)$. This is the case whenever ψ is Lipschitz around p.

III. Flux Functions

In the case of Lighthill-Whitham-Richards partial differential equations when $X := \mathbb{R}$, *the flux function is defined by a concave function* ψ_0 *vanishing at density* 0 *and at a jam density* $\omega > 0$. The function $\varphi_0(p)$ is defined by $\varphi_0(p) := \psi_0(-p)$. Note that its domain is not necessarily compact. Its Fenchel conjugate is defined by

$$\varphi_0^*(u) := \sup_{p\in\mathrm{Dom}(\psi_0)} [\langle p,u\rangle + \psi_0(p)]$$

Definition 3.1: A flux fonction is a concave function defined on a neighborhood of the interval $[0,\omega]$ and satisfying

$$\psi_0(0) = \psi_0(\omega) = 0$$

Its Fenchel conjugate is defined by

$$\varphi_0^*(u) := \sup_{p\in\mathrm{Dom}(\psi_0)} [\langle p,u\rangle + \psi_0(p)]$$

and satisfies

$$\forall\, u \in \mathbb{R}, \;\; \varphi_0^*(u) \geq \max(0,\omega u)$$

Furthermore

$$\varphi_0^*(0) = \sup_{p\in\mathbb{R}} \psi_0(p) \geq 0 \text{ is the maximal flux}$$

The subdifferential $\partial_-\varphi_0^*(0) \subset [0,\omega]$ is the critical density interval, where the flux function achieves its maximum. Since the superdifferential is a convex subset, it is an interval $\partial_-\varphi_0^*(0) =: [\gamma^\flat,\gamma^\sharp]$. We say that $\gamma^\flat$ is the lower critical density and $\gamma^\sharp$ the upper critical density. We shall assume that

$$0 < \gamma^\flat \leq \gamma^\sharp < \omega$$

The superdifferential map $p \rightsquigarrow \partial_+\psi_0(p)$ plays an important role and is monotone decreasing in the sense that

$$\forall\ p_i \text{ and } u_i \in \partial_+\psi_0(p_i),\ i=1,2,\ \ (u_1-u_2)(p_1-p_2)\ \leq\ 0$$

We set

$$\left\{\begin{array}{l} (i)\nu^\flat\ :=\ \sup(u \in \partial_+\psi_0(0)) \\ \ \ (\nu^\flat\ :=\ \psi_0'(0) \text{ if } \psi_0 \text{ is differentiable at } 0) \\ (ii)\nu^\sharp\ :=\ -\sup(u \in \partial_+\psi_0(\omega)) \\ (\nu^\sharp\ :=\ -\psi_0'(\omega) \text{ if } \psi_0 \text{ is differentiable at } \omega) \end{array}\right.$$

We observe that both flow velocities $\nu^\sharp$ and $\nu^\flat$ are strictly positive.

For physical reasons, only the nonnegative values of ψ_0 when p ranges over the density interval $[0,\omega]$ do matter, so that any concave flux fonction ψ which coincides with ψ_0 on the density interval $[0,\omega]$ can be used instead of ψ_0. Hence we may have to choose another function which satisfies the following property: *The domain of the Fenchel transform*

$$\varphi^*(u)\ :=\ \sup_{p\in\text{Dom}(\varphi)}\ [\langle p,u\rangle - \varphi(p)]$$

is compact. This property is mandatory for both mathematical and numerical reasons. The next Proposition thus constructs a function ψ which coincides with ψ_0 on the set where ψ_0 is positive, such that the corresponding conjugate φ^* has a compact domain.

Proposition 3.2: Let us consider a concave flux function ψ_0 defined on a neighborhood of the interval $[0,\omega]$ and satisfying

$$\psi_0(0)\ =\ \psi_0(\omega)\ =\ 0$$

We associate with it the continuous concave function ψ defined by

$$\psi(p)\ = \left\{\begin{array}{lll} \nu^\flat p & \text{if} & p \leq 0 \\ \psi_0(p) & \text{if} & p \in [0,\omega] \\ \nu^\sharp(\omega - p) & \text{if} & p \geq \omega \end{array}\right.$$

Then the Fenchel transform φ^* satisfies

$$\varphi^*(u)\ = \left\{\begin{array}{lll} \varphi_0^*(u) & \text{if} & u \in [-\nu^\flat, +\nu^\sharp] \\ +\infty & \text{if} & u \notin [-\nu^\flat, +\nu^\sharp] \end{array}\right.$$

Example: trapezoidal flux function — In this example, we fix the following data: The jam density ω, the coefficients $\nu^\flat > 0$ and $\nu^\sharp > 0$ and the maximal flux $\delta \leq \frac{\omega\nu^\flat\nu^\sharp}{\nu^\flat+\nu^\sharp}$. The lower and upper critical densities are equal to

$$\gamma^\flat\ :=\ \frac{\delta}{\nu^\flat} \text{ and } \gamma^\sharp\ :=\ \frac{\nu^\sharp\omega - \delta}{\nu^\sharp}$$

Then the trapezoidal flux function (such as the one proposed by Daganzo [4], [5]) is defined by

$$\psi_0(p)\ = \left\{\begin{array}{lll} \nu^\flat p & \text{if} & p \leq \gamma^\flat \\ \delta & \text{if} & p \in [\gamma^\flat, \gamma^\sharp] \\ \nu^\sharp(\omega - p) & \text{if} & p \geq \gamma^\sharp \end{array}\right.$$

satisfies the prerequisites of a flux function, In this case, $\psi = \psi_0$ and its Fenchel transform is equal to

$$\varphi^*(u)\ = \left\{\begin{array}{lll} \frac{\delta}{\nu^\flat}u + \delta & \text{if} & u \in [-\nu^\flat, 0] \\ \frac{(\omega\nu^\sharp - \delta)}{\nu^\sharp}u + \delta & \text{if} & u \in [0, +\nu^\sharp] \\ +\infty & \text{if} & u \notin [-\nu^\flat, +\nu^\sharp] \end{array}\right.$$

It is piecewise affine (affine on $[-\nu^\flat, 0]$ and $[0, +\nu^\sharp]$) and satisfies $\varphi^*(0) = \delta$. The superdifferential $\partial_+\psi(p)$ is equal to

$$\partial_+\psi_0(p)\ = \left\{\begin{array}{lll} \nu^\flat & \text{if} & p \leq \gamma^\flat \\ 0 & \text{if} & p \in [\gamma^\flat, \gamma^\sharp] \\ -\nu^\sharp & \text{if} & p \geq \gamma^\sharp \end{array}\right.$$

and thus, piecewise constant. When the maximal flux is equal to the upper bound $\delta^* := \frac{\omega\nu^\flat\nu^\sharp}{\nu^\flat+\nu^\sharp}$, then the flux function is triangular, the lower and upper critical densities collapse to the critical density $\gamma := \dfrac{\nu^\sharp\omega}{\nu^\flat + \nu^\sharp}$ and the conjugate function is defined on $[-\nu^\flat, +\nu^\sharp]$ by the affine function

$$\varphi^*(u)\ =\ \frac{\nu^\sharp\omega}{\nu^\flat + \nu^\sharp}(u + \nu^\flat)$$

Example: Greenshield flux function — The Greenshield function reads:

$$\psi_0(p)\ :=\ \frac{\nu}{\omega}p(\omega - p)$$

It vanishes at 0 and ω and reaches its critical density at $\gamma = \gamma^\flat = \gamma^\sharp := \frac{\omega}{2}$. We observe

$$\psi_0'(p)\ =\ \frac{\nu}{\omega}(\omega - 2p)$$

is affine and that $\nu^\flat = \nu^\sharp = \nu$. The maximum flux is equal to $\varphi_0^*(0) = \frac{\omega\nu}{4}$ because the Fenchel conjugate is equal to

$$\varphi_0^*(p)\ =\ \frac{\omega}{4\nu}(u+\nu)^2$$

Hence, the associated function ψ is equal to

$$\psi(p)\ = \left\{\begin{array}{lll} \nu p & \text{if} & p \leq 0 \\ \frac{\nu}{\omega}(\omega - p) & \text{if} & p \in [0,\omega] \\ \nu(\omega - p) & \text{if} & p \geq \omega \end{array}\right.$$

and its Fenchel transform φ^* is equal to

$$\varphi^*(u)\ = \left\{\begin{array}{lll} \frac{\omega}{4\nu}(u+\nu)^2 & \text{if} & u \in [-\nu, +\nu] \\ +\infty & \text{if} & u \notin [-\nu, +\nu] \end{array}\right.$$

IV. Hamilton-Jacobi Equations with Concave Flux on the Whole Space

A. *The Viability Hyposolution*

We introduce the following set-valued map $F_0 : \mathbb{R}_+ \times X \times \mathbb{R} \rightsquigarrow \mathbb{R} \times X \times \mathbb{R}$:

$$F_0(\tau, x, y)\ :=\ \{(-1, u, -\mathbf{l}(\tau, x) + \varphi^*(u))\}_{u\in\text{Dom}(\varphi^*)} \qquad (9)$$

Then the differential inclusion

$$(\tau'(t), x'(t), y'(t))\ \in\ F_0(\tau(t), x(t), y(t))$$

is defined as the characteristic control system

$$\left\{\begin{array}{lcl}\tau'(t) & = & -1 \\ x'(t) & = & u(t) \text{ where } u(t) \in \text{Dom}(\varphi^*) \\ y'(t) & = & -\mathbf{l}(\tau(t), x(t)) + \varphi^*(u(t))\end{array}\right. \quad (10)$$

that we shall use for characterizing the solution to Hamilton-Jacobi partial differential equation (6) with concave flux ψ. We associate with the initial condition $\mathbf{N}_0(x)$ defined on X the following function $\mathbf{c}_\downarrow$, which encodes the initial condition of the problem:

$$\mathbf{c}_\downarrow(t,x) = \left\{\begin{array}{ll}\mathbf{N}_0(x) & \text{if} \quad t=0,\ x \in X \\ -\infty & \quad\quad \text{otherwise}\end{array}\right. \quad (11)$$

Definition 4.1: We call viability hyposolution to the Hamilton-Jacobi equation (6) the function $\mathbf{N}(\cdot,\cdot)$ defined by:

$$\mathbf{N}(t,x) := \sup_{(t,x,y)\in \text{Capt}_{(10)}(\mathbb{R}_+ \times X \times \mathbb{R}, \mathcal{H}yp(\mathbf{c}_\downarrow))} y \quad (12)$$

The capture basin Capt is defined in [3] based on [1]. The subscript (10 means that the considered dynamics is (10. In the present case, the constraint set is $\mathbb{R}_+ \times X \times \mathbb{R}$ and the target is $\mathcal{H}yp(\mathbf{c}_\downarrow)$).

Theorem 4.2: The viability hyposolution to the Hamilton-Jacobi equation (6) is given by formula

$$\begin{array}{l}\mathbf{N}(t,x) := \sup_{u(\cdot)\in L^1(0,+\infty;\text{Dom}(\varphi^*))} \\ \left(\mathbf{N}_0\left(x + \int_0^t u(r)dr\right) + \right. \\ \left.\int_0^t \left(\mathbf{l}\left(t-r, x + \int_0^r u(s)ds\right) - \varphi^*(u(r))\right) dr\right)\end{array} \quad (13)$$

and thus, satisfies the initial condition: $\forall\, x \in X,\ \ \mathbf{N}(0,x) = \mathbf{N}_0(x)$.

Corollary 4.3 (A posteriori Estimates): The viability hyposolution satisfies the following estimate

$$\mathbf{N}(t,x) \leq \|\mathbf{N}_0\|_\infty + [\|\mathbf{l}\|_\infty - \psi(0)]\, t$$

and

$$\mathbf{N}(t,x) \leq \|\mathbf{N}_0\|_\infty + \|\mathbf{l}\|_\infty t$$

when furthermore $\psi(0) = 0$.

Remark: — Since $\mathbf{N}(t,x)$ represents the cumulative vehicle count in traffic theory, an interpretation for the highway problem is that the growth of the number of vehicles on the highway is bounded by an upper bound $\|\mathbf{N}_0\|_\infty$ on the initial number of vehicles plus an upper bound $\|\mathbf{l}\|_\infty$ on the maximum inflow times the duration t of the time range considered.

B. The Hyposolution is the Unique Frankowska-Barron/Jensen Solution

Theorem 4.4: Let us assume that the function $\mathbf{l}$ is upper semicontinuous, bounded below and with linear growth ($-a \leq \mathbf{l}(\tau,x) \leq c(1+\tau+\|x\|)$), that the domain of φ^* is compact, that the convex function φ^* is bounded above and that ψ is finite (and thus, continuous and superdifferentiable). Then $\mathbf{N}$ is the **largest** upper semicontinuous solution satisfying

$$\begin{array}{l}\forall\, t \geq 0,\ \forall\, x \in X,\ \ \mathbf{N}(0,x) = \mathbf{N}_0(x) \\ \forall t > 0,\ \forall \in X,\ \forall (p_t,p_x) \in \partial_+\mathbf{N}(t,x), \\ \quad p_t + \psi(p_x) \leq +\mathbf{l}(t,x) \\ \forall t > 0,\ \forall x \in X,\ \forall (p_t,p_x) \in (\text{Dom}(D_\downarrow \mathbf{N}(t,x)))^-, \\ \quad p_t - \sigma(\text{Dom}(\varphi^*), p_x) \leq 0\end{array} \quad (14)$$

and the initial condition $\mathbf{N}(0,x) = \mathbf{N}_0(x)$, where we set $\sigma(\text{Dom}(\varphi^*), p_x) := \sup_{u\in \text{Dom}(\varphi^*)} \langle p_x, u\rangle$. If l is Lipschitz, if φ^* is Lipschitz, then $\mathbf{N}$ is the **smallest** upper semi continuous solution satisfying

$$\left\{\begin{array}{l}\forall\, t \geq 0,\ \forall\, x \in X,\ \ \mathbf{N}(0,x) = \mathbf{N}_0(x) \\ \forall t \geq 0,\ \forall x \in X,\ \forall (p_t,p_x) \in \partial_+\mathbf{N}(t,x), \\ \quad p_t + \psi(p_x) \geq \mathbf{l}(t,x) \\ \forall t \geq 0,\ \forall x \in X,\ \forall (p_t,p_x) \in (\text{Dom}(D_\downarrow \mathbf{N}(t,x)))^-, \\ \quad -p_t + \sigma(\text{Dom}(\varphi^*), p_x) \leq 0\end{array}\right. \quad (15)$$

and the initial condition $\mathbf{N}(0,x) = \mathbf{N}_0(x)$. Under both assumptions, $\mathbf{N}$ is the **unique** upper semicontinuous solution of

$$\left\{\begin{array}{l}\forall\, t \geq 0,\ \forall\, x \in X,\ \ \mathbf{N}(0,x) = \mathbf{N}_0(x) \\ \forall t > 0,\ \forall x \in X,\ \forall (p_t,p_x) \in \partial_+\mathbf{N}(t,x), \\ \quad p_t + \psi(p_x) = +\mathbf{l}(t,x) \\ \forall t > 0,\ \forall x \in X,\ \forall (p_t,p_x) \in (\text{Dom}(D_\downarrow \mathbf{N}(t,x)))^-, \\ \quad -p_t + \sigma(\text{Dom}(\varphi^*), p_x) = 0\end{array}\right. \quad (16)$$

with initial condition $\mathbf{N}(0,x) = \mathbf{N}_0(x)$.

V. The Regulation Map

Instead of characterizing capture basins in terms of normal cones and translate them in terms of Frankowska-Barron/Jensen solutions, we can characterize them in terms of tangent cones and translate them in termes of Frankowska hyposolutions. This allows us to derive a regulation law to govern the optimal évolutions to the control problem.

Theorem 5.1: Let us assume that the function $\mathbf{l}$ is upper semicontinuous, bounded below and with linear growth ($-a \leq \mathbf{l}(\tau,x) \leq c(1+\tau+\|x\|)$), that the domain of φ^* is compact, that the convex function φ^* is bounded above and that ψ is finite (and thus, continuous and superdifferentiable). Then $\mathbf{N}$ is the **largest** upper semicontinuous solution satisfying

$$0 \leq \sup_{u\in \text{Dom}(\varphi^*)} (D_\downarrow \mathbf{N}(t,x)(-1,u) - \varphi^*(u)) + \mathbf{l}(t,x) \quad (17)$$

and the initial condition $\mathbf{N}(0,x) = \mathbf{N}_0(x)$.
The associated regulation map R for regulating the evolutions maximizing the underlying optimal control problem is defined by $\forall\, t > 0,\ x \in X$,

$$R(t,x) := \{u |\ 0 \le D_{\downarrow}\mathbf{N}(t,x)(-1,u) - \varphi^*(u) + \mathbf{l}(t,x)\}$$

One can prove that the optimal solutions of the control problem are governed by the control system

$$\left\{\begin{array}{lcl} \tau'(t) & = & -1 \\ x'(t) & = & u(t) \in\ R(T-t, x(t)) \\ y'(t) & = & -\mathbf{l}(\tau(t),x(t)) + \varphi^*(u(t)) + \pi(t) \end{array}\right.$$

where $\pi(t) \in [0, \alpha + \mathbf{l}(\tau(t),x(t)) - \varphi^*(u(t))]$ and $\alpha := \sup_{u \in \mathrm{Dom}(\varphi^*)} \varphi^*(u) - \inf_{\tau\ge 0,\ x \in X} \mathbf{l}(\tau,x)$. Note that α is finite. This motivates a further study of the regulation map. If the solution $\mathbf{N}$ is differentiable, the regulation map can be written in the form

$$R(t,x) := \left\{u \ |\ 0 \ \le\ -\frac{\partial \mathbf{N}(t,x)}{\partial t} + \frac{\partial \mathbf{N}(t,x)}{\partial x} u - \varphi^*(u) + \mathbf{l}(t,x)\right\}$$

The elements u maximizing the right-hand side are the elements belonging to $-\partial_+\psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right)$, Consequently,

$$-\partial_+\psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \ \subset\ R(t,x)$$

VI. The Capture Basin Algorithm

Numerical solutions of the result presented in the sections above can be obtained with an adaptation of the viability algorithm [10]. In order to discretize with respect to time the characteristic control system (10), we introduce the discrete time line $h\mathbb{N} := \{0, h, \cdots, nh, \cdots\}$. We can approximate the viability hyposolution $\mathbf{N}$ to the Hamilton-Jacobi equation with concave flux by viability hyposolutions $\mathbf{N}_h$ to the discrete Hamilton-Jacobi equation with concave flux

$$\sup_{u\in \mathrm{Dom}(\varphi^*)} [\mathbf{N}^h((n-1)h, x+hu) - \mathbf{N}^h(nh,x) - h\varphi^*(u)] + h\mathbf{l}(nh,x) \ =\ 0 \qquad (18)$$

satisfying the initial condition $\mathbf{N}^h(0,x) = \mathbf{N}_0(x)$. We introduce the set-valued map
$\Phi : h\mathbb{N} \times X \times \mathbb{R} \rightsquigarrow h\mathbb{Z} \times X \times \mathbb{R}$ defined by:

$$\Phi(nh,x,y) := \{((n-1)h, x+hu, y - h\mathbf{l}(\tau,x) + h\varphi^*(u))\}_{u \in \mathrm{Dom}(\varphi^*)} \qquad (19)$$

Then the finite-difference inclusion

$$((n-1)h, x_{n+1}, y_{n+1}) \ \in\ \Phi(nh, x_n, y_n)$$

is nothing else than the discrete characteristic control system

$$\left\{\begin{array}{l} (n+1)h \ =\ nh - h \\ x_{n+1} \ =\ x_n + hu_n \text{ where } u_n \ \in\ \mathrm{Dom}(\varphi^*) \\ y_{n+1} \ =\ y_n - h\mathbf{l}(nh,x_n) + h\varphi^*(u_n) \end{array}\right. \qquad (20)$$

Let us associate with the initial data $\mathbf{N}_0$ the target $\mathcal{C} := \mathcal{H}yp(\mathbf{c}_0)$ contained in the environment $\mathcal{K} := h\mathbb{N}_+ \times X \times \mathbb{R}$, where $\mathbf{c}_0(0,x) := \mathbf{N}_0(x)$ and $\mathbf{c}_0(nh,x) := -\infty$ if $n \ge 1$.

Definition 6.1: We call *discrete viability hyposolution* to the Hamilton-Jacobi equation (VI) the function $\mathbf{N}^h(\cdot,\cdot)$ defined by:

$$\mathbf{N}^h(nh,x) \ :=\ \sup_{(nh,x,y)\in \mathrm{Capt}_{(20)}(h\mathbb{N}_+\times X\times\mathbb{R}, \mathcal{H}yp(\mathbf{c}_0))} y \qquad (21)$$

Theorem 6.2: The viability hyposolution $\mathbf{N}^h$ is the largest fixed point of the following discrete Hamilton-Jacobi functional equation (VI), which can be written

$$\mathbf{N}^h(nh,x) \ =\ h\mathbf{l}(nh,x) + \sup_{u\in \mathrm{Dom}(\varphi^*)} [\mathbf{N}^h((n-1)h, x+hu) - h\varphi^*(u)] \qquad (22)$$

Proposition 6.3: The Capture Basin Algorithm states that the discrete viability hyposolution is the supremum of a sequence of functions computed recursively in the following way

$$\mathbf{N}^h(nh,x) \ =\ \sup_{0\le j\le n} \mathbf{c}_j(jh,x)$$

where $\mathbf{c}_j(nh,x) = -\infty$ if $n \ne j$, $\mathbf{c}_0(0,x) := \mathbf{N}_0(x)$, and for $j \ge 1$,

$$\mathbf{c}_j(jh,x) \ :=\ h\mathbf{l}(h,x) + \sup_{u\in \mathrm{Dom}(\varphi^*)} [\mathbf{c}_{(j-1)}((j-1)h, x+hu) - h\varphi^*(u)]$$

Proofs are available upon request from the authors.

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