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Stabilizing Controller Parametrization of Fault Tolerant Control Systems

Hongbin Li[1], Qing Zhao[1*]

Abstract— **This paper presents a stabilizing controller parametrization method of Fault Tolerant Control Systems (FTCS) based on the stochastic exponential mean square (EMS) stability of the Markovian model of FTCS. The free parameters of the controller parametrization are real matrices or scalars therefore it is easy to implement numerically. Furthermore, a relation between the necessary existence conditions of stabilizing controllers and the classical Linear Quadratic Regulator (LQR) is shown and can be used to direct the parameter selection. As a possible application, this parametrization result can be used with the randomized algorithms for controller design.**

I. INTRODUCTION

Fault Tolerant Control Systems (FTCS) aim to maintain acceptable system performance when component faults occur in a control system and thereby to avoid total failure and to achieve high reliability [1], [2]. FTCS usually employ the Fault Detection and Isolation (FDI) scheme and the reconfigurable controller to eliminate the effects of the presumed faults, usually called as active FTCS [1], [2]. But the false alarm, missing detection and detection delays of FDI may corrupt the overall stability and performance of the FTCS [3]. By modelling the fault and FDI behaviors as two separate Markov processes, the FDI and controller can be analyzed and designed in an integrated manner, such as the effects of FDI false alarms on stochastic stability and controller design with the consideration of FDI imperfectness [4], [5]. In this paper, we study the stabilizing controller parametrization problem of active FTCS based on the Markovian model.

Controller parametrization plays an important role in systems and control theory, which can facilitate the design of optimal controller by using Linear Matrix Inequality (LMI) or other classical optimization techniques. These techniques usually require the parametrization expression and the design objective function are affine with respect to the free parameters. The randomized algorithms can be used to find an approximately optimal controller given any parametrization method by assuming bounded parameters with known probability distribution [6], [7]. For linear systems, many controller parametrization results have been reported, such as the Youla parametrization [8], H_∞ controller parametrization by Riccati equations and by Linear Matrix Inequalities (LMI) [9], [10], [11], covariance controller parametrization [12], [13], and stabilizing controller parametrization by quadratic Lyapunov functions [14]. However, to the best of our knowledge, no controller parametrization results have been reported for FTCS.

This paper studies state-feedback stabilizing controller parametrization of active FTCS in Markovian models. The stability is in the sense of stochastic exponential mean square (EMS) stability [4]; the necessary existence conditions are given in the form of LMI and the free parameters are real matrices and scalars. If the two Markov processes are combined into one integrated process and the system model is converted into a Jump Linear System (JLS) [15], then the existence conditions of stabilizing controllers are equivalent to the Algebraic Recatti Equations (ARE) of the Linear Quadratic Regulator (LQR) problem for JLS [16]; if the Markov state of the fault process are available for controller design and the number of controller equals the number of Markov states, the central controller of the parametrization set can be deemed as the LQR optimal controller in JLS. This connection gives an interpretation of the parametrization result which is helpful for selecting some of the free parameters. The parametrization result combined with the randomized algorithms can be used to search for the approximately optimal controller that achieve some high-level performance criterion of FTCS, such as the system reliability [6], [17].

The remaining of the paper is organized as follows. Section II states the Markovian model and problem formulation; section III gives the mathematical preliminaries; section IV-VI presents the main results: the necessary existence conditions, connections with LQR problem and generation of controllers; a numerical example is given in section VII followed by a conclusion in section VIII.

We will use the following notations. A^{-T} means $(A^T)^{-1}$. A^+ is the Moore Penrose inverse of A. $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range space of A, respectively. $A^\perp$ denotes a matrix with the following properties: $\mathcal{N}(A^\perp) = \mathcal{R}(A)$ and $A^\perp A^{\perp T} > 0$. The norm $\|A\|$ is the largest singular value of A.

II. PROBLEM FORMULATION

Consider the following Markovian model of FTCS

$$\dot{x}(t) = A(\zeta(t))x(t) + B(\zeta(t))u(\eta(t), t), \quad (1)$$
$$y(t) = C(\zeta(t))x(t) + D(\zeta(t))u(\eta(t), t). \quad (2)$$

where $x(t) \in \mathbb{R}^n, u(\eta(t), t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^l$ and $w(t) \in \mathbb{R}^h$ denote the system state, control input, output and exogenous input respectively. $A(\zeta(t))$, $B(\zeta(t))$, $C(\zeta(t))$ and $D(\zeta(t))$ are system matrices with appropriate dimensions.

[1] The authors are with the Department of Electrical & Computer Engineering, University of Alberta, Edmonton, Alberta, Canada, T6G 2V4.
* The corresponding author. Tel: (780)492-5792; Fax: (780)492-1811; Email: qingzhao@ece.ualberta.ca.

As in many references [3], [4], [5], $\zeta(t)$ and $\eta(t)$ are assumed to be two separate continuous-time Markov processes with finite state spaces $S_1 = \{0, 1, 2, \cdots, N_1\}$ and $S_2 = \{0, 1, 2, \cdots, N_2\}$ to represent system faults and FDI results. $\zeta(t)$ is a homogeneous process while the transition rates of $\eta(t)$ depend on the current state of $\zeta(t)$.

For notational simplicity, for $\zeta(t) = i, \eta(t) = j, i \in S_1, j \in S_2$, denote

$$A_i = A(\zeta(t)), B_i = B(\zeta(t)), C_i = C(\zeta(t)), D_i = D(\zeta(t)),$$
$$E_i = E(\zeta(t)), F_i = F(\zeta(t)), u_j(t) = u(\eta(t), t).$$

Let us begin with the basic case that the state spaces of $\zeta(t)$ and $\eta(t)$ are equal and both take values from $\{0, 1\}$, where '0' denotes fault-free situation and '1' the faulty state. The behavior of $\zeta(t)$ is governed by its generator matrix G [19]; when $\zeta(t) = 0$ or 1, the behavior of $\eta(t)$ is determined by the corresponding generator matrix H^0 or H^1. The generator matrices of the basic case have the following form:

$$F = \begin{bmatrix} -\alpha_{01} & \alpha_{01} \\ \alpha_{10} & -\alpha_{10} \end{bmatrix},$$

$$H^0 = \begin{bmatrix} -\beta^0_{01} & \beta^0_{01} \\ \beta^0_{10} & -\beta^0_{10} \end{bmatrix}, \quad H^1 = \begin{bmatrix} -\beta^1_{01} & \beta^1_{01} \\ \beta^1_{10} & -\beta^1_{10} \end{bmatrix},$$

where $\alpha_{ij} \geq 0$ and $\beta^k_{ij} \geq 0$ are the transition rates of $\zeta(t)$ and $\eta(t)$. For simplicity, given FTCS (1)-(2), consider a static state feedback controller. For the single fault case, the controller is composed of two static gains K_0 and K_1 and denoted as (K_0, K_1). When FDI results indicate the fault mode 1, K_1 is switched in; otherwise, K_0 is in use for normal system mode.

We consider the EMS stability of $x = 0$ in FTCS [4], meaning that for any initial Markov states, $\zeta(0)$ and $\eta(0)$, and some $\delta(\zeta(0), \eta(0)) > 0$, there exist $a > 0$ and $b >$ such that when $\|x(0)\| \leq \delta(\zeta(0), \eta(0))$, we have the following inequality for $t \geq 0$.

$$E\{\|x(t)\|^2\} \leq b\|x(0)\|^2 e^{-at},$$

where $E\{\cdot\}$ denotes the mathematical expectation.

***Lemma** 2.1 (Stochastic stability [4], [18]):* The FTCS in (1)-(2) is stabilized by the following static state feedback control law in the sense of EMS stability

$$u_i(t) = K_i x(t), i \in S_2,$$

if and only if for any given $k \in S_1$ and $i \in S_2$, there exist positive-definite matrices $P_{ik} > 0$, satisfying

$$\tilde{A}^T_{ik} P_{ik} + P_{ik}\tilde{A}_{ik} + \sum_{j \neq i}^{M} \beta^k_{ij} P_{jk} + \sum_{j \neq k}^{N} \alpha_{kj} P_{ij} < 0,$$

where

$$\tilde{A}_{ik} = A_k + B_k K_i - 0.5 \sum_{j \neq i}^{\mu} \beta^k_{ij} - 0.5 \sum_{j \neq k}^{v} \alpha_{kj}.$$

We consider the stabilizing controller parametrization problem in the sense of Lemma 2.1. The objective is to find a parametrization method to generate all the stabilizing static state feedback controllers of FTCS. The parametrization method should use matrices or scalars as free parameters for easy calculation; it should also have bounded free parameters and so the randomized algorithms can be applied for controller design.

III. Preliminaries

***Lemma** 3.1 (Finsler's theorem [11], [13]):* Let matrices $M \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{n\times n}$ be given. Suppose rank$(M) < n$ and $Q = Q^T$. Let (M_R, M_L) be any full rank factor of M, i.e., $M = M_L M_R$ and rank(M_L) = rank(M_R) = rank(M), and define $N := (M_R M_R^T)^{(-1/2)}$. Then

$$\mu M M^T - Q > 0$$

holds for some $\mu \in \mathbb{R}$ if and only if

$$P := M^\perp Q M^{\perp T} < 0$$

holds, in which case, all such μ are given by

$$\mu > \mu_{\min} := \lambda_{\max}[N(Q - QM^{\perp T} P^{-1} M^\perp Q)N^T],$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue.

***Lemma** 3.2 (Projection lemma and parametrization set [13]):* Let matrices $M \in \mathbb{R}^{n\times m}$, and $Q = Q^T \in \mathbb{R}^{n\times n}$ be given. Then the following statements are equivalent.

(i) There exists a matrix X satisfying

$$MX + (MX)^T + Q < 0. \quad (3)$$

(ii) The following conditions holds:

$$M^\perp Q M^{\perp T} < 0 \ \text{ or } \ MM^T > 0.$$

Suppose the above statements hold, then all matrices X in statement (i) are given by

$$X = -\rho^{-1} M^T + \rho^{-1/2} L(\rho^{-1} MM^T - Q)^{1/2}, \quad (4)$$

where L is any matrix such that $\|L\| < 1$ and ρ is a positive scalar in the interval $(0, \rho_{\max})$. $\rho_{\max} = a^{-1}$ is given by solving the following LMI problem.

$$\text{Min}_{\{a,X\}} a$$

subject to

$$a > 0,$$
$$\begin{bmatrix} -aI & X \\ X^T & MX + (MX)^T + Q \end{bmatrix} < 0.$$

For a given inequality in the form of (3), Lemma 3.2 gives its solvability condition and the following parametrization set of all its solutions.

$$\mathcal{G}_{M,Q}(L, \rho) = \{X | X = -\rho^{-1} M^T + \rho^{-1/2} L$$
$$(\rho^{-1} MM^T - Q)^{1/2}, \|L\| < 1, \rho \in (0, \rho_{\max})\}. \quad (5)$$

IV. Necessary existence conditions

For the basic case of FTCS in (1)-(2), suppose the state feedback law when $\eta(t)=i$ is represented by $u_i(t)=K_i x(t), i\in S_2=\{0,1\}$. By Lemma 2.1, (K_0,K_1) stabilizes the FTCS in the sense of EMS stability if and only if there exist positive definite matrices $P_{ik}, i\in S_2, k\in S_1$, such that the following inequalities hold simultaneously.

$$\begin{aligned} P_{00}B_0K_0+(P_{00}B_0K_0)^T+Q_{00} &< 0, & (6)\\ P_{10}B_0K_1+(P_{10}B_0K_1)^T+Q_{10} &< 0, & (7)\\ P_{01}B_1K_0+(P_{01}B_1K_0)^T+Q_{01} &< 0, & (8)\\ P_{11}B_1K_1+(P_{11}B_1K_1)^T+Q_{11} &< 0, & (9)\end{aligned}$$

where $Q_{ik}, i\in S_2, k\in S_1$, is defined as follows.

$$Q_{ik}=(A_k-0.5\beta^k_{i(1-i)}-0.5\alpha_{k(1-k)})^T P_{ik}+P_{ik}(A_k-0.5\beta^k_{i(1-i)}-0.5\alpha_{k(1-k)})+\beta^k_{(1-i)k}P_{i(1-k)}+\alpha_{k(1-k)}P_{i(1-k)}. \quad (10)$$

The set of all stabilizing controllers can be captured naturally by posing a matrix inequality problem to solve (6)-(9) for (K_0,K_1). In other words, one can solve (6) and (8) simultaneously for K_0 and (7) and (9) for K_1.

***Lemma** 4.1:* For the basic case of FTCS in (1)-(2), assume that B_0 and B_1 are row rank deficient, then there exists a stabilizing state feedback law in the sense of EMS stability only if there exist positive-definite matrices P_{ik}, $k\in S_1=\{0,1\}$, $i\in S_2=\{0,1\}$, such that

$$\begin{aligned} (P_{00}B_0)^\perp Q_{00}(P_{00}B_0)^{\perp T} &< 0, & (11)\\ (P_{10}B_0)^\perp Q_{10}(P_{10}B_0)^{\perp T} &< 0, & (12)\\ (P_{01}B_1)^\perp Q_{01}(P_{01}B_1)^{\perp T} &< 0, & (13)\\ (P_{11}B_1)^\perp Q_{11}(P_{11}B_1)^{\perp T} &< 0, & (14)\end{aligned}$$

where, for $k\in S_1=\{0,1\},\ i\in S_2=\{0,1\}, Q_{ik}$ is defined in (10). If B_0 has full row rank, (11) and (12) are removed from the conditions; if B_1 has full row rank, (13) and (14) are removed.

Proof: Based on Lemma 3.2, each one of the inequalities (6)-(9) has solution if and only if the the corresponding condition in (11)-(14) holds. Considering the (6)-(9) must hold simultaneously for system stability, (11)-(14) are only necessary conditions. If B_0 has full row rank, (6) and (7) always have feasible solutions for any P_{00}, P_{10}, Q_{00} and Q_{10} so (11) and (12) are removed; similarly, if B_1 has full row rank, (13) and (14) are removed. ■

***Theorem** 4.1:* For the basic case of FTCS in (1)-(2), if B_0 and B_1 are row rank deficient and all the transition rates of the fault and FDI process are nonzero, there exist stabilizing state feedback controllers in the sense of EMS stability only if there exist positive-definite matrices P_{ik}, positive scalars $\mu_{ik}, k\in S_1=\{0,1\}, i\in S_2=\{0,1\}$, such that the following inequalities hold.

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^T+\bar{A}_{00}P_{00}^{-1}-\mu_{00}B_0B_0^T & P_{00}^{-1} & P_{00}^{-1}\\ P_{00}^{-1} & -\frac{1}{\beta^0_{01}}P_{10}^{-1} & 0\\ P_{00}^{-1} & 0 & -\alpha_{01}^{-1}P_{01}^{-1}\end{bmatrix} < 0, \quad (15)$$

$$\begin{bmatrix} P_{10}^{-1}\bar{A}_{10}^T+\bar{A}_{10}P_{10}^{-1}-\mu_{10}B_0B_0^T & P_{10}^{-1} & P_{10}^{-1}\\ P_{10}^{-1} & -\frac{1}{\beta^0_{10}}P_{00}^{-1} & 0\\ P_{10}^{-1} & 0 & -\alpha_{01}^{-1}P_{11}^{-1}\end{bmatrix} < 0, \quad (16)$$

$$\begin{bmatrix} P_{01}^{-1}\bar{A}_{01}^T+\bar{A}_{01}P_{01}^{-1}-\mu_{01}B_1B_1^T & P_{01}^{-1} & P_{01}^{-1}\\ P_{01}^{-1} & -\frac{1}{\beta^1_{01}}P_{11}^{-1} & 0\\ P_{01}^{-1} & 0 & -\alpha_{10}^{-1}P_{00}^{-1}\end{bmatrix} < 0, \quad (17)$$

$$\begin{bmatrix} P_{11}^{-1}\bar{A}_{11}^T+\bar{A}_{11}P_{11}^{-1}-\mu_{11}B_1B_1^T & P_{11}^{-1} & P_{11}^{-1}\\ P_{11}^{-1} & -\frac{1}{\beta^1_{10}}P_{01}^{-1} & 0\\ P_{11}^{-1} & 0 & -\alpha_{10}^{-1}P_{10}^{-1}\end{bmatrix} < 0, \quad (18)$$

where $\bar{A}_{00}=A_0-0.5\beta^0_{01}-0.5\alpha_{01}$, $\bar{A}_{10}=A_0-0.5\beta^0_{10}-0.5\alpha_{01}$, $\bar{A}_{01}=A_1-0.5\beta^1_{01}-0.5\alpha_{10}$, $\bar{A}_{11}=A_1-0.5\beta^1_{10}-0.5\alpha_{10}$. In case that B_0 has full row rank, (15) and (16) are removed from the conditions; if B_1 has full row rank, (17) and (18) are removed. If some transition rates are zero, the corresponding rows and columns containing those transition rates are removed from the above matrices.

Proof: Take (11) as an example and the derivations are similar for the other three inequalities. As $P_{00}>0$ and $(P_{00}B_0)^\perp((P_{00}B_0)^\perp)^T>0$, both $(P_{00}B_0)^\perp$ and $(P_{00}B_0)^\perp P_{00}$ have full row rank. Considering $(P_{00}B_0)^\perp P_{00}B_0=0$ and $(P_{00}B_0)^\perp P_{00}=B_0^\perp$, we have

$$(P_{00}B_0)^\perp = B_0^\perp P_{00}^{-1}.$$

So (11) is equivalent to

$$B_0^\perp P_{00}^{-1}Q_{00}P_{00}^{-1}B_0^{\perp T}<0.$$

Substitute Q_{00} and denote $\bar{A}_{00}=A_0-0.5\beta^0_{01}-0.5\alpha_{01}$, we have

$$B_0^\perp(P_{00}^{-1}\bar{A}_{00}^T+\bar{A}_{00}P_{00}^{-1}+\beta^0_{01}P_{00}^{-1}P_{10}P_{00}^{-1}+\alpha_{01}P_{00}^{-1}P_{01}P_{00}^{-1})B_0^{\perp T}<0.$$

By Lemma 3.1, this inequality is equivalent to

$$P_{00}^{-1}\bar{A}_{00}^T+\bar{A}_{00}P_{00}^{-1}+\beta^0_{01}P_{00}^{-1}P_{10}P_{00}^{-1}+\alpha_{01}P_{00}^{-1}P_{01}P_{00}^{-1}<\mu_{00}B_0B_0^T,\quad \mu_{00}\in\mathbb{R}. \quad (19)$$

Pre- and post-multiply P_{00}, we have

$$\bar{A}_{00}^TP_{00}+P_{00}\bar{A}_{00}+\beta^0_{01}P_{10}+\alpha_{01}P_{01}<\mu_{00}P_{00}B_0B_0^TP_{00}. \quad (20)$$

According to Lemma 3.1, all feasible μ_{00} are given by $\mu_{00}>\mu_{00\min}$, where $\mu_{00\min}$ can be calculated by the parameters in the inequality. Therefore, if the feasible set of μ_{00} is non-empty, there must be a feasible $\mu_{00}>0$. Furthermore, we only need to consider the positive case of μ_{00} to obtain all the feasible P_{ij} due to the following reasons.

Suppose for any two feasible values of μ_{00}, $\mu_1\le 0$ and $\mu_2>0$, all the corresponding feasible solutions of P_{ij} in (19), $i,j\in\{0,1\}$, are denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. For every element $P_{ij}\in\mathcal{P}_1$, $i,j\in\{0,1\}$, (19) holds for this P_{ij} and μ_1. Again, based on Lemma 3.1, this element P_{ij}, $i,j\in$

$\{0,1\}$, is also feasible for (19) corresponding to μ_2 as $\mu_2 > \mu_1$ and thereby belongs to $\mathcal{P}_2$. Therefore, $\mathcal{P}_1 \subseteq \mathcal{P}_2$, which means the feasible solution of P_{ij}, $i,j \in \{0,1\}$, for (19) when $\mu \leq 0$ is a subset of those when $\mu > 0$ and we only need to consider this positive case.

Suppose the transition rates $\beta_{01}^0 > 0$ and $\alpha_{01} > 0$. By Schur's complement lemma, (19) is equivalent to

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^T + \bar{A}_{00}P_{00}^{-1} - \mu_{00}B_0B_0^T & P_{00}^{-1} & P_{00}^{-1} \\ P_{00}^{-1} & -\frac{1}{\beta_{01}^0}P_{10}^{-1} & 0 \\ P_{00}^{-1} & 0 & -\alpha_{01}^{-1}P_{01}^{-1} \end{bmatrix} < 0. \quad (21)$$

If some transition rate is zero, the corresponding term in (19) is removed and so are the corresponding row and column in (21). For example, if $\alpha_{01} = 0$, (21) becomes

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^T + \bar{A}_{00}P_{00}^{-1} - \mu_{00}B_0B_0^T & P_{00}^{-1} \\ P_{00}^{-1} & -\frac{1}{\beta_{01}^0}P_{10}^{-1} \end{bmatrix} < 0.$$

Similarly, (12)-(14) can also be converted to LMI that are convex in $P_{00}^{-1}, P_{01}^{-1}, P_{10}^{-1}, P_{11}^{-1}, \mu_{00}, \mu_{01}, \mu_{10}$ and μ_{11}. ■

The above results are for the basic case of FTCS. For the multiple fault mode cases, the above procedure and results are readily modified. For example, if $S_1 = S_2 = \{0,1,2\}$, to ensure stochastic stability, there are 9 inequalities in the necessary conditions of Theorem 4.1 and a typical one is given in (22).

The necessary conditions may hold for infinite sets of $P_{ij}, i \in S_2, j \in S_1$. For each set of P_{ij}, there may exist a corresponding set of stabilizing controllers. The freedom of P_{ij} can be exploited by the connection of the necessary existence conditions with the LQR problem, presented in the next section.

V. Connection with LQR problem

The necessary existence conditions in section IV have some connections with the LQR problem of JLS. To see this relationship, convert the FTCS model into the form of JLS by representing the behaviors of two Markov processes into one, called the integrated Markov process $\phi(t)$ [15]; then, compare the ARE of LQR problem in this JLS form with the results in section IV.

For the basic case of FTCS, $S_1 = \{0,1\}$ and $S_2 = \{0,1\}$ in the above model, the augmented state space of the integrated Markov process is $S_3 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}$, where the first element represents the FDI mode in S_2 and the second the fault mode in S_1. Suppose that the generator matrix of $\phi(t)$ is represented by $G = [\gamma_{(ij)(kl)}]_{4\times 4}$, then we have:

$$\gamma_{(ij)(kl)} = \begin{cases} \alpha_{ii} + \beta_{jj}^i, & i = k, j = l; \\ \beta_{jl}^i, & i = k, j \neq l; \\ \alpha_{ik}, & i \neq k, j = l; \\ 0, & i \neq k, j \neq l. \end{cases} \quad (23)$$

For the basic case, the generator matrix G is given below.

$$\begin{bmatrix} -(\alpha_{01} + \beta_{01}^0) & \alpha_{01} & \beta_{01}^0 & 0 \\ \alpha_{10} & -(\alpha_{10} + \beta_{01}^1) & 0 & \beta_{01}^1 \\ \beta_{10}^0 & 0 & -(\alpha_{01} + \beta_{10}^0) & \alpha_{01} \\ 0 & \alpha_{10} & \beta_{10}^1 & -(\alpha_{10} + \beta_{10}^1) \end{bmatrix}$$

The FTCS model becomes a standard JLS model if we substitute $\phi(t)$ into (1)-(2) and let $w(t) = 0$.

$$\dot{x}(t) = A(\phi(t))x(t) + B(\phi(t))u(\phi(t),t), \quad (24)$$
$$y(t) = C(\phi(t))x(t) + D(\phi(t))u(\phi(t),t). \quad (25)$$

The infinite time LQR problem aims to find a state feedback controller to minimize the following objective.

$$J(t_0, x(t_0), u(t)) = E\{\int_{t_0}^{\infty} [x^T(t)S(\phi(t))x(t) + u^T(\phi(t),t)R(\phi(t))u(\phi(t),t)]dt | x(t_0), \phi(t_0)\}. \quad (26)$$

This problem was solved by the Theorem 5 in [16], which stated that the optimal steady state control is

$$u_i(t) = -R_i^{-1}B_i^T P_i x(t), i \in S_3,$$

where $u_i(t)$ denotes $u(\phi(t),t)$ when $\phi(t) = i$. Here $i = 1 \sim 4$ is the index of the elements of S_3. P_i is the set of positive definite solutions of the coupled ARE

$$A_i^T P_i + P_i A_i - P_i A_i - P_i B_i R_i^{-1} B_i^T P_i - \lambda_i P_i + \sum_{j \in S_3, j \neq i} \lambda_{ij} P_j + Q_i = 0, \quad i \in S_3. \quad (27)$$

In (27), the diagonal transition rates of $\phi(t)$ is represented by λ_i and off-diagonal λ_{ij}; the system matrices are represented by A_i and B_i corresponding to a particular state of $\phi(t)$.

For notational consistency, denote the elements of S_3 as (i,j), $i \in S_2, j \in S_1$. When $\phi(t) = (i,j)$, suppose the weighting matrices for state and control input are S_{ij} and R_{ij} respectively and substitute the system parameters into (27) to obtain 4 coupled ARE. Take the following ARE for state $\phi(t) = (0,0)$ as an example and use the same notation P_{ij} for the variables of ARE for $\phi(t) = (i,j)$.

$$A_0^T P_{00} + P_{00}A_0 - P_{00}B_0R_{00}^{-1}B_0^T P_{00} - (\alpha_{01} + \beta_{01}^0)P_{00} + \beta_{01}^0 P_{01} + \alpha_{01}P_{10} + Q_{00} = 0. \quad (28)$$

Use $\bar{A}_{00} = A_0 - 0.5\beta_{01}^0 - 0.5\alpha_{01}$ defined in Theorem 4.1 to simplify this equation, we have

$$\bar{A}_{00}^T P_{00} + P_{00}\bar{A}_{00} + \beta_{01}^0 P_{01} + \alpha_{01}P_{10} + S_{00} = P_{00}B_0R_{00}^{-1}B_0^T P_{00}. \quad (29)$$

Compare (29) with (20), these two equations are actually equivalent if we let R_{00} be the scalar μ_{00}^{-1} and add a slack variable $S_{00} > 0$ to convert the inequality (20) into equality.

For this JLS model, it has 4 states in the integrated Markov process, hence there are correspondingly 4 controllers whose static state feedback gains are given by

$$K_{ij} = -R_{ij}^{-1}B_i^T P_{ij}, i \in S_2, j \in S_1. \quad (30)$$

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^T P_{00} P_{00}^{-1}\bar{A}_{00} P_{00}^{-1} - \mu_{00} B_0 B_0^T & P_{00}^{-1} & P_{00}^{-1} & P_{00}^{-1} & P_{00}^{-1} \\ P_{00}^{-1} & -\frac{1}{\beta_{01}^0} P_{10}^{-1} & 0 & 0 & 0 \\ P_{00}^{-1} & 0 & -\frac{1}{\beta_{02}^0} P_{10}^{-1} & 0 & 0 \\ P_{00}^{-1} & 0 & 0 & \frac{1}{\alpha_{01}} P_{01}^{-1} & 0 \\ P_{00}^{-1} & 0 & 0 & 0 & \frac{1}{\alpha_{02}} P_{02}^{-1} \end{bmatrix} < 0. \quad (22)$$

If using 4 controllers for the FTCS in the basic case, one typical subset of controllers are given by [20]

$$K_{ij} = -\rho_{ij} B_i^T P_{ij}, i \in S_2, j \in S_1, \quad (31)$$

where $\rho_{ij} > 0$ is a scalar to make $\rho_{ij} P_{ij} B_i B_i P_{ij} - Q_{ij} > 0$. Obviously, ρ_{ij} may take the value of μ_{ij} because we have

$$\mu_{ij} P_{ij} B_i B_i P_{ij} > Q_{ij}, i \in S_2, j \in S_1. \quad (32)$$

To see this inequality, substitute the definition of Q_{ij} in (10) and let $i = 0$ and $j = 0$, then (32) becomes (20), which has been proved in Theorem 4.1. When i and j take other values, (32) holds similarly. Thus, (31) and (30) are equivalent and the central controller in (31) matches the LQR controller in (30). When assuming the controller K only depends on the FDI modes $j \in S_2$, for the basic case, only two controllers, K_0 and K_1, need to be designed. Each controller is required to satisfy two inequalities simultaneously and the parametrization set of FTCS can be considered to be around the LQR optimal controller.

In summary, the necessary existence conditions of the parametrization set, stated in Theorem 4.1, are equivalent to the ARE of LQR problem. One particular feasible P_{ij}, $i \in S_2, j \in S_1$, represents a particular solution to a LQR problem corresponding to some specific weighting matrix. We may use this connection to select appropriate P_{ij} and generate controllers for this P_{ij}.

VI. CONTROLLER GENERATION

The stabilizing static state feedback controllers of FTCS in the sense of EMS stability are the common solutions of algebraic inequalities (11)-(14). The necessary existence condition has been established in the form of LMI for P_{ik}, $i \in S_2$, $k \in S_1$. We have also shown how to select P_{ik} in section V. For the selected P_{ik}, if the controllers exist, a set of stabilizing controllers can be generated using a parameterization and testing procedure.

Take K_0 as an example. If K_0 satisfies (6) and (8), it is a stabilizing controller. Denote the set of common solutions of (6) and (8) as S. Considering that $W \triangleq \mathcal{G}_{P_{00}B_0,Q_{00}}(L,\rho)$ is the parameterization set of all the solutions of (6), S is a subset of W.

Let the free parameters L and ρ of W be uniformly distributed random variables and the elements of W can be generated by samples of L and ρ. The elements in S are those of G which satisfy (8).

When S is nonempty, the following probability is positive.

$$\Pr\{K_0 \in S | K_0 \in W\} = \Pr\{K_0 \text{ satisfies (8)} | K_0 \in W\}.$$

Define the indicator function

$$I(L,\rho) = \begin{cases} 1, & K_0 \in S \text{ given } K_0 \in W \text{for some } (L,\rho); \\ 0, & \text{otherwise,} \end{cases}$$

then $\Pr\{I(L,\rho) = 1\} = \Pr\{K_0 \in S | K_0 \in W\}$. According to the Chernoff's bound [6], when generating $N \geq \frac{\ln(2/\delta)}{2\epsilon^2}$ samples for $\delta > 0$ and $\epsilon > 0$, the following statistic gives an estimate of this probability.

$$\hat{P}_N = \frac{\sum_{i=1}^N I(L_i,\rho_i)}{N}.$$

Furthermore, this estimate satisfies

$$\Pr\{|\Pr\{I(L,\rho) = 1\} - \hat{P}_N| \leq \epsilon\} \geq 1 - \delta.$$

Suppose that ϵ and δ are so small that we can use the estimate $\hat{P}_N$ as the true probability of $\Pr\{K_0 \in S | K_0 \in W\}$. If $\hat{P}_N > 0$, the stabilizing controller K_0 exists and can be generated by testing (8) for the elements of W. There remains the question of deciding the sample size, i.e., the number of elements of W to be tested. For instance, suppose we want to obtain N_2 stabilizing controllers K_0, how many elements of W need to be tested?

Define a new random variable $Y = I(L,\rho)$, then $\Pr\{Y = 1\} = \hat{P}_N$ and $\Pr\{Y = 0\} = 1 - \hat{P}_N$. For N_1 identically independently distributed samples L_i, ρ_i, denote $Y_i = I(L_i, \rho_i)$, $i = 1 \sim N_1$. $\sum_{i=1}^{N_1} Y_i$ is the number of $K_0 \in S$ and subject to the Binomial distribution.

$$\Pr\{\sum_{i=1}^{N_1} Y_i \geq N_2\} = \sum_{k=N_2}^{N_1} \binom{N_1}{k} \hat{P}_{N_1}^k (1 - \hat{P}_{N_1})^{N_1-k}. \quad (33)$$

Set a confidence level δ_3 and select N_1 to make $\Pr\{\sum_{i=1}^{N_1} Y_i \geq N_2\} \geq 1 - \delta_3$. When testing N_1 samples of L_i, ρ_i, with probability $1 - \delta_3$, we get N_2 samples of K_0^i, the common solutions of (6) and (8). The procedures of generating N_2 controllers are summarized as follows.

1) For K_0, estimate $\Pr\{K_0 \in S | K_0 \in W\}$ for some small parameters ϵ_2 and δ_2. If the estimated probability is 0, no stabilizing controller exists and stop.
2) For a small confidence level δ_3, select N_3 to make the following inequality hold.
$$\sum_{k=N_2}^{N_1} \binom{N_3}{k} \hat{P}_N^k (1 - \hat{P}_N)^{N_3-k} \geq 1 - \delta_3.$$
3) Generate N_1 samples in set $G = \mathcal{G}_{P_{00}B_0,Q_{00}}(L,\rho)$. Test (8) for each sample and record those that satisfies (8) which are stabilizing controllers in set S.
4) If less than N_1 members in S are generated, repeat 3).
5) For K_1, follow similar steps of 1)-4) to generate the controller samples.

VII. AN ILLUSTRATIVE EXAMPLE

Consider a system in the form of (1)-(2) with the following parameters.

$$A_0 = \begin{bmatrix} 7 & 3 \\ 6 & 18 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 9 \\ 14 & 5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 13 & 12 \\ 5 & 14 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 10 & 4 \\ 8 & 15 \end{bmatrix}, G = \begin{bmatrix} -0.5 & 0.5 \\ 0 & 0 \end{bmatrix},$$

$$C_0 = [3 \quad 8], \quad C_1 = [13 \quad 10].$$

$$H^0 = \begin{bmatrix} -0.02 & 0.02 \\ 3.90 & -3.90 \end{bmatrix}, \quad H^1 = \begin{bmatrix} -2.99 & 2.99 \\ 0.05 & -0.05 \end{bmatrix}.$$

Construct the conditions of Theorem 4.1 and solve them for P_{ij} as follows.

$$P_{00} = \begin{bmatrix} 7.07 & -8.55 \\ -8.55 & 29.48 \end{bmatrix}, P_{01} = \begin{bmatrix} 22.63 & -1.46 \\ -1.46 & 18.05 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 73.00 & -90.16 \\ -90.16 & 300.710 \end{bmatrix}, P_{11} = \begin{bmatrix} 16.4347 & -2.6097 \\ -2.6097 & 5.5386 \end{bmatrix}.$$

These P_{ij} corresponds to minimize the LQR objective (26) with state weighting matrices S_{ij} and control weighting $R_{ij} = \mu_{ij}^{-1}$ as follows.

$$S_{00} = \begin{bmatrix} 42.01 & -108.88 \\ -108.88 & 347.90 \end{bmatrix}, S_{01} = \begin{bmatrix} 162.96 & -117.19 \\ -117.19 & 364.11 \end{bmatrix},$$

$$S_{10} = \begin{bmatrix} 940 & -1936 \\ -1936 & 14887 \end{bmatrix}, S_{11} = \begin{bmatrix} 278.46 & -129.46 \\ -129.46 & 79.66 \end{bmatrix}.$$

$$R_{00} = 70.99, \quad R_{01} = 182.35, \quad R_{10} = 404.40, \quad R_{11} = 63.12.$$

The maximum value of positive ρ can be calculated as $\rho_{1\max} = 757$ and $\rho_{2\max} = 1340.4$ respectively. For fixed $P_{ij}, i, j \in \{0, 1\}$, the free parameters of the controllers are: $0 < \rho_i \leq \rho_{i\max}, \|L_i\|_2 < 1, i = 1, \sim 2$. Assume these parameters to be uniformly distributed in their bounded sets and 194 set of controllers are generated, one of which is given below. The state trajectory of the closed-loop system for one sample path of the Markov process is given in Fig. 1. Obviously, the closed loop system is stable for this sample path.

$$K_0 = \begin{bmatrix} -1.60 & -0.43 \\ 1.13 & -11.64 \end{bmatrix}, K_1 = \begin{bmatrix} -6.97 & -2.72 \\ 3.68 & -18.30 \end{bmatrix}.$$

VIII. CONCLUSION

A stabilizing controller parametrization result is presented in this paper which is derived based on the EMS stability of FTCS. The free parameters in the expression of the controllers are real matrices and scalars, which is easy for numerical implementation. The existence conditions are given in the form of LMI and they are shown to be equivalent to the ARE of LQR problem in JLS. So the central controller in the parametrization set is a optimal LQR controller of JLS. This method can be used with the randomized algorithms to design the approximately optimal controller of FTCS.

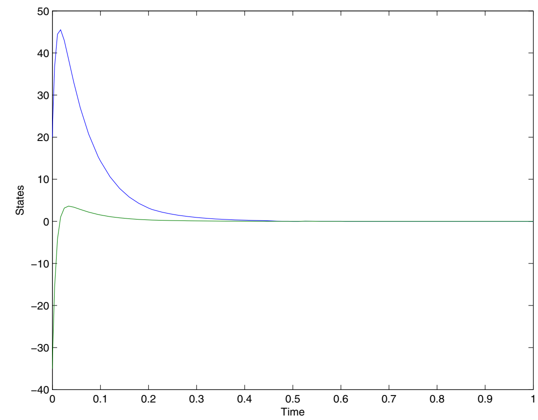

Fig. 1. State trajectory of the closed-loop system.

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