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Identification of Parameters in Neutral Functional Differential Equations with State-Dependent Delays

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***Abstract*— We introduce a parameter identification algorithm and establish its theoretical convergence on initial value problems governed by neutral functional differential equations with state-dependent delays. The discretization of the differential equation is based on an Euler-type approximation method using equations with piecewise constant arguments. Numerical examples are included.**

I. INTRODUCTION

In this paper, making use of a general framework for parameter identification in distributed parameter systems (see e.g., [1], [2], [3], [20], and the references therein), we study convergence properties of numerical schemes producing approximate solutions of parameter estimation problems for a class of neutral functional differential equations with state-dependent delays (SD-NFDEs) of the form

$$\dot{x}(t) = f(t, x(t), x(t - \sigma(t, x(t))), \dot{x}(t - \tau(t, x(t)))). \quad (1)$$

Existence, uniqueness and numerical approximations of solutions for such equations were considered in [4], [7], [8], [11] and [19] and the references therein. Parameter estimation problems were studied in [14] for equations of the form

$$\Big(x(t)+q(t)x(t-\tau(t, x(t)))\Big)' = f\Big(t, x(t), x(t-\sigma(t, x(t)))\Big) \quad (2)$$

(the so-called implicit neutral case). Note that (1) represents a more general class of SD-NFDEs than (2).

In this paper, extending and refining our earlier results (see [17] for retarded equations with state-dependent delays and [12]–[14] for NFDEs with constant, time- and state-dependent delays), we define a parameter estimation method using an approximation scheme based on equations with piecewise-constant arguments (EPCAs), show its theoretical convergence, and study its applicability on numerical examples.

The remaining part of the paper is organized as follows: In Section 2 we specify the class SD-NFDEs we will study and obtain some basic properties of the solutions. In Section 3 we briefly recall the general identification framework used in the current study. In Section 4 we introduce a simple EPCA-based numerical approximation scheme, and discuss the key step of the general identification procedure, namely, the convergence of the approximate problem under a certain double limiting process. Section 5 contains a few numerical examples.

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Note that EPCAs were used first in [9] to obtain numerical approximation schemes and to prove the convergence of the approximation method for linear delay and neutral differential equations with constant delays, and later in [10] and [15] for nonlinear delay and neutral equations with state-dependent delays.

II. A CLASS OF SD-NFDES

Consider the nonlinear SD-NFDE

$$\dot{x}(t) = f(t, x(t), x(t - \sigma(t, x(t), \lambda)), \dot{x}(t - \tau(t, x(t), \xi)), \theta), \quad (3)$$

for $t \geq 0$, and the associated initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \quad (4)$$

We assume throughout the paper that Λ, Ξ and Θ are normed linear spaces, where the norms are denoted by $|\cdot|_\Lambda$, $|\cdot|_\Xi$ and $|\cdot|_\Theta$, respectively.

A fixed norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. The Banach-space of continuous real-valued functions defined on $[-r, 0]$ with the norm $|\psi|_C = \max\{|\psi(s)| : s \in [-r, 0]\}$ will be denoted by C. The Banach-space of absolutely continuous functions on $[-r, 0]$ is denoted by $W^{1,\infty}$, where $|\psi|_{W^{1,\infty}} = \max\{|\psi|_C, \operatorname{ess\,sup}\{|\dot{\psi}(s)| : s \in [-r, 0]\}\}$.

We will assume

(H1) $f \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta; \mathbb{R}^n)$ is locally Lipschitz continuous in its first, second, third and fifth arguments, and globally Lipschitz continuous in its fourth argument, i.e., there exists $L_2 \geq 0$ and for every $M \geq 0$ there exists $L_1 = L_1(M) \geq 0$ such that $|f(t, x, y, z, \theta) - f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{\theta})| \leq L_1\Big(|t - \bar{t}| + |x - \bar{x}| + |y - \bar{y}| + |\theta - \bar{\theta}|_\Theta\Big) + L_2|z - \bar{z}|$, for $t, \bar{t} \in [0, T]$, $x, \bar{x}, y, \bar{y}, z, \bar{z} \in \mathbb{R}^n$, $\theta, \bar{\theta} \in \Theta$ such that $|x|$, $|\bar{x}|$, $|y|$, $|\bar{y}|$, $|\theta|_\Theta$, $|\bar{\theta}|_\Theta \leq M$;

(H2) $\tau, \sigma \in C([0, T] \times \mathbb{R}^n; \mathbb{R})$ are such that

(i) there exist $r > 0$ and $r_0 > 0$ such that

$$0 \leq \sigma(t, x, \lambda) \leq r \quad \text{and} \quad 0 < r_0 \leq \tau(t, x, \xi) \leq r$$

for $t \in [0, T]$, $x \in \mathbb{R}^n, \lambda \in \Lambda$, $\xi \in \Xi$;

(ii) σ and τ are locally Lipschitz continuous in their all arguments, i.e., for every $M \geq 0$ there exists constants $L_3 = L_3(M) \geq 0$ and $L_4 = L_4(M) \geq 0$ such that $|\sigma(t, x, \lambda) - \sigma(\bar{t}, \bar{x}, \bar{\lambda})| \leq L_3(|t - \bar{t}| + |x - \bar{x}| + |\lambda - \bar{\lambda}|_\Lambda)$ and $|\tau(t, x, \xi) - \tau(\bar{t}, \bar{x}, \bar{\xi})| \leq L_4(|t - \bar{t}| + |x - \bar{x}| + |\xi - \bar{\xi}|_\Xi)$, for $t, \bar{t} \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$,

$\lambda, \bar{\lambda} \in \Lambda$, $\xi, \bar{\xi} \in \Xi$ such that $|x|$, $|\bar{x}| \le M$, $|\lambda|_\Lambda$, $|\bar{\lambda}|_\Lambda \le M$, $|\xi|_\Xi$, $|\bar{\xi}|_\Xi \le M$;

(H3) $\varphi \in W^{1,\infty}$, (i.e., φ is Lipschitz continuous), and its derivative $\dot{\varphi}$ is Lipschitz continuous on $[-r, 0]$, i.e., there exists $L_5 \ge 0$ such that $|\dot{\varphi}(t) - \dot{\varphi}(\bar{t})| \le L_5|t - \bar{t}|$ for a.e. $t \in [-r, 0]$.

By a solution of IVP (3)-(4) on $[-r, \alpha]$ we mean a continuous function x, which is almost everywhere (a.e.) differentiable and satisfies (3) for a.e. $t \in [0, \alpha]$, and (4) for all $t \in [-r, 0]$.

In (3)-(4) we consider $\gamma = (\varphi, \lambda, \xi, \theta)$ as parameters of the IVP, and define $\Gamma = W^{1,\infty} \times \Lambda \times \Xi \times \Theta$ as the space of the parameters with the norm $|\gamma|_\Gamma = |\varphi|_{W^{1,\infty}} + |\lambda|_\Lambda + |\xi|_\Xi + |\theta|_\Theta$. We assume that these parameters (or some of them) are unknown, but values $(X_0, X_1, \ldots, X_l)$ of the solution, $x(t)$, are available via measurements at discrete time values $(t_0, t_1, \ldots, t_l)$. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion

$$J(\gamma) = \sum_{i=0}^{l} |x(t_i; \gamma) - X_i|^2,$$

where γ belongs to an admissible set Δ contained in the parameter space Γ. (Denote this problem by $\mathcal{P}$).

Since by (H2) (i) $t - \tau(t, x(t), \lambda) \le 0$ for $t \in [0, r_0]$, it follows that on this interval (3) is equivalent to the state-dependent delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \sigma(t, x(t), \lambda)), \dot{\varphi}(t - \tau(t, x(t), \xi)), \theta).$$

A result in [18] implies the existence of a unique solution of (3) on $[0, r_0]$ (or on a shorter interval). Repeating the argument successively on intervals $[kr_0, (k+1)r_0]$ $(k \in \mathbb{N})$ we get that there exists $\alpha > 0$ such that IVP (3)-(4) has a unique solution x on $[0, \alpha]$. Of course, if all Lipschitz constants in (H1)–(H2) are global, then it is easy to argue that the unique solution exists on the whole interval $[0, T]$. Lemma 1 shows that both x and $\dot{x}$ are Lipschitz continuous on $[0, \alpha]$.

Lemma 1: Assume (H1)–(H3), and let $\gamma = (\varphi, \lambda, \xi, \theta) \in \Gamma$ be fixed, and x be a solution of (3) corresponding to γ defined on a finite interval $[0, \alpha]$. Then there exist constants $M_2 = M_2(\alpha) > 0$ and $M_3 = M_3(\alpha) > 0$ such that

$$|x(t) - x(\bar{t})| \le M_2|t - \bar{t}|, \qquad t, \bar{t} \in [-r, \alpha],$$

and

$$|\dot{x}(t) - \dot{x}(\bar{t})| \le M_3|t - \bar{t}|, \qquad \text{a.e. } t, \bar{t} \in [-r, \alpha].$$

Proof: Let $M_1 = \max\{|x(t)| : t \in [0, \alpha]\}$, $L_i = L_i(M_1)$ $(i = 1, 3, 4)$ be the Lipschitz constants from (H1) and (H2), respectively. Assumption (H1) implies

$$\begin{aligned}
&|\dot{x}(t)| \\
&\quad \le |f(t, x(t), x(t - \sigma(t, x(t), \lambda)), \dot{x}(t - \tau(t, x(t), \xi)), \theta) \\
&\qquad - f(0, 0, 0, 0, \theta)| + |f(0, 0, 0, 0, \theta)| \\
&\quad \le L_1(t + |x(t)| + |x(t - \sigma(t, x(t), \lambda))| + |\theta|_\Theta) \\
&\qquad + L_2|\dot{x}(t - \tau(t, x(t), \xi))| + |f(0, 0, 0, 0, \theta)|
\end{aligned}$$

for a.e. $t \in [0, \alpha]$. Therefore there exists $K > 0$ such that

$$|\dot{x}(t)| \le K + L_2 \operatorname{ess\,sup}\{|\dot{x}(s)| : s \in [-r, t - r_0]\},$$

which, using (H3) and the method of steps, yields that there exists $M_2 > 0$ such that $|\dot{x}(t)| \le M_2$ for a.e. $t \in [0, \alpha]$, hence x is Lipschitz continuous. This proves the first part of this lemma.

To prove the second inequality, consider

$$\begin{aligned}
&|\dot{x}(t) - \dot{x}(\bar{t})| \\
&\quad = |f(t, x(t), x(t - \sigma(t, x(t), \lambda)), \dot{x}(t - \tau(t, x(t), \xi)), \theta) \\
&\qquad - f(\bar{t}, x(\bar{t}), x(\bar{t} - \sigma(\bar{t}, x(\bar{t}), \lambda)), \dot{x}(\bar{t} - \tau(\bar{t}, x(\bar{t}), \xi)), \theta)| \\
&\quad \le L_1(|t - \bar{t}| + |x(t) - x(\bar{t})| \\
&\qquad + |x(t - \sigma(t, x(t), \lambda)) - x(\bar{t} - \sigma(\bar{t}, x(\bar{t}), \lambda))|) \\
&\qquad + L_2|\dot{x}(t - \tau(t, x(t), \xi)) - \dot{x}(\bar{t} - \tau(\bar{t}, x(\bar{t}), \xi))| \\
&\quad \le L_1(|t - \bar{t}| + M_2|t - \bar{t}| \\
&\qquad + M_2|t - \bar{t}| + M_2|\sigma(t, x(t), \lambda) - \sigma(\bar{t}, x(\bar{t}), \lambda)|) \\
&\qquad + L_2|\dot{x}(t - \tau(t, x(t), \xi)) - \dot{x}(\bar{t} - \tau(\bar{t}, x(\bar{t}), \xi))|.
\end{aligned}$$

For $t \in [0, r_0]$ it follows $t - \tau(t, x(t), \xi)$ and $\bar{t} - \tau(\bar{t}, x(\bar{t}), \xi)$ belong to $[-r, 0]$, therefore, using (H3), we get

$$\begin{aligned}
|\dot{x}(t) - \dot{x}(\bar{t})| &\le K_0|t - \bar{t}| + L_2L_5|t - \bar{t}| \\
&\quad + L_2L_5|\tau(t, x(t), \xi) - \tau(\bar{t}, x(\bar{t}), \xi)| \\
&\le (K_0 + L_5K_1)|t - \bar{t}|, \quad \text{a.e. } t, \bar{t} \in [0, r_0],
\end{aligned}$$

where $K_0 = L_1(1 + 2M_2 + M_2L_3 + M_2^2L_3)$ and $K_1 = L_2(1 + L_4 + L_4M_2)$. Similarly, for a.e. $t, \bar{t} \in [r_0, 2r_0]$ we get

$$|\dot{x}(t) - \dot{x}(\bar{t})| \le (K_0 + K_0K_1 + L_5K_1^2)|t - \bar{t}|.$$

Therefore, repeating these steps finitely many times, we get that there exists $M_3 > 0$ satisfying the second part of the statement. ■

We will use the following elementary estimate in the sequel. In the proof of the lemma and throughout this paper $[\cdot]$ denotes the greatest integer function.

Lemma 2: Let $a, b \ge 0$, and $g : [0, \alpha] \to [0, \infty)$ and $u : [-r_0, \alpha] \to [0, \infty)$ be monotone increasing functions satisfying

$$u(t) \le a + g(t) + bu(t - r_0), \qquad t \in [0, \alpha]$$

and $u(0) \le a$. Then

$$u(t) \le (a + g(t))(1 + b + b^2 + \cdots + b^{m-1}) + b^m a, \quad t \in [0, \alpha],$$

where $m = [\alpha / r_0]$.

Proof: The assumptions imply

$$u(t) \le a + g(t) + bu(0) \le a + g(t) + ba, \qquad t \in [0, r_0].$$

Therefore, using the monotonicity of g and u, we get

$$\begin{aligned}
u(t) &\le a + g(t) + bu(r_0) \\
&\le a + g(t) + b(a + g(r_0) + ba) \\
&\le (a + g(t))(1 + b) + b^2a, \qquad t \in [r_0, 2r_0].
\end{aligned}$$

Then the lemma follows by induction. ■

III. A General Parameter Estimation Method

In this section we briefly recall a general method frequently used to identify parameters in various classes of differential equations (see, e.g., [1], [3], [20], and also [12]–[17]). We will apply this framework for our IVP (3)-(4) in the next sections.

The general method consists of the following steps:

Step 1) First take finite dimensional approximations of the parameters, γ_N, (i.e., $\gamma_N \in \Delta_N \subset \Gamma_N \subset \Gamma$, $\dim \Gamma_N < \infty$, $\gamma_N \to \gamma$ as $N \to \infty$).

Step 2) Consider a sequence of approximate IVPs corresponding to a discretization of IVP (3)-(4) for some fixed parameter $\gamma_N \in \Gamma_N$ with solutions $y_{M,N}(\cdot;\gamma_N)$ satisfying $y_{M,N}(t,\gamma_N) \to x(t,\gamma)$ as $M, N \to \infty$, uniformly on compact time intervals, and $\gamma_N \in \Delta_N$.

Step 3) Define the least square minimization problems $(\mathcal{P}_{M,N})$: for each $N, M = 1, 2, \ldots$, i.e., find $\gamma_{M,N} \in \Delta_N \subset \Gamma_N$, which minimizes the least squares fit-to-data criterion

$$J_{M,N}(\gamma_N) = \sum_{i=0}^{l} |y_{M,N}(t_i;\gamma_N) - X_i|^2, \qquad \gamma_N \in \Delta_N.$$

Often Δ_N is the projection of Δ to Γ_N, and we restrict our discussion to this case.

Step 4) Assuming that Δ is a compact subset of Γ, and the approximate solution, $y_{M,N}(t;\gamma_N)$, depends continuously on the parameter, γ_N, we get, that $J_{M,N}(\cdot)$ is continuous for each M, N. Hence the finite dimensional minimization problems, $\mathcal{P}_{M,N}$, have a solution, $\bar\gamma_{M,N}$. Since $\bar\gamma_{M,N} \in \Delta$, the sequence $\bar\gamma_{M,N}$ $(M, N = 1, 2, \ldots)$ has a convergent subsequence, say $\bar\gamma_{M_j,N_j}$, with limit $\bar\gamma \in \Gamma$.

Step 5) It follows from Step 2 that $J_{M_j,N_j}(\bar\gamma_{M_j,N_j}) \to J(\bar\gamma)$ as $j \to \infty$. Let $\gamma \in \Delta$ be fixed, and let $\gamma_N \to \gamma$ satisfying Step 1. Then, in particular, $\gamma_{N_j} \to \gamma$ as $j \to \infty$. Using that $\bar\gamma_{M_j,N_j}$ is a solution of $\mathcal{P}_{M_j,N_j}$, Step 2 implies

$$J(\bar\gamma) = \lim_{j\to\infty} J_{M_j,N_j}(\bar\gamma_{M_j,N_j}) \le \lim_{j\to\infty} J_{M_j,N_j}(\gamma_{N_j}) = J(\gamma),$$

therefore $\bar\gamma$ is the solution of the minimization problem $\mathcal{P}$.

In practice we take "large enough" N and M, and use the solution of $\mathcal{P}_{M,N}$ as an approximate solution of $\mathcal{P}$. Note that Step 4 and 5 yield that the limit of any convergent subsequence of $\gamma_{M,N}$ is a solution of $\mathcal{P}$ (with the same cost). It is possible that the minimizer of $J(\gamma)$ is not unique (see, e.g., Example 5.4 in [16]). For results on identifiability of parameters, i.e., the uniqueness of the parameter minimizing the cost function $J(\gamma)$ (for simpler classes of delay equations) is discussed e.g., in [2] and [21].

In our examples, we will use linear spline approximation to discretize the parameters φ, ξ and θ, in the case of nonconstant functions, in Step 1. In the next section we introduce a set of approximate IVPs corresponding to IVP (3)-(4) we use in Step 2, and show uniform convergence of the scheme, as required in Step 2 and Step 4, respectively.

IV. A Parameter Estimation Method and Its Theoretical Convergence

Throughout this section we will use the notation $[t]_h = [t/h]h$, where $[\cdot]$ is the greatest integer part function. The graph of the function $[t]_h$ can be seen in Figure 1.

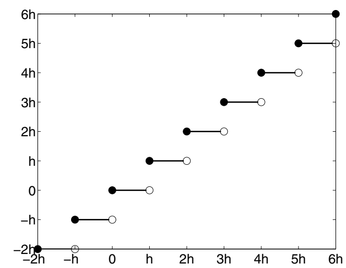

Fig. 1. Graph of $[t]_h$

It is easy to check that

$$t - h < [t]_h \le t, \qquad t \in \mathbb{R}, \quad h > 0, \tag{5}$$

therefore $[t]_h \to t$ as $h \to 0+$, uniformly in t. A fractional part of a real number x is denoted by $\{x\} = x - [x]$.

Motivated by our earlier works [12]–[17], we now consider the approximating equation

$$\begin{aligned}\dot y_{h,N}(t)& \\ = f\Big(&[t]_h, y_{h,N}([t]_h), y_{h,N}([t]_h - \sigma([t]_h, y_{h,N}([t]_h), \lambda_N)), \\ &\dot y_{h,N}([t]_h - \tau([t]_h, y_{h,N}([t]_h), \xi_N)), \theta_N\Big)\end{aligned} \tag{6}$$

for $t \in [0, T]$, with the initial condition

$$y_{h,N}(t) = \varphi_N(t), \qquad t \in [-r, 0]. \tag{7}$$

A solution of IVP (6)-(7) is a continuous function, satistying (6) for all $t \ge 0$ except at $t = kh$, $k = 0, 1, \ldots$. It is easy to see that $\dot y_{h,N}$ is constant on the intervals $[kh, (k+1)h)$, therefore $y_{h,N}$ is a piecewise-linear continuous function for $t \ge 0$. Integrating (6) from kh to t, where $t \in [kh, (k+1)h)$, and then taking a limit as $t \to (k+1)h-$ it is easy to obtain the following recursive formula for the computation of the solution of IVP (6)-(7): Let $a(k) = y_{h,N}(kh)$ and $b(k) = \dot y_{h,N}(kh-)$. Then

$$\begin{aligned}\sigma_k &= \sigma(kh, a(k), \lambda_N)\\ \tau_k &= \tau(kh, a(k), \xi_N)\\ u_k &= a\Big(k - \Big[\frac{\sigma_k}{h}\Big] - 1\Big)\Big\{\frac{\sigma_k}{h}\Big\}\\ &\quad + a\Big(k - \Big[\frac{\sigma_k}{h}\Big]\Big)\Big(1 - \Big\{\frac{\sigma_k}{h}\Big\}\Big)\\ v_k &= b\Big(k - \Big[\frac{\tau_k}{h}\Big] - 1\Big)\Big\{\frac{\tau_k}{h}\Big\}\\ &\quad + b\Big(k - \Big[\frac{\tau_k}{h}\Big]\Big)\Big(1 - \Big\{\frac{\tau_k}{h}\Big\}\Big)\\ b(k+1) &= f(kh, a(k), u_k, v_k, \theta_N)\\ a(k+1) &= a(k) + hb(k+1)\end{aligned}$$

for $k = 0, 1, \ldots, [T/h]$ will define the sequence $a(k)$ explicitly.

Next we prove our main result about the convergence of this approximation method.

Theorem 1: Assume (H1)–(H3), and that IVP (3)-(4) has a unique Lipschitz continuous solution, $x(t)$, on $[-r,T]$. Let $\gamma_N=(\varphi_N,\lambda_N,\xi_N,\theta_N)\in\Gamma$ be such that $|\gamma-\gamma_N|_\Gamma\to 0$ as $N\to\infty$. Then the solution, $y_{h,N}$, of IVP (6)-(7) converges uniformly on $[0,T]$ to the solution, x, of IVP (3)-(4) as $h\to 0+$ and $N\to\infty$, i.e.,

$$\lim_{\substack{h\to 0+\\ N\to\infty}} \max_{0\le t\le T} |x(t)-y_{h,N}(t)|=0. \tag{8}$$

Proof: Define $M_1\equiv\max\{|\lambda|_\Lambda,|\xi|_\Xi,|\theta|_\Theta,\max\{|x(t)|: t\in[-r,T]\}\}+\varepsilon$ for some $\varepsilon>0$. We assume, without loss of generality, that N is large enough that $|\varphi_N-\varphi|_C<\varepsilon$, $|\lambda_N-\lambda|_\Lambda<\varepsilon$, $|\xi_N-\xi|_\Xi<\varepsilon$, and $|\theta_N-\theta|_\Theta<\varepsilon$. Let $0<\alpha_{h,N}\le T$ be the largest number such that $|y_{h,N}(t)|<M_1$ for $t\in[0,\alpha_{h,N})$. ($\alpha_{h,N}$ is well-defined since $|\varphi_N(0)|<M_1$ by our assumptions.) Substracting (6) from (3) and applying (H1) with $L_1=L_1(M_1)$ we get

$$\begin{aligned}
&|\dot x(t)-\dot y_{h,N}(t)|\\
&\quad= \Big|f\Big(t,x(t),x(t-\sigma(t,x(t),\lambda)),\\
&\qquad\quad \dot x(t-\tau(t,x(t),\xi)),\theta\Big)-f\Big([t]_h,y_{h,N}([t]_h),\\
&\qquad\quad y_{h,N}([t]_h-\sigma([t]_h,y_{h,N}([t]_h),\lambda_N),\\
&\qquad\quad \dot y_{h,N}([t]_h-\tau([t]_h,y_{h,N}([t]_h),\xi_N)),\theta_N\Big)\Big|\\
&\quad\le L_1\Big(h+|x(t)-y_{h,N}([t]_h)|+|x(t-\sigma(t,x(t),\lambda))\\
&\qquad\quad -y_{h,N}([t]_h-\sigma([t]_h,y_{h,N}([t]_h),\lambda_N))|\\
&\qquad\quad +|\theta-\theta_N|_\theta\Big)+L_2|\dot x(t-\tau(t,x(t),\xi))\\
&\qquad\quad -\dot y_{h,N}([t]_h-\tau([t]_h,y_{h,N}([t]_h),\xi_N))|.
\end{aligned}\tag{9}$$

Next we will estimate the terms on the right-hand side of (9) separately. Introduce the notation $w_{h,N}(t):=\max\{|x(s)-y_{h,N}(s)|:\ -r\le s\le t\}$. Then (5) and Lemma 1 yield

$$\begin{aligned}
&|x(t)-y_{h,N}([t]_h)|\\
&\quad\le |x(t)-x([t]_h)|+|x([t]_h)-y_{h,N}([t]_h)|\\
&\quad\le M_2h+w_{h,N}(t).
\end{aligned}\tag{10}$$

We have assumed that $|\lambda-\lambda_N|_\Lambda<\varepsilon$ and $|\xi-\xi_N|_\Xi<\varepsilon$, therefore $|\lambda_N|_\Lambda\le M_1$ and $|\xi_N|_\Xi\le M_1$. Let $L_3=L_3(M_1)$ and $L_4=L_4(M_1)$ the Lipschitz constants from (H2). Hence (H2) (ii), (5), Lemma 1 and the monotonicity of $w_{h,N}$ yield

$$\begin{aligned}
&|x(t-\sigma(t,x(t),\lambda))-y_{h,N}([t]_h-\sigma([t]_h,y_{h,N}([t]),\lambda_N))|\\
&\quad\le |x(t-\sigma(t,x(t),\lambda))-x([t]_h-\sigma([t]_h,y_{h,N}([t]),\lambda_N))|\\
&\qquad +|x([t]_h-\sigma([t]_h,y_{h,N}([t]),\lambda_N))\\
&\qquad\quad -y_{h,N}([t]_h-\sigma([t]_h,y_{h,N}([t]),\lambda_N))|\\
&\quad\le M_2\Big(h+|\sigma(t,x(t),\lambda)-\sigma([t]_h,y_{h,N}([t]_h),\lambda_N)|\Big)\\
&\qquad +w_{h,N}([t]_h-\sigma([t]_h,y_{h,N}([t]),\lambda_N))\\
&\quad\le M_2\Big(h+L_3h+L_3|x(t)-y_{h,N}([t]_h)|\\
&\qquad +L_3|\lambda-\lambda_N|_\Lambda\Big)+w_{h,N}(t)\\
&\quad\le M_2\Big((1+L_3+L_3M_2)h+L_3w_{h,N}(t)\\
&\qquad +L_3|\lambda-\lambda_N|_\Lambda\Big)+w_{h,N}(t)
\end{aligned}\tag{11}$$

for $t\in[0,\alpha_{h,N}]$.

Introduce $z_{h,N}(t)=\operatorname{ess\,sup}\{|\dot x(s)-\dot y_{h,N}(s)|:\ s\in[-r,t]\}$. Then, similarly to (11), it is easy to obtain

$$\begin{aligned}
&|\dot x(t-\tau(t,x(t),\xi))-\dot y_{h,N}([t]_h-\tau([t]_h,y_{h,N}([t]),\xi_N))|\\
&\quad\le M_3\Big(h+L_4h+L_4M_2h+L_4w_{h,N}(t)+L_4|\xi-\xi_N|_\Xi\Big)\\
&\qquad +z_{h,N}(t-r_0)
\end{aligned}\tag{12}$$

for a.e. $t\in[0,\alpha_{h,N}]$. Here we also used that $r_0\le\tau([t]_h,y_{h,N}([t]_h),\xi_N)$ for $t\in[0,\alpha_{h,N}]$ by (H2), and $z_{h,N}$ is monotone increasing.

Combining (9), (10), (11) and (12) we get

$$|\dot x(t)-\dot y_{h,N}([t])|\le g_{h,N}+K_1w_{h,N}(t)+L_2z_{h,N}(t-r_0) \tag{13}$$

for a.e. $t\in[0,\alpha]$, where $g_{h,N}=L_1(1+2M_2+M_2L_3+M_2^2L_3)h+L_2M_3(1+L_4+L_4M_4)h+L_1|\theta-\theta_N|_\Theta+L_1M_2L_3|\lambda-\lambda_N|_\Lambda+L_2M_3L_4|\xi-\xi_N|_\Xi$ and $K_1=L_1(2+M_2L_3)+L_2M_3L_4$. Since $w_{h,N}$ and $z_{h,N}$ are monotone increasing, (13) implies

$$|\dot x(s)-\dot y_{h,N}([s])|\le g_{h,N}+K_1w_{h,N}(t)+L_2z_{h,N}(t-r_0)$$

for a.e. $0\le s\le t\le\alpha_{h,N}$. On the other hand $|\dot x(s)-\dot y_{h,N}([s])|\le|\varphi-\varphi_N|_{W^{1,\infty}}$ for a.e. $s\in[-r,0]$. Therefore

$$z_{h,N}(t)\le g_{h,N}+|\varphi-\varphi_N|_{W^{1,\infty}}+K_1w_{h,N}(t)+L_2z_{h,N}(t-r_0)$$

holds for all $t\in[0,\alpha_{h,N}]$. Then Lemma 2 implies

$$z_{h,N}(t)\le K_2(g_{h,N}+|\varphi-\varphi_N|_{W^{1,\infty}})+K_3w_{h,N}(t), \tag{14}$$

for $t\in[0,\alpha_{h,N}]$, where $K_2=1+L_2+\cdots+L_2^m$, $K_3=K_1(1+L_2+\cdots+L_2^{m-1})+L_2^m$ and $m=[\alpha/r_0]$.

Integrating (3) and (6) from 0 to t and substracting the two equations, and using estimates (9), (10), (11), (12) and (14) we get for $t\in[0,\alpha_{h,N}]$

$$\begin{aligned}
&|x(t)-y_{h,N}(t)|\\
&\quad\le |\varphi(0)-\varphi_N(0)|\\
&\qquad +\int_0^t\Big|f\Big(s,x(s),x(s-\sigma(s,x(s),\lambda)),\\
&\qquad\quad \dot x(s-\tau(s,x(s),\xi)),\theta\Big)-f\Big([s]_h,y_{h,N}([s]_h),\\
&\qquad\quad y_{h,N}([s]_h-\sigma([s]_h,y_{h,N}([s]_h),\lambda_N),\\
&\qquad\quad \dot y_{h,N}([s]_h-\tau([s]_h,y_{h,N}([s]_h),\xi_N)),\theta_N\Big)\Big|ds\\
&\quad\le |\varphi-\varphi_N|_{W^{1,\infty}}+\alpha g_{h,N}+K_1\int_0^t w_{h,N}(s)\,ds\\
&\qquad +L_2\int_0^t z_{h,N}(s-r_0)\,ds\\
&\quad\le G_{h,N}+K_4\int_0^t w_{h,N}(s)\,ds,
\end{aligned}\tag{15}$$

where $G_{h,N}=|\varphi-\varphi_N|_{W^{1,\infty}}+\alpha g_{h,N}+L_2K_2(g_{h,N}+|\varphi-\varphi_N|_{W^{1,\infty}})\alpha$ and $K_4=K_1+L_2K_3$. Since $|x(t)-y_{h,N}(t)|\le G_{h,N}$ for $t\in[-r,0]$, (15) implies

$$w_{h,N}(t)\le G_{h,N}+K_4\int_0^t w_{h,N}(s)\,ds,\quad t\in[0,\alpha_{h,N}].$$

Hence Gronwall's inequality yields

$$w_{h,N}(t) \le G_{h,N} e^{K_4 \alpha}, \quad t \in [0, \alpha_{h,N}].$$

Clearly, $G_{h,N} \to 0$ as $h \to 0+$, $N \to \infty$, hence $\max\{|x(s) - y_{h,N}(s)| \colon -r \le s \le \alpha_{h,N}\} \to 0$ as $h \to 0+$, $N \to \infty$. Consequently $\alpha_{h,N} = T$ for small enough h and large enough N, and the statement of the theorem follows. ∎

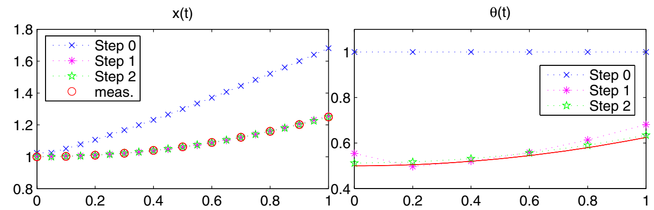

Fig. 2. Estimation of $\theta(t)$: $N = 6$, $h = 0.05$

V. Numerical Examples

In this section we present some numerical examples to illustrate our identification method. The general method is the following: consider an IVP with unknown parameters. If the parameters are infinite dimensional, use linear spline approximation of the parameters. Then, for a fixed small $h > 0$, consider IVP (6)-(7), and solve the corresponding finite dimensional least-square minimization problem, $\mathcal{P}_{h,N}$ (see Step 3 in Section 2). If h is small and N is large, use the solution of $\mathcal{P}_{h,N}$ as an approximate solution of the identification method.

To solve $\mathcal{P}_{h,N}$, we used a nonlinear least square minimization code, based on a secant method with Dennis-Gay-Welsch update, combined with a trust region technique. See Section 10.3 in [6] for detailed description of this method.

Example 1: Consider the SD-NFDE

$$\dot{x}(t) = \dot{x}\left(t - \frac{t^2}{t^2+4}|x(t)| - 1\right) + \theta(t), \quad t \in [0,1], \quad (16)$$

and the associated initial condition

$$x(t) = \frac{1}{4}t^2 + 1, \qquad t \le 0. \qquad (17)$$

In this example the unknown parameter is the function $\theta(t)$ in the right-hand-side of (16). It is easy to check that the solution of IVP (16)-(17), corresponding to $\bar{\theta}(t) = \frac{1}{8}t^2 + \frac{1}{2}$ is $x(t) = \frac{1}{4}t^2 + 1$. We used this solution to generate measurements $X_i = x(t_i)$ at $t_i = 0.1i$, $i = 0, \ldots, 10$, and considered a 6-dimensional linear spline approximation of θ on $[0,1]$. Figure 2 displays the first 3 steps of the numerical solution of the corresponding finite dimensional minimization problem $\mathcal{P}_{0.05,6}$ (the discretization step $h = 0.05$) starting from the constant initial parameter value $\theta_0(t) = 1$. On the right hand side of Figure 2 the first three iterates of the finite dimensional minimization method and the true parameter value (solid line) can be seen. The figure shows that the iterates converge to the "true" parameter value $\bar{\theta}(t)$. On the left the measurements (circles) and the numerical solutions of IVP (16)-(17) corresponding to these parameter values are plotted.

Example 2: Consider again the SD-NFDE investigated in Example 1, where the time dependent coefficient ξ of the state-dependent delay function is a parameter to be identified:

$$\dot{x}(t) = \dot{x}\Big(t - \xi(t)|x(t)| - 1\Big) + \frac{1}{8}t^2 + \frac{1}{2}, \quad t \in [0,1]. \quad (18)$$

We associate initial condition (17) to the equation. The solution of IVP (18)-(17) corresponding to $\bar{\xi}(t) = \frac{t^2}{t^2+4}$ equals to that of the previous example, so we use the same measurements here. In this numerical run we employ an 11-dimensional spline approximation of ξ with $h = 0.05$. Figure 3 shows the first four iterates of the solution of the corresponding finite dimensional minimization problem. We can see that the fourth iterate of $\xi(t)$ is close to the "true" parameter $\bar{\xi}(t)$.

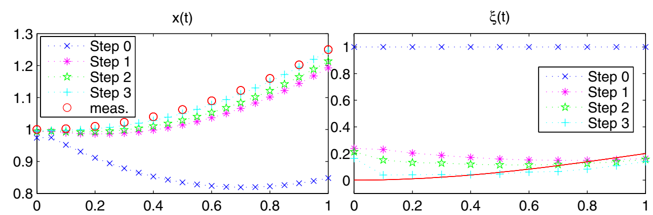

Fig. 3. Estimation of $\xi(t)$: $N = 11$, $h = 0.05$

Example 3: Consider again the SD-NFDE investigated in Example 1, where this time an other part of the delay, a constant $\xi \in \mathbb{R}$ in (19) is the unknown parameter:

$$\dot{x}(t) = \dot{x}\left(t - \frac{t^2}{t^2+4}|x(t)| - \xi\right) + \frac{1}{8}t^2 + \frac{1}{2}, \quad t \in [0,1]. \quad (19)$$

The associated initial condition is again (17). The solution of (19) corresponding to $\bar{\xi} = 1$ is again to that of Example 1, so we use the same measurements. The parameter is one-dimensional, so now we discretize only the equation. The first 5 steps of the iterates of ξ starting from $\xi_0 = 5$ initial guess and the corresponding least-square cost are listed in Table I for step-sizes $h = 0.1$, 0.05 and 0.01, respectively. In Figure 4 the measurements and the solutions corresponding to the first 4 parameter values are plotted. We can see good recovery of the parameter.

Example 4: Consider again the SD-NFDE investigated in the previous examples:

$$\dot{x}(t) = \dot{x}\left(t - \frac{t^2}{t^2+4}|x(t)| - 1\right) + \frac{1}{8}t^2 + \frac{1}{2}, \quad t \in [0,1], \quad (20)$$

with unknown initial condition of the form

$$x(t) = \begin{cases} \varphi(t), & t \in [-r, 0] \\ \varphi(-r), & t < -r. \end{cases} \quad (21)$$

TABLE I

	$h = 0.1$		$h = 0.05$		$h = 0.01$	
i	J_h	ξ_i	J_h	ξ_i	J_h	ξ_i
0	7.731720	5.0000	7.635218	5.0000	6.778048	5.0000
1	0.348809	1.8161	0.343652	1.8344	0.301280	1.8443
2	0.026832	1.1915	0.025915	1.2116	0.022147	1.2254
3	0.000475	0.9837	0.000432	1.0063	0.000347	1.0240
4	0.000001	0.9526	0.000000	0.9765	0.000000	0.9958
5	0.000000	0.9518	0.000000	0.9759	0.000000	0.9952

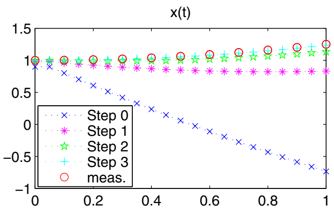

Fig. 4. Estimation of ξ: $h = 0.05$

As we have seen before, the solution corresponding to initial function $\bar{\varphi}(t) = \frac{1}{4}t^2+1$ with $r = 1$ equals to $x(t) = \frac{1}{4}t^2+1$. We use again the measurements of Example 1.

The main difficulty in this example is that the exact initial interval, i.e., $-\bar{r} = \min\{t - \frac{t^2}{t^2+4}|x(t)| - 1 : t \in [0,1]\}$ depends on the solution. Our numerical results show that if r is selected large enough, then at the first few node points the initial guess of the spline approximation of the initial function is not updated during the iteration. Therefore we pick "large enough r", execute the minimization routine, and locate the last node point where the initial guess for the spline is not modified. Then changing $-r$ to this value and repeating the run we can find a good approximation of $-r$ and the initial function. For similar examples and more detailed discussions we refer the reader to [14] and [17].

In Figure 4 we plotted the first 3 steps of the numerical solution of the finite dimensional minimization problem corresponding to an $N = 5$ dimensional approximation of the initial function on the initial interval $[-1.5, 0]$ and the discretization step $h = 0.05$. The second iteration already shows a really good recovery of the initial function. This run suggests that $\bar{r} < 1.5$.

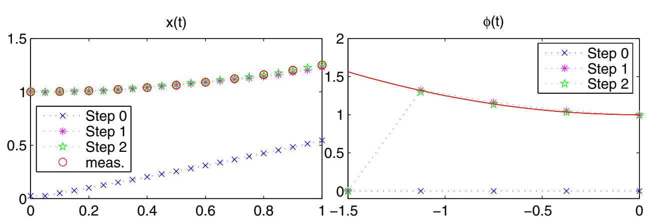

Fig. 5. Estimation of $\varphi(t)$: $N = 5$, $h = 0.05$

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