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A note on common quadratic Lyapunov functions for linear inclusions : Exact results and Open Problems

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Abstract— **We prove several exact results on approximability of joint spectral radius by matrix norms induced by Euclidean norms. We point out, perhaps for the first time in this context, a difference between complex and real cases. New connections of joint spectral radius to convex geometry and combinatorics are established. Several open problems are posed.**

I. The layout of the paper

We introduce our main results and the necessary notation in the first two sections, treating the complex and the real domains correspondingly.

The third section introduces and treats a very useful notion of an entropy of a set of matrices.

The fourth section mentions several open problems.

In Appendix **A** an extremal example of sets of rank-1 matrices is introduced and analyzed.

The proofs of the main results are presented in Appendix **B**.

II. The complex domain

A. Norms in C^n

Recall that C^n stands for a linear space of complex n-dimensional vectors $Z = (x_1 + iy_1, ..., x_n + iy_n)$. There is a natural embedding $f(\cdot)$. of C^n into R^{2n}, by taking $f(Z) = (x_1, y_1; ...; x_n, y_n)$. A norm $\|\cdot\|$ in C^n is called a C-norm if $\|aZ\| = |a|\|Z\|$ for all complex numbers a and vectors $Z \in C^n$. Denote the set of all C-norms by $CN(n)$. A norm $\|\cdot\|_*$ in C^n is an R-norm if $\|Z\|_* = \|f(Z)\|$, where $\|\cdot\|$ is some norm in R^{2n}. We denote by $RN(n)$ the set of all R-norms in C^n. An Euclidean norm on C^n is simply $< Pf(Z), f(Z) >^{\frac{1}{2}}$, where P is a real symmetric positive-definite matrix of order $2n$. We denote the set of all Euclidean norms on C^n by $EN(n)$. It is easy to see that a Euclidean norm $\|\cdot\|$ on C^n is a C-norm iff $\|Z\| =< QZ, Z >^{\frac{1}{2}}$ for some Hermitian positive-definite matrix Q of order n with complex entries. We will call such norms Q-norms and denote the set of all Q-norms in C^n by $QN(n)$.

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The following Lemma is a slight generalization of a famous theorem of F. John.

Lemma 2.1: **:** Let $\|\cdot\|$ be a C-norm in C^n . Then there exists a Hermitian positive-definite matrix Q of order n such that $< QZ, Z >^{\frac{1}{2}} \leq \|Z\| \leq \sqrt{n} < QZ, Z >^{\frac{1}{2}}$, for any $Z \in C^n$.

Remark 2.2: **:** A "naive" application of John's theorem gives the estimate of the Lemma with a worse constant: $\sqrt{2n}$ instead of $\sqrt{n}$. ∎

Consider a finite set S of finite square matrices of order n with complex entries $S = \{A_1, \dots, A_k\}$. In other words $S \subset M(n)$, $Card(S) = k$.

Definition 2.3: **:** The joint spectral radius $\bar{\rho}(S) = \inf_{\|\cdot\| \in RN(n)} \max_{1\leq j\leq k} \|A_j\|$, where $\|\cdot\|$ is an R-norm on C^n and $\|A\|$ is the corresponding induced operator norm. The Euclidean spectral radius $E\rho(S) = \inf_{\|\cdot\| \in EN(n)} \max_{1\leq j\leq k} \|A_j\|$. (It is easy to see that if $\bar{\rho}(S) = 0$ then $E\rho(S) = 0$ also.)
Define

$$CLV(k, n) = \sup_{\bar{\rho}(S)=1, S\subset M(n), Card(S)=k} E\rho(S).$$

∎

The following Proposition is "fairly" obvious.

Proposition 2.4: **:**

1) $\bar{\rho}(S) = \inf_{\|\cdot\| \in CN(n)} \max_{1\leq j\leq k} \|A_j\|$ and $E\rho(S) = \inf_{\|\cdot\| \in QN(n)} \max_{1\leq j\leq k} \|A_j\|$.
2) $\bar{\rho}(S)$ is a continuous function of S (a known result); $E\rho(S)$ is a continuous function of S.

Combining Lemma 2.1 and Proposition 2.4 we see that $CLV(k, n) \leq \sqrt{n}$ for all k. It had been proved in [1] that $CLV(k, n) \leq \sqrt{k}$. Therefore $CLV(k, n) \leq \min(\sqrt{n}, \sqrt{k})$.

B. The main results for the complex case

Definition 2.5: **:** A complex matrix W of order n is *a Hadamard matrix* if $WW^* = nI$ and $|W(i, j)| = 1$ for all $1 \leq i, j \leq n$. (It is well known that complex Hadamard matrices exist for all n. Consider, for instance, the matrix of the discrete Fourier transform.) ∎

Lemma 2.6: **:**

1) Consider a subset $S \subset M(n)$ given by $S = \{A_1, ..., A_n\}$, where $A_i = We_ie_i^T$, for $1 \le i \le n$. Here W is a Hadamard matrix and $\{e_i, 1 \le i \le n\}$ is the canonical coordinate basis in C^n . We will call such subsets n-Hadamard . Then $\bar{\rho}(S) = 1$ and $E\rho(S) = \sqrt{n}$.
2) If there exists a Hermitian matrix $Q \succ 0$ for which $\|A_i\|_Q \le \sqrt{n}$ for $1 \le i \le n$ (equivalently $Q \succeq nA_iQA_i^*$) then $Q = aI$ for some positive real number a.

Theorem 2.7::

1) $CLV(k, n) = \min(\sqrt{n}, \sqrt{k})$.
2) Consider $S = \{A_1, ..., A_n\} \subset M(n)$. If $\bar{\rho}(S) = 1$ and $E\rho(S) = \sqrt{n}$ then there exist a Hadamard matrix W and a nonsingular matrix B such that $A_i = BWe_ie_i^TB^{-1}$ for all $1 \le i \le n$.

III. THE REAL CASE

In the following discussion the notation $Mat(n)$ will stand for the space of $n \times n$ matrices with real entries. We define the real Euclidean spectral radius for a set $S = \{A_1, ..., A_k\}$ of k matrices of order n as

$$RE\rho(S) = \inf_{P \in Mat(n), P \succ 0} \max_{1 \le j \le k} \|A_j\|_P.$$

And similarly to the complex case, we define

$$RLV(k, n) = \sup_{\bar{\rho}(S)=1, S \subset M(n), Card(S)=k} RE\rho(S).$$

As above, we have the inequality $RLV(k, n) \le \min(\sqrt{n}, \sqrt{k})$. The problem is that real Hadamard matrices do not exist for all n and it is a well-known open problem to describe the set of all 'Hadamard' orders n.

Theorem 3.1:: $RLV(n, n) = \sqrt{n}$ if and only if there exists a real Hadamard matrix of order n.

The next theorem is a slightly more refined geometric statement of Theorem 3.1.

Theorem 3.2:: The following conditions are equivalent.

1) $RLV(n) = \sqrt{n}$.
2) There exists a real Hadamard matrix of order n.
3) There exists a norm $|| \cdot ||$ and two orthonormal bases $\{x_1, ..., x_n\}$, $\{y_1, ..., y_n\}$ in R^n such that
   - A $\langle x, x \rangle \le ||x||^2 \le n \cdot \langle x, x \rangle$, for all $x \in R^n$.
   - B $||x_i|| = ||y_i|| = 1$; and $\langle x_i, x_i \rangle = 1$, $\langle y_i, y_i \rangle = n$ for all $1 \le i \le n$.
   - C $\sum_{1 \le i \le n} x_ix_i^T = I$ and $\sum_{1 \le i \le n} y_iy_i^T = nI$.

IV. ENTROPIES OF SETS OF MATRICES

Definition 4.1: : Let $S = \{A_1, ..., A_k\}$ be a set of square matrices of equal order. Define $S^{(l)} = \left\{\prod_{1 \le j \le l} B_j : B_j \in S, 1 \le j \le l\right\}$. We define the entropy of S as $Ent(S) = \lim_{l \to \infty} \frac{\log_k\left(Card\ S^{(l)}\right)}{l}$. (It is easy to see that the limit exists.) Clearly $0 \le Ent(S) \le 1$.

We will call a set of square matrices $S = \{A_1, ..., A_k\}$ *almost nonsingular* if the equality $YA_i = XA_i$ implies that $X = Y$ provided $X, Y \in S^{(l)}$ for some $l \ge 1$. Note that if all the matrices in S are nonsingular then S is almost nonsingular.

For a word $\omega = (\omega_1, ..., \omega_l) \in \{1, 2, ..., k\}^l$ we define $A_\omega = A_{\omega_l}...A_{\omega_1}$. We also set $|\omega| = l$, i.e. $|\omega|$ is the length of the word $|\omega|$. ∎

Let us recall how the inequality $CLV(k, n) \le \sqrt{k}$ was proved in [1]. Assume that there exists a norm such that the induced norms $\|A_i\| \le b < 1$ and $b^2k < 1$, i.e. the joint spectral radius $\bar{\rho}(S) < \frac{1}{k}$. Then the following matrix series converge absolutely:

$$P = \sum_{|\omega| < \infty} A_\omega^* A_\omega.$$

(Here we take $A_\emptyset = I$.) Indeed, there is a constant C for which $\sum_{|\omega|=l} ||A_\omega^* A_\omega||_I \le C(b^2k)^l$.

The matrix P above is positive definite and $\|A_i\|_P < 1$.

Remark 4.2:: Notice that $P = I + \sum_{1 \le l < \infty} T_S^l(I)$, where the action of the completely positive operator T_S is defined by $T_S(X) = \sum_{1 \le i \le k} A_i^* X A_i$.
Since $\rho(T_S) = \lim(||T_S^l(I)||_I)^{\frac{1}{l}}$,this implies the inequality $\rho(T_S) \le (\bar{\rho}(S))^2 Card(S)$, which was rediscovered by Blondel and Nesterov [2]. ∎

As the reader can see, this proof seems to be too loose to produce an exact bound. Nevertheless, it does for k complex matrices of order k. One might also think, that to get an exact bound, we need that matrices $\{A_1, ..., A_k\}$ generate a free semi-group. It happens not to be the case either. The next theorem sheds a bit of extra light.

Theorem 4.3::

1) Suppose that the set $S = \{A_1, ..., A_k\}$ is *almost nonsingular*. If $Ent(S) = \alpha$ then $E\rho(S) \le \sqrt{k}^\alpha \cdot \bar{\rho}(S)$.
2) Consider a sequence of *almost nonsingular*sets $S_m = \{A_{1,m}, ..., A_{n,m}\} \subset M(n)$. Namely, each S_m consists of n complex matrices of order n. If there exists a Hadamard matrix W such that $\lim_{m \to \infty} A_{i,m} = We_ie_i^T$ for all $1 \le i \le n$ then $\lim_{m \to \infty} Ent(S_m) = 1$.

Proof:

1) The proof of the first part of the Theorem is similar to the corresponding proof in [1] .
   Let $S^{(l)} = \left\{A_{\omega_{l,1}}, ..., A_{\omega_{l,m(l)}}\right\}$ be the set of all distinct products of length l. Let the cardinality of $S^{(l)}$ be denoted by $m(l)$. We take $S^{(0)} = \{I\}$. Suppose $\bar{\rho}(S) < \frac{1}{\sqrt{k}^\alpha}$. Then there exists a norm $|| \cdot ||$ such that the induced norms $\|A_i\| \le b < \frac{1}{\sqrt{k}^\alpha}$,

for $1 \leq i \leq k$. Therefore there exists a constant C such that

$$||A_{\omega}^{*}A_{\omega}||_{I}=||A_{\omega}||_{I}^{2}\leq C||A_{\omega}||^{2}\leq Cb^{2|\omega|}.$$

By the same reasoning as above, the following matrix series converge absolutely:

$$Q=\sum_{l<\infty}\sum_{1\leq j\leq Card(S^{(l)})}A_{\omega_{l,j}}^{*}A_{\omega_{l,j}}.$$

The matrix Q is positive definite. Fix a vector z. Then

$$||A_{i}z||_{Q}^{2}=\langle A_{i}^{*}QA_{i}z,z\rangle=$$

$$\sum_{l<\infty}\sum_{1\leq j\leq Card(S^{(l)})}\left\langle (A_{\omega_{l,j}}A_{i})^{*}(A_{\omega_{l,J}}A_{i})z,z\right\rangle.$$

It follows from *almost nonsingularity* that RHS is strictly smaller than $\langle Qz,z\rangle=||z||_{Q}^{2}$.
In other words, $||A_{i}||_{Q}<1$ for all $1\leq i\leq k$, completing the proof.

2) Using continuity of the joint spectral radius and of the Euclidean radius, we see that $\lim_{m\rightarrow\infty}\bar{\rho}(S_{m})=1$ and $\lim_{m\rightarrow\infty}E\rho(S_{m})=\sqrt{n}$. Since $E\rho(S_{m})\leq\sqrt{n}^{Ent(S_{m})}\cdot\bar{\rho}(S_{m})$ and $Ent(S_{m})\leq1$, for all $m\geq1$, this implies $\lim_{m\rightarrow\infty}Ent(S_{m})=1$.

∎

Remark 4.4:: Note that the entropy $Ent(S)=0$ for any set S of rank one matrices. Part 1 of Lemma 2.6 shows that *almost nonsingularity* (or an alternative additional assumption) in Theorem 4.3 is unavoidable. Note also that Part 2 of Theorem 4.3 implies that $Ent(S)$ is *not* a continuous function of S. ∎

V. Open Problems

Question 5.1:: For a finite subset $S=\{A_{1},...,A_{k}\}\subset M(n)$ define the following subset of $n^{r}\times n^{r}$ matrices $S^{(r)}=\{A_{1}\otimes...\otimes A_{1},...,A_{k}\otimes...\otimes A_{k}\}$ (namely r-th tensor powers of $A_{1},...,A_{k}$).

Define

$$CLV(k,n;r)=\sup_{\bar{\rho}(S)=1,S\subset M(n),Card(S)=k}E\rho(S^{(r)}).$$

The real analogue $RLV(k,n;r)$ is defined in the same way. Note that algorithms for approximation of the joint spectral radius from [1] , [2] are based on the bound $CLV(k,n;r)\leq\sqrt{k}$. On the other hand, it is easy to see that if W is a real $n\times n$ matrix and $S=\left\{We_{1}e_{1}^{T},...,We_{n}e_{n}^{T}\right\}$ then $E\rho(S^{(2)})=1$.

Give tight (tighter) estimates for $CLV(k,n;r)$ and $RLV(k,n;r)$. ∎

Question 5.2:: Find good estimates for $RLV(k,n)$.

We know that if $k\leq n$ is a Hadamard number, namely there exists a real Hadamard matrix of order k, then $RLV(k,n)=\sqrt{k}$. By similar reasoning $RLV(n,n)\geq\sqrt{h(n)}$, where $h(n)\leq n$ is the largest Hadamard number not exceeding n.

Another lower bound stems from Proposition A.4. Let $Sign(n)\subset Mat(n)$ be the set of all $n\times n$ matrices with $+1,-1$ entries . Define $Had(n)$ to be $max_{B\in Sign(n)}s_{1}(B)$, where $s_{1}(B)$ is the smallest singular value of B. Then $RLV(n,n)\geq Had(n)$. ∎

Question 5.3:: Is the inequality $E\rho(S)\leq\sqrt{k}^{\alpha}\cdot\bar{\rho}(S)$ in Theorem 4.3 sharp? Is there an alternative (not using joint spectral radius) simple proof of the second part of Theorem 4.3? ∎

Appendix

A. Joint spectral radius and scaling

Consider a finite set of rank one complex matrices of order k: $Z=\{x_{1}y_{1}^{*},...,x_{m}y_{m}^{*}\}$. Define an $m\times m$ matrix of inner products $M_{Z}=\{<x_{i},y_{j}>:1\leq i,j\leq m\}$. We also define for any square, possibly infinite, matrix B the following quantity:

$$\gamma(B):=sup_{i_{1},i_{2},...,i_{k}}\left|B(i_{1},i_{2})B(i_{2},i_{3})...B(i_{k},i_{1})\right|^{\frac{1}{k}}.$$

Since the spectral radius of $\rho(xy^{*})$ is $|<x,y>|$, we obtain the following identity:

$$\bar{\rho}(Z)=\gamma(M_{Z}).$$

The following result follows from [3] (an "infinite" generalization and an alternative proof are presented in [1]).

Proposition 1.1:: The joint spectral radius $\bar{\rho}(Z)$ is smaller than a positive number a if and only if there exist positive real numbers ("scaling factors") $d_{1},...,d_{m}$ such that $\left|d_{i}<x_{i},y_{j}>d_{j}^{-1}\right|\leq a$ for all $1\leq i,j\leq m$.

Corollary 1.2:: Let $Z=\{x_{1}y_{1}^{*},...,x_{k}y_{k}^{*}\}$ with $x_{i},y_{i}\in C^{k}$. Suppose that the vectors $y_{1},...,y_{k}$ are linearly independent. Let A be a non-singular matrix with rows $y_{1}...y_{k}$. Then $\bar{\rho}(Z)\leq1$ if and only if the vectors $r_{i}=Ax_{i}$ are in the unit ball of l_{∞}.

B. Euclidean spectral radius of sets of rank one matrices

Let $Q\succ0$ be s positive definite complex Hermitian matrix of order k. Let $||z||_{Q}=\sqrt{<Qz,z>},z\in C^{k}$. Correspondingly one defines an induced operator norm $||A||_{Q}$ for $k\times k$ complex matrices A. It is well known (and easy to prove) that $||A||_{Q}=a$ iff $a^{2}Q\succeq A^{*}QA$ and $\det(a^{2}Q-A^{*}QA)=0$. Or, equivalently, iff the spectral radius $\rho\left(Q^{-1}A^{*}QA\right)=a^{2}$. It follows that $||A||_{Q}=||A^{*}||_{Q^{-1}}$. For rank one matrices $A=xy^{*}$ we get that $||xy^{*}||_{Q}^{2}=\langle Qx,x\rangle\left\langle Q^{-1}y,y\right\rangle$.

Theorem 1.3::

1) Let $Z=\{x_{1}y_{1}^{*},...,x_{m}y_{m}^{*}\}$ with x_{i},y_{i} in C^{k}. Suppose that $||x_{i}||_{I}=||y_{i}||_{I}=1$. Then the

Euclidean spectral radius $E\rho(Z) = 1$ if and only if the convex hull of the matrices $x_1x_1^* - y_1y_1^*$, $1 \le i \le m$, contains the zero matrix. Namely, there exist nonnegative $\alpha_1...\alpha_m$ summing to 1 such that $\sum_{1\le i\le m} \alpha_i (x_ix_i^* - y_iy_i^*) = 0$.
2) Let G be a maximal under inclusion subset of $\{1...m\}$ such that there are positive numbers $\alpha_i : i \in G$ with the property $\sum_{i\in G} \alpha_i(x_ix_i^* - y_iy_i^*) = 0$. Define $Ad(G) = \{x_iy_i^*, y_ix_i^* : i \in G\}$. Define $L(Z) := \{Q \succ 0 : ||x_iy_i^*||_Q \le 1,\ for\ 1 \le i \le m\}$. Then $L(Z) = \{aI : a > 0\}$ if and only if $Ad(G)$ is irreducible, i.e. there is no nontrivial common invariant linear subspace.

Proof:

1) If the convex hull $CO(\{x_ix_i^* - y_iy_i^* :,\ 1 \le i \le m\})$ does not contain the zero matrix then by the Hanh-Banach theorem there exists a traceless Hermitian matrix T such that $tr(T(x_ix_i^* - y_iy_i^*)) > 0$ for all $1 \le i \le m$. Define $Q_\epsilon = I - \epsilon T$. Clearly, $Q_\epsilon \succ 0$ for a sufficiently small ϵ. Since $||xy^*||^2_{Q_\epsilon} = \langle Q_\epsilon x, x\rangle \left\langle Q_\epsilon^{-1} y, y\right\rangle$, we get $||x_iy_i^*||^2_{Q_\epsilon} = 1 - \epsilon\cdot(\langle Tx_i, x_i\rangle - \langle Ty_i, y_i\rangle) + O\left(\epsilon^2\right) = 1 - \epsilon \cdot tr(T(x_ix_i^* - y_iy_i^*)) + O\left(\epsilon^2\right)$. Thus, for a sufficiently small positive ϵ holds $|x_iy_i^*||^2_{Q_\epsilon} < 1$ for all $1 \le i \le m$. In other words, $E\rho(Z) < 1$. This proves the "only if" part.
Suppose now that the convex hull does contain the zero matrix. Namely, there is a convex combination $\sum_{1\le i\le m} \alpha_i (x_ix_i^* - y_iy_i^*) = 0$. If for $Q \succ 0$ we have the inequalities $Q \succ y_ix_i^*Qx_iy_i^*$, for $1 \le i \le m$, then also $\langle Qy_i, y_i\rangle - \langle Qx_i, x_i\rangle = -tr(Q(x_ix_i^* - y_iy_i^*)) > 0$, for $1 \le i \le m$. Summing the last inequalities with nonnegative weights α_i we obtain $\sum_{1\le i\le m} \alpha_i tr(Q(x_ix_i^* - y_iy_i^*)) < 0$, contradicting the equation $\sum_{1\le i\le m} \alpha_i(x_ix_i^* - y_iy_i^*) = 0$. This proves the "if" part.
2) The proof is based on ideas similar to the first part and left to the reader.

∎

The following proposition is a slight generalization of the first part of Theorem A.3.

Proposition 1.4:: Let $Z = \{x_1y_1^*, ..., x_my_m^*\}$ with x_i, y_i in C^k. Suppose that $||x_i||_I = ||y_i||_I = 1$ and there exist positive numbers α_i , $1 \le i \le m$ such that $\sum_{1\le i\le m} \alpha_i x_ix_i^* \ge a \cdot \sum_{1\le i\le m} \alpha_i y_iy_i^*$, for $0 < a \le 1$. Then $E\rho(Z) \ge a$.

C. Maximal / Minimal volume ellipsoids and rank one matrices

We are concerned here only with symmetric convex bodies, i.e. convex compact sets $K = -K \in R^N$ with nonempty interior. Such symmetric convex bodies are simply the unit balls of norms in R^N. Recall that a symmetric ellipsoid in R^N is defined as $EL_Q = \left\{x \in R^N : ||x||_Q = \sqrt{< Qx, x >} \le 1\right\}$, for a positive definite matrix Q. The volume $Vol(EL_Q) = v_N \det(Q)^{-\frac{1}{2}}$, where v_N is the volume of unit Euclidean ball in R^N. Let $Ball(||\cdot||, 1) = \{x \in R^N : ||x|| \le 1\}$ be a unit ball of some norm $||\cdot||$ in R^N. Then $EL_Q \subset Ball(||\cdot||, 1)$ iff $||x|| \le ||x||_Q$ and $Ball(||\cdot||, 1) \subset EL_Q$ iff $||x|| \ge ||x||_Q$. It was proved by Fritz John that for any norm $||\cdot||$ in R^N there exists a unique maximal volume ellipsoid, called John's ellipsoid, $EL_{Q_{in}} \subset Ball(||\cdot||, 1)$. By duality there also exists a unique minimal volume ellipsoid, called Loewner's ellipsoid, $Ball(||\cdot.||, 1) \subset EL_{Q_{out}}$.
The uniqueness of maximal (minimal) volume ellipsoid implies that if $A : R^N \to R^N$ is a linear isometry with respect to the norm $||\cdot||$ then A is also an isometry with respect to both Euclidean norms $||\cdot||_{Q_{in}}$ and $||\cdot||_{Q_{out}}$. This simple observation is sometimes very useful.

Example 1.5:: Let us clarify the notions we have just introduced via the proof of Lemma 2.1.

Consider a C-norm $||\cdot||$ in C^n. Its unit ball $Ball(||\cdot||, 1) \subset C^n$ can be viewed, via the embedding $f(x_1 + iy_1, ..., x_n + iy_n) = (x_1, y_1; ...; x_n, y_n)$, as a convex symmetric body $f(Ball(||\cdot.||, 1))$ in R^{2n}. Let $EL_{Q_{in}} \subset f(Ball(||\cdot||, 1))$ be the maximal volume ellipsoid inscribed in $f(Ball(||\cdot||, 1))$, where $Q_{in} \succ 0$ is a positive-definite symmetric real matrix of order $2n$. Write Q_{in} in the following block form:

$$Q_{in} = \begin{pmatrix} Q_{in}^{(1,1)} & Q_{in}^{(1,2)} & \dots & Q_{in}^{(1,n)} \\ Q_{in}^{(2,1)} & Q_{in}^{(2,2)} & \dots & Q_{in}^{(2,n)} \\ \dots & \dots & \dots & \dots \\ Q_{in}^{(n,1)} & Q_{in}^{(n,2)} & \dots & Q_{in}^{(n,n)} \end{pmatrix}$$

The blocks $Q_{in}^{(k,l)}$ are real 2×2 matrices.

Also define the following block-diagonal matrices:

$$R(\phi) = \begin{pmatrix} Rot(\phi) & 0 & \dots & 0 \\ 0 & Rot(\phi) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Rot(\phi) \end{pmatrix},$$

where $Rot(\phi)$ is a 2×2 unitary matrix representing the rotation by an angle ϕ. Since $||\cdot||$ is a C-norm, $R(\phi)(f(Ball(||\cdot||, 1))) = f(Ball(||\cdot||, 1))$ for all $0 \le \phi < 2\pi$. Therefore, by uniqueness of the maximal volume ellipsoid, $R(\phi)(EL_{Q_{in}}) = EL_{Q_{in}}$ or, equivalently,

$$R(\phi)Q_{in}R(\phi)^T = Q_{in}.$$

Therefore, $Rot(\phi)Q_{in}^{(k,l)}Rot(\phi)^T = Q_{in}^{(k,l)}$ for all $1 \le k, l \le n$ and $0 \le \phi < 2\pi$. In other words, there exists a complex Hermitian positive definite matrix P of order n such that $Q_{in} = \bar{P}$, where the block matrix $\bar{P}$ is defined as:

$$Q_{in}^{(k,l)} = \begin{pmatrix} A(k,l) & B(k,l) \\ -B(k,l) & A(k,l) \end{pmatrix}.$$

It follows that $||f(Z)||_{Q_{in}} = \sqrt{< PZ, Z >}$ for all $Z \in C^n$. Now we can apply a standard technique for the proof of a real version of John's theorem (see, for instance, [4]). By a linear change of variables $Z = P^{-\frac{1}{2}} V$, we can assume the maximal volume ellipsoid inscribed in $Ball(||\cdot||, 1) \subset C^n$ to be $Ball(||.||_I, 1)$. (Namely, we consider a "scaled" norm $||V||_{(sc)} = ||P^{-\frac{1}{2}} V||$, $V \in C^n$.)

The optimality of unit Euclidean ball among inscribed ellipsoids is equivalent to existence of $k < \infty$ contact points $V_1, ..., V_k$ such that $||V_i||_{(sc)} = ||V_i||_I = 1$ (given, of course, the minorization condition $||V||_{(sc)} \leq ||V||_I$, for all $V \in C^n$), and such that $I = \sum_{1 \leq i \leq k} a_i V_i V_i^*$ for some nonnegative constants a_i.

Similarly to the real case, we have $\sum_{1 \leq i \leq k} a_i = n$ (equating the traces) and $|\langle V_i, V \rangle| \leq 1$ for all V with $||V||_{(sc)} \leq 1$ and all $1 \leq i \leq k$. This gives the following complex analog of John's theorem:

$$||V||_I^2 = \sum_{1 \leq i \leq k} a_i |\langle V_i, V \rangle|^2 \leq n ||V||_{(sc)}^2.$$

A similar result (with a similar proof) holds also in H^N, i.e. in the quaternionic case.

∎

Lemma 1.6:: Let $||\cdot||$ be a C-norm in C^n (alternatively $||\cdot||$ is a norm in R^n); Let EL_I be the John ellipsoid of its unit $Ball(||\cdot||, 1)$ (or the Loewner ellipsoid of the unit ball). If A is a complex (real) $n \times n$ matrix such that $||A|| = 1$ and $||A||_I = \sqrt{n}$ then $Rank(A) = 1$.

Proof: Among all the variants above, we will deal with the complex domain case, and with the inscribed (John's) ellipsoid, other cases are proved similarly. Recall that EL_I is John's ellipsoid of $Ball(||\cdot||, 1)$ iff

1) $||Z|| \leq ||Z||_I = \sqrt{< Z, Z >}$ for all $Z \in C^n$.
2) There exists a finite set of vectors $Z_1 ... Z_K$ and positive numbers $a_1 ... a_K$ such that $||Z_i||_I = ||Z_i|| = 1$ for $1 \leq i \leq K$ and $\sum_{1 \leq i \leq K} a_i Z_i Z_i^* = I$. (Note that in this case the vectors $\{Z_1, ..., Z_K\}$ span C^n).

It follows that $\sum_{1 \leq i \leq K} a_i = n$ and $|\langle Z_i, Z \rangle| \leq 1$ for all $1 \leq i \leq K$ and all $Z \in Ball(||\cdot||, 1)$. Define a new norm $|||Z||| = \max_{1 \leq i \leq K} |\langle Z_i, Z \rangle|$. Clearly $|||Z||| \leq ||Z||$. Note that if $||Z||_I = \sqrt{n} \cdot ||Z||$ then $|||Z||| = ||Z||$. Let A be a complex $n \times n$ matrix such that $||A|| = 1$ and $||A||_I = \sqrt{n}$. This means that there exist two vectors $U, V \in C^n$ such that

$$||U|| = ||V|| = 1; \quad ||U||_I = 1, \quad ||V||_I = \sqrt{n}; \quad AU = V.$$

It follows that $|\langle Z_i, AU \rangle| = |\langle Z_i, V \rangle| = 1$, for $1 \leq i \leq K$.

Suppose that $Rank(A) > 1$, i.e. there exists a vector $W \in C^n$ such that $AW \neq 0$ and $\langle U, W \rangle = 0$. Choose any index j such $\langle Z_j, AW \rangle \neq 0$ (it exists since the vectors $\{Z_1, ..., Z_K\}$ span C^n). Set $z = \frac{\langle Z_j, V \rangle}{\langle Z_j, AW \rangle}$. We get the following inequalities for all sufficiently small $\epsilon > 0$:

$$||U + \epsilon z W|| \leq ||U + \epsilon z W||_I = \sqrt{1 + |\epsilon z|^2} \leq 1 + O(\epsilon^2);$$

$$||A(U + \epsilon z W)|| \geq |||A(U + \epsilon z W)||| \geq |\langle Z_j, V + \epsilon z A W \rangle| = 1 + \epsilon.$$

Therefore, if ϵ is sufficiently small, then

$$\frac{||A(U + \epsilon z W)||}{||U + \epsilon z W||} \geq \frac{1 + \epsilon}{1 + O(\epsilon^2)} > 1.$$

We got a desired contradiction. Thus $Rank(A) = 1$. ∎

D. Basic properties of extremal sets of matrices

Let $CQ(n)$ be the smallest integer such that there exists a set of complex $n \times n$ matrices $S = \{A_1, .., A_{CQ(n)}\}$ with $\bar{\rho}(S) = 1$ and $E\rho(S) = \sqrt{n}$. We will call such sets of matrices $n - extremal$. We define a real analog $RQ(n)$ similarly.

Proposition 1.7::

1) $CQ(n) = n$; $RQ(n) = n$ if there exists a real Hadamard matrix of order n.
2) $n \leq RQ(n) \leq \frac{n(n+1)}{2}$

Proof: Part 1: Let W be a Hadamard matrix of order n, and let $e_1 ... e_k$ be the standard basis of C^n. Consider $S = \{A_1, ..., A_n\}$, where $A_i = W e_i e_i^T$. Clearly $\bar{\rho}(S) = 1$. It follows from Part 1 of Theorem A.3 that $E\rho(S) = \sqrt{n}$. Therefore $CQ(n) \leq n$. The inequality $n \leq CQ(n)$ follows from the inequality $CLV(k, n) \leq \sqrt{k}$ (see [1]). The proof of the real case is similar.

Part 2: We only need to prove the upper bound. Consider the following set of block-diagonal real symmetric matrices of order $2n$, $Bool(n) = \{L_{i,j} = e_i e_i^T \oplus b_j b_j^T : 1 \leq i \leq n, 1 \leq j \leq 2^n\}$, where the set $\{b_j : 1 \leq j \leq 2^n\}$ consists of all 2^n vectors with $(+1, -1)$ coordinates. Clearly, the convex hull $CO(Bool(n))$ contains $\frac{1}{n} I \oplus I$. Using the Caratheodory Theorem (and carefully estimating the dimension) we see that there exists a set of pairs of indices (i_k, j_k), $1 \leq k \leq \frac{n(n+1)}{2}$, and positive numbers a_k, $1 \leq k \leq \frac{n(n+1)}{2}$, such that

$$\sum_{1 \leq k \leq \frac{n(n+1)}{2}} a_k \cdot e_{i_k} e_{i_k}^T \oplus b_{j_k} b_{j_k}^T = \frac{1}{n} I \oplus I.$$

Consider the set of rank one $n \times n$ real matrices $S = \left\{ b_{j_k} e_{i_k}^T, \ 1 \leq k \leq \frac{n(n+1)}{2} \right\}$. Then $\bar{\rho}(S) = 1$ and, using Part 1 of Theorem A.3, $E\rho(S) = \sqrt{n}$. ∎

Lemma 1.8::

1) In both complex and real cases, $n - extremal$ sets of matrices are irreducible, i.e. they don't have a common nontrivial linear subspace . Moreover, $E\rho(W) \leq \sqrt{n-1} \cdot \bar{\rho}(W)$ for any bounded reducible subset of $n \times n$ complex matrices $W \subset M(n)$.
2) Consider, a necessary irreducible, $n - extremal$ set of complex matrices $S = \{A_1, .., A_n\}$. (Or an $n - extremal$ set of real matrices $S = \{A_1, .., A_{RQ(n)}\}$). Since $\bar{\rho}(S) = 1$, in this irreducible case there exists a C-norm $||\cdot||$ such that $||A_i|| \leq 1$ for all i. Assume, without loss of generality, that John's ellipsoid of the unit ball $Ball(||\cdot||, 1)$ is EL_I. Then $||A_i||_I = \sqrt{n}$ (and thus $||A_i|| = 1$).

3) In both complex and real cases, $n - extremal$ sets of matrices consist of rank one matrices.

Proof: Part 1 is simple and is left to the reader. Let us prove Part 2 in the real case (complex case is analogous). First, it follows from John's theorem that $||A_i||_I \leq \sqrt{n}$ for all $1 \leq i \leq RQ(n)$. Assume that $||A_{RQ(n)}||_I = a < \sqrt{n}$. By the definition of $RQ(n)$, there exists a positive definite matrix Q such that $|A_i||_Q \leq c < \sqrt{n}$ for all $1 \leq i \leq RQ(n) - 1$. Define $Q_\epsilon = (1 - \epsilon)I + \epsilon Q$. Then for all $0 < \epsilon \leq 1$ the norms $||A_i||_{Q_\epsilon}$ are strictly smaller than $\sqrt{n}$ for $1 \leq i \leq RQ(n) - 1$, and for all sufficiently small $\epsilon > 0$ the norm $||A_{RQ(n)}||_{Q_\epsilon}$ is strictly smaller than $\sqrt{n}$. In other words, there exists $0 < \epsilon \leq 1$ such that $||A_i||_{Q_\epsilon} < \sqrt{n}$ for all $1 \leq i \leq RQ(n)$. And this contradicts the definition of $RQ(n)$.

Now, apply Lemma 4.6 to prove Part 3. ∎

E. The Final Proofs

Proof: (Proof of Lemma 2.6)
Part 1 follows from Part 1 of Theorem A.3, Part 2 follows from Part 2 of Theorem A.3. ∎

Proof: (Proof of Theorem 2.7)
Part 1 of Lemma 2.6 together with Lemma 2.1 prove that $CLV(k, n) = \sqrt{n}$ for all $k \geq n$. For $k < n$ take any k-Hadamard subset $S_{(k)} = \{A_1, ..., A_k\} \subset M(k)$ and define $S_{(k,n)} = \{A_1 \oplus 0, ..., A_k \oplus 0\} \subset M(n)$. Then $\bar{\rho}(S_{(k,n)}) = 1$ and $E\rho(S_{(k,n)}) = \sqrt{k}$. This, together with [1] proves that $CLV(k, n) = \sqrt{k}$ for all $k < n$.

We break our proof of Part 2 of Theorem 1.7 into a few small steps:

Step 1 Let $S = \{A_1, ..., A_n\} \subset M(n)$, $\bar{\rho}(S) = 1$ and $E\rho(S) = \sqrt{n}$. It follows from Part 2 of Lemma B.2 that $Rank(A_i) = 1$, for $1 \leq i \leq n$. Therefore $A_i = x_i y_i^*; x_i, y_i \in C^n$.

Step 2 It follows from Part 1 of Lemma B.2 that the set $\{x_1 y_1^*, ..., x_n y_n^*\}$ is irreducible. Therefore, $\{x_1, ..., x_n\}$ as well as $\{y_1, ..., y_n\}$ are linearly independent. We can now apply Corollary A.2 : there exists a nonsingular matrix A such that $A x_i y_i^* A^{-1} = r_i e_i^*$, for $1 \leq i \leq k$. Here $\{e_i, 1 \leq i \leq k\}$ is the canonical coordinate basis in C^n and $||r_i||_\infty \leq 1$, for $1 \leq i \leq k$.

Step 3 Note that $||r_i e_i^*||_\infty \leq 1$ and $||r_i e_i^*||_I \leq \sqrt{n}$. Since John's ellipsoid of the unit ball $Ball(|| \cdot ||_\infty, 1)$ is EL_I, using Part 2 of Lemma B.2, we get $||r_i e_i^*||_\infty = 1$ and $||r_i e_i^*||_I = \sqrt{n}$ for all $1 \leq i \leq n$. In other words, the magnitude of all coordinates of vectors r_i, $1 \leq i \leq n$ equals 1.

Step 4 Consider the following set of rank one matrices $Z = \{\sqrt{n}^{-1} r_1 e_1^*, ..., \sqrt{n}^{-1} r_n e_n^*\}$. It follows that $E\rho(Z) = \sqrt{n}^{-1} \cdot E\rho(S) = 1$. Using Part 1 of Theorem A.3 we get that there exist nonnegative real numbers $a_1 ... a_n$ summing to one, such that

$$\sum_{1 \leq i \leq n} a_i (\frac{1}{n} r_i r_i^* - e_i e_i^*) = 0.$$

Note that diagonals of matrices $r_i r_i^*$, $1 \leq i \leq n$, consist of all ones. Therefore, $a_i = \frac{1}{n}$, for $1 \leq i \leq n$. Which means that the matrix $W = [r_1, ..., r_n]$ satisfies $WW^* = nI$ and thus W is Hadamard.

∎

Remark 1.9: **:** Assume, without loss of generality, in the proof of Theorem 2.7 above that there exists a C-norm $|| \cdot ||$ such that $||A_i|| = 1$ for all $1 \leq i \leq n$ and that EL_I is the John ellipsoid for $Ball(|| \cdot ||, 1)$. Then the nonsingular matrix A in Step 2 is proportional to a unitary matrix, i.e. $AA^* = cI$. Indeed, we have that

$$||\sqrt{n}^{-1} A_i||_I = ||A^{-1}(\sqrt{n}^{-1} r_i e_i^*)A||_I = 1, \quad 1 \leq i \leq n.$$

Therefore $||\sqrt{n}^{-1} r_i e_i^*)||_{(A^{-1})^* A^{-1}} = 1$ for $1 \leq i \leq n$. It follows from Part 2 of Lemma 2.6 that $A^{-1})^* A^{-1} = aI$. ∎

Proof: (Proof of Theorem 3.1)
Suppose that there exists a sequence of subsets of real $n \times n$ matrices $S_l = \{A_{l,1}, ..., A_{l,n}\} \subset Mat(n)$ such that $\bar{\rho}(S_l) = 1$ for $l \geq 1$ and $\lim_{l \to \infty} E\rho(S_l) = \sqrt{n}$. Since for reducible subsets $E\rho(W) \leq \sqrt{n-1} \cdot \bar{\rho}(W)$, we can assume, without loss of generality, that S_l is irreducible for all $l \geq 1$. Therefore there exist norms $|| \cdot ||_l$ in R^n, such that $||A_{l,i}||_l \leq 1$ for $l \geq 1$, and $1 \leq i \leq n$. Using John's theorem, we conclude that there exist real nonsingular matrices B_l such that $||B_l(A_{l,i})B_l^{-1}||_I \leq 1$ for $l \geq 1$ and $1 \leq i \leq n$. Define $Z_l = \left\{B_l(A_{l,1})B_l^{-1}, ..., B_l(A_{l,n})B_l^{-1}\right\} \subset Mat(n)$. Clearly, $\bar{\rho}(Z_l) = \bar{\rho}(S_l) = 1$ and; $E\rho(Z_l) = E\rho(S_l)$ for $l \geq 1$.

The sequence of subsets $Z_l, l \geq 1$ is bounded. Therefore it has a limit point $Z = \{C_1, ..., C_l\}$. It follows from Part 2 of Proposition 2.4 that $\bar{\rho}(Z) = 1$ and $E\rho(Z) = \sqrt{n}$.
Now, by the same reasoning as in the proof of Part 2 of Theorem 2.7, we get that, up to a similarity transformation, $C_i = W e_i e_i^T$, for $1 \leq i \leq n$, where W is a real Hadamard matrix of order n. Thus if $RLV(n, n) = \sqrt{n}$ then there exists a real Hadamard matrix of order n. And we already proved that existence of real Hadamard matrices of order n implies $RLV(n, n) = \sqrt{n}$. ∎

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