Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Convergence Properties of the Newton-Kantorovich Iteration for the Hamilton-Jacobi-Bellman Equation

Costas Kravaris and Dimitra Mousavere

***Abstract*—This work applies Kantorovich's method to construct an iterative algorithm for the solution of the Hamilton-Jacobi-Bellman equation. At each step of the iteration, a Zubov partial differential equation is solved. Convergence properties of the algorithm are developed independently of Kantorovich's theorem. The results are illustrated in a numerical example.**

I. Introduction

THE Hamilton-Jacobi-Bellman equation plays a central role in the theory of optimal control [7]. In the pioneer work of Al'Brekht [1] and Lukes [8], the Hamilton-Jacobi-Bellman equation has been studied as the means of generating a state feedback law, which is optimal with respect to a quadratic performance index over an infinite time horizon. A power series solution approach has been formulated to generate the positive-definite solution to the Hamilton-Jacobi-Bellman PDE, which could potentially be used to compute the optimal state feedback law. A variant of the Hamilton-Jacobi-Bellman equation arises in nonlinear H_∞ control [11] and has been studied in a similar spirit.

There have been some engineering applications of optimal state feedback designed through the Hamilton-Jacobi-Bellman equation, the most notable one being the work of Garrard and Jordan [3] in aircraft control. The key difficulty that prevented the widespread use of the Hamilton-Jacobi-Bellman equation as a design tool has been the enormous computational effort and complexity in implementing Lukes' method. However, with the explosive increase in computing power in recent years, along with the availability of user-friendly and powerful symbolic computation software such as MAPLE, it is now becoming realistic to use the Hamilton-Jacobi-Bellman equation as a design tool for nonlinear control applications.

The idea of applying a Newton-Kantorovich iteration for the numerical solution of the Hamilton-Jacobi-Bellman equation was first proposed in a recent paper [10], using a power series algorithm for the solution of the resulting Zubov equation at each step of the iteration. The results in [10] have indicated significant computational advantages of the Newton-Kantorovich iteration over Lukes' method, and this motivates further investigation of the theoretical properties of the iteration.

The purpose of the present paper is to study the convergence properties of the Newton-Kantorovich iteration. After giving some brief necessary background in Section II, Section III will develop the Newton-Kantorovich iteration for the Hamilton-Jacobi-Bellman equation. Section IV will provide the main results of the present paper regarding the properties of the iteration. Finally, in Section V, the results will be illustrated with a numerical example.

II. Background

A. The Hamilton-Jacobi-Bellman equation

Consider a nonlinear system with the following representation

$$\dot{x} = f(x) + g(x)u \\ y = h(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}^m$ is the vector of input variables, $y \in \mathbb{R}^m$ is the vector of output variables, $f: \mathbb{R}^n \to \mathbb{R}^n$, $g(x) = \left[g_1(x), g_2(x), \ldots, g_m(x)\right]$ with $g_i: \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, m$ and $h: \mathbb{R}^n \to \mathbb{R}^m$ are real analytic vector functions. Without loss of generality, it can be assumed that the origin $x_0 = 0$ is an equilibrium for system (1) corresponding to $u_0 = 0$.

Consider now the following quadratic performance index

$$J = \int_0^\infty \left\{[h(x)]^{\mathrm{T}} Q[h(x)] + u^{\mathrm{T}} Ru\right\} dt \tag{2}$$

where $Q = diag\{q_1, q_2, \ldots, q_m\}$, $R = diag\{r_1, r_2, \ldots, r_m\}$ with $q_i > 0$ and $r_i > 0$, $i = 1, \ldots, m$.

Also, consider the optimal control problem of minimizing the performance index (2) subject to the dynamics of (1). Associated with this optimal control problem, the Hamilton-Jacobi-Bellman equation is

$$\frac{\partial V}{\partial x}(x) f(x) - \frac{1}{4}\frac{\partial V}{\partial x}(x) g(x) R^{-1}\left[g(x)\right]^{\mathrm{T}}\left[\frac{\partial V}{\partial x}(x)\right]^{\mathrm{T}} \\ + \left[h(x)\right]^{\mathrm{T}} Q\left[h(x)\right] = 0 \tag{3}$$

Manuscript received March 7, 2005.
Costas Kravaris and Dimitra Mousavere are with the Department of Chemical Engineering, University of Patras, 26500 Patras, Greece.
phone: +30 2610 996339; fax: +30 2610 993070;
e-mail: kravaris@chemeng.upatras.gr.

0-7803-9568-9/05/$20.00 ©2005 IEEE

Lukes [8] proved that, if $\left(\frac{\partial f}{\partial x}(0), g(0)\right)$ is a stabilizable pair and $\left(\frac{\partial h}{\partial x}(0), \frac{\partial f}{\partial x}(0)\right)$ an observable pair, (3) admits a unique locally positive definite solution $V(x)$ in a neighborhood of the origin. Moreover, the solution is locally analytic around the origin.

Once the positive definite solution of (3) is computed, the optimal state feedback control law is obtained from

$$u^*(x) = -\frac{1}{2} R^{-1}[g(x)]^{\mathrm{T}} \left[\frac{\partial V}{\partial x}(x)\right]^{\mathrm{T}} \tag{4}$$

B. The Newton-Kantorovich iteration

Consider a nonlinear operator equation

$$\mathcal{N}(v) = 0 \tag{5}$$

The Newton-Kantorovich iteration [4] is a generalization of the Newton-Raphson iteration, which is commonly used for the numerical solution of nonlinear algebraic equations, to general nonlinear operator equations, where the operator $\mathcal{N}$ maps a Banach space to another. At the N-th step of the Newton-Kantorovich iteration, the following linear operator is solved for v_{N+1}

$$\mathcal{N}'(v_N)\cdot(v_{N+1} - v_N) = -\mathcal{N}(v_N) \tag{6}$$

where v_N is the result of the (N–1)-th step and $\mathcal{N}'(v)\cdot\delta v$ is the Fréchet differential of the operator $\mathcal{N}$.

Kantorovich's theorem [4]: *Let* $\mathcal{N}: \mathrm{X} \to \mathrm{Y}$*, where* X, Y *are Banach spaces. Assume that*

i) $\mathcal{N}$ *is twice Fréchet differentiable for all* $v \in \mathrm{X}$.
ii) there is a $v_0 \in \mathrm{X}$ *such that* $\mathcal{N}'(v_0)$ *is invertible with bounded inverse and*
$\left\| [\mathcal{N}'(v_0)]^{-1} \right\| \le \beta_0 \quad , \quad \left\| [\mathcal{N}'(v_0)]^{-1} \mathcal{N}(v_0) \right\| \le \eta_0$
iii) $\| \mathcal{N}''(v) \| \le K$ *for all* v *such that* $\| v - v_0 \| < 2\eta_0$
iv) $h = \beta_0 \eta_0 K < \frac{1}{2}$
Then the sequence

$$v_{N+1} = v_N - [\mathcal{N}'(v_N)]^{-1} \mathcal{N}(v_N), \quad N = 0,1,2,\ldots$$

is well-defined for all $N = 0,1,2,\ldots$ *and converges to a solution* v^* *of the equation* $\mathcal{N}(v) = 0$.
Moreover,

$$\| v_N - v^* \| \le \frac{\eta_0}{h} \frac{\left(1 - \sqrt{1-2h}\right)^{2^N}}{2^N}$$

Kantorovich's theorem essentially states that if the iteration (6) is initialized at v_0 sufficiently close to a root v^* of (5), the iteration (6) converges to v^* and convergence is quadratic.

III. Newton-Kantorovich Iteration for the Solution of the Hamilton-Jacobi-Bellman Equation

Let $\mathrm{C}^0(\Omega)$ denote the space of continuous functions defined on an open set $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}$. Endowed with the ∞–norm, $\mathrm{C}^0(\Omega)$ is a Banach space. Also, let $\mathrm{C}^1(\Omega)$ denote the space of continuously differentiable functions from $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}$. Endowed with the norm

$$\| v \| = \max\left\{ \| v \|_\infty, \left\| \frac{\partial v}{\partial x_1} \right\|_\infty, \ldots, \left\| \frac{\partial v}{\partial x_n} \right\|_\infty \right\},$$

$\mathrm{C}^1(\Omega)$ is a Banach space.

Consider now the nonlinear operator $\mathcal{N}: C^1(\Omega) \to C^0(\Omega)$ defined by

$$\mathcal{N}(v) = \frac{\partial V}{\partial x}(x) f(x) - \frac{1}{4} \frac{\partial V}{\partial x}(x) g(x) R^{-1} [g(x)]^{\mathrm{T}} \left[\frac{\partial V}{\partial x}(x)\right]^{\mathrm{T}} + [h(x)]^{\mathrm{T}} Q [h(x)] \tag{7}$$

The operator $\mathcal{N}$ is Fréchet differentiable at every $v \in C^1(\Omega)$. Its Fréchet derivative $\mathcal{N}'(v) \in \mathcal{L}\left(C^1(\Omega), C^0(\Omega)\right)$ assigns to each $\delta v \in C^1(\Omega)$ the element $\mathcal{N}'(v)\cdot\delta v \in C^0(\Omega)$ given by

$$\mathcal{N}'\cdot\delta v = \frac{\partial(\delta v)}{\partial x}\left[f(x) - \frac{1}{2} g(x) R^{-1} [g(x)]^{\mathrm{T}} \left[\frac{\partial v}{\partial x}(x)\right]^{\mathrm{T}} \right] \tag{8}$$

Thus, the Newton-Kantorovich iteration (6) takes the form

$$\frac{\partial(v_{N+1} - v_N)}{\partial x}(x)\left[f(x) - \frac{1}{2} g(x) R^{-1} [g(x)]^{\mathrm{T}} \left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}} \right] =$$
$$= -\frac{\partial v_N}{\partial x}(x) f(x) + \frac{1}{4} \frac{\partial v_N}{\partial x}(x) g(x) R^{-1} [g(x)]^{\mathrm{T}} \left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}} - -[h(x)]^{\mathrm{T}} Q h(x)$$

or equivalently

$$\frac{\partial v_{N+1}}{\partial x}(x)\left[f(x)-\frac{1}{2}g(x)R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}}\right]=$$
$$-\left[\frac{1}{2}R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}}\right]^{\mathrm{T}}R\left[\frac{1}{2}R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}}\right]-$$
$$-[h(x)]^{\mathrm{T}}Qh(x) \tag{9}$$

The above is a Zubov equation of the form

$$\frac{\partial v_{N+1}}{\partial x}(x)F_N(x)=-Q_N(x) \tag{10}$$

where

$$F_N(x)=f(x)-\frac{1}{2}g(x)R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}} \tag{11}$$

$$Q_N(x)=[h(x)]^{\mathrm{T}}Qh(x)+$$
$$+\left[\frac{1}{2}R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}}\right]^{\mathrm{T}}R\left[\frac{1}{2}R^{-1}[g(x)]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}}\right] \tag{12}$$

Thus, at each step of the iteration, the Zubov equation (10) must be solved, whose right hand side is a negative definite function.

The reader is reminded that the Zubov partial differential equation

$$\frac{\partial v}{\partial x}(x)F(x)=-Q(x) \tag{13}$$

where $Q(x)$ a positive definite function, has been studied extensively in the literature, primarily as the means of generating Lyapunov functions [6]. It is known ([9],[12]) that if $\frac{\partial F}{\partial x}(0)$ is Hurwitz, the Zubov equation (13) admits a unique solution $v(x)$ in a neighborhood of the origin and that the solution is locally positive definite. Moreover, the solution $v(x)$ can be represented as $v(x)=\int_0^\infty Q\left(\Phi_F(t,x)\right)dt$, where $\Phi_F(t,x)$ is the flow of $F(x)$.

We see, therefore, that the inverse of the Fréchet derivative operator $\left[\mathcal{N}'(v_N)\right]^{-1}$ involved in the Newton-Kantorovich iteration is implicitly defined through the solution of Zubov equation (10). As long as this inverse is well-defined and bounded at the initial condition of the iteration, and the initial condition is sufficiently close to the true solution, Kantorovich's theorem guarantees convergence of the iteration.

IV. Specific Convergence Properties

Consider the Newton-Kantorovich iteration for the Hamilton-Jacobi-Bellman equation, developed in the previous section. In a more compact form, it can be written as follows:

$$\begin{aligned}&\frac{\partial v_{N+1}}{\partial x}\left[f(x)-g(x)k_N(x)\right]=\\&=-\left\{\left[h(x)\right]^{\mathrm{T}}Q\,h(x)+\left[k_N(x)\right]^{\mathrm{T}}R\,k_N(x)\right\},\quad N=1,2,\ldots\end{aligned} \tag{14}$$

where

$$k_N(x)=\frac{1}{2}R^{-1}\left[g(x)\right]^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}(x)\right]^{\mathrm{T}} \tag{15}$$

Convergence of the algorithm is guaranteed by Kantorovich's theorem as long as the iteration is appropriately initialized, so that

i) the initial condition $v_1(x)$ is such that the Zubov equation (14) is solvable
ii) the initial condition $v_1(x)$ is sufficiently close to the solution $V(x)$ of (3)

In the present section, we will prove some stronger properties that characterize the progress of the Newton-Kantorovich iteration for the Hamilton-Jacobi-Bellman equation. These will lead to a proof of convergence of the Newton-Kantorovich iteration, which will be pointwise in x, as opposed to norm-convergence guaranteed by Kantorovich's theorem.

A standing assumption will be made in this section regarding the initialization of the iteration (14)-(15):

$k_1(x)=\frac{1}{2}R^{-1}\left[g(x)\right]^{\mathrm{T}}\left[\frac{\partial v_1}{\partial x}(x)\right]^{\mathrm{T}}$ *is such that the dynamics* $\dot{x}=f(x)-g(x)k_1(x)$ *is locally asymptotically stable.*

Property 1:

i) The dynamics $\dot{x}=f(x)-g(x)k_N(x)$ *is locally asymptotically stable* $\forall N\in\mathbb{N}$
ii) $v_{N+1}(x)$ *is locally positive definite* $\forall N\in\mathbb{N}$

Proof: Assuming property holds for N, we will show that it holds for $N+1$.

Define $k_{N+1}(x)=\frac{1}{2}R^{-1}\left[g(x)\right]^{\mathrm{T}}\left[\frac{\partial v_{N+1}}{\partial x}(x)\right]^{\mathrm{T}}$, where $v_{N+1}(x)$ is the solution of

$$\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)=-\left\{h^{\mathrm{T}}Q\,h+k^{\mathrm{T}}R\,k_N\right\}$$

Since $\dot{x}=f(x)-g(x)k_N(x)$ has been assumed to be locally asymptotically stable, the solution $v_{N+1}(x)$ to the above Zubov equation is locally positive definite.

Now we form

$$\frac{\partial v_{N+1}}{\partial x}\left(f-gk_{N+1}\right)=\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)+\frac{\partial v_{N+1}}{\partial x}g\left(k_N-k_{N+1}\right)=$$
$$=-\left\{h^{\mathrm{T}}Qh+k_N^{\mathrm{T}}Rk_N\right\}+\frac{\partial v_{N+1}}{\partial x}g\left(k_N-k_{N+1}\right)=$$
$$=-\left(h^{\mathrm{T}}Qh+\left(k_N-k_{N+1}\right)^{\mathrm{T}}R\left(k_N-k_{N+1}\right)+k_{N+1}^{\mathrm{T}}2R\left(k_N-k_{N+1}\right)\right)+$$
$$+\frac{\partial v_{N+1}}{\partial x}g\left(k_N-k_{N+1}\right)$$

Since $k_{N+1}^{\mathrm{T}}=\frac{1}{2}\frac{\partial v_{N+1}}{\partial x}gR^{-1}\Leftrightarrow k_{N+1}^{\mathrm{T}}2R=\frac{\partial v_{N+1}}{\partial x}g$ it follows that

$$\frac{\partial v_{N+1}}{\partial x}\left(f-gk_{N+1}\right)=$$
$$=-\left(h^{\mathrm{T}}Qh+k_{N+1}^{\mathrm{T}}Rk_{N+1}+\left(k_N-k_{N+1}\right)^{\mathrm{T}}R\left(k_N-k_{N+1}\right)\right)$$

(negative definite)

which proves that the locally positive definite function $v_{N+1}(x)$ is a Lyapunov function for the dynamics $\dot{x}=f(x)-g(x)k_{N+1}(x)$. □

<u>Property 2:</u> *$\left(v_N(x)-v_{N+1}(x)\right)$ is locally positive definite, for* $N=2,3,\ldots$

Proof: The equation

$$\frac{\partial v_N}{\partial x}\left(f-gk_{N-1}\right)=-\left\{h^{\mathrm{T}}Qh+k_{N-1}^{\mathrm{T}}Rk_{N-1}\right\}$$

upon rearrangement, can be written as

$$\frac{\partial v_N}{\partial x}\left(f-gk_N\right)-\left\{\frac{\partial v_N}{\partial x}g-k_N^{\mathrm{T}}2R\right\}\left(k_{N-1}-k_N\right)=$$
$$=-\left\{h^{\mathrm{T}}Qh+k_N^{\mathrm{T}}Rk_N+\left(k_{N-1}-k_N\right)^{\mathrm{T}}R\left(k_{N-1}-k_N\right)\right\}$$

But $\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)=-\left\{h^{\mathrm{T}}Qh+k_N^{\mathrm{T}}Rk_N\right\}$ and
$k_N^{\mathrm{T}}=\frac{1}{2}\frac{\partial v_N}{\partial x}gR^{-1}$

Thus

$$\frac{\partial v_N}{\partial x}\left(f-gk_N\right)=\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)-\left(k_{N-1}-k_N\right)^{\mathrm{T}}R\left(k_{N-1}-k_N\right)$$

or

$$\frac{\partial}{\partial x}\left(v_N-v_{N+1}\right)\left(f-gk_N\right)=-\left(k_{N-1}-k_N\right)^{\mathrm{T}}R\left(k_{N-1}-k_N\right)$$

Since, by Property 1, $\dot{x}=f(x)-g(x)k_N(x)$ is locally asymptotically stable, the solution $\left(v_N(x)-v_{N+1}(x)\right)$ to the above Zubov PDE is locally positive definite. □

<u>Property 3:</u> *Let $V(x)$ be the positive definite solution of the Hamilton-Jacobi-Bellman equation (3). Then $\left(v_{N+1}(x)-V(x)\right)$ is locally positive definite $\forall N\in\mathbb{N}$.*

Proof: Setting $k^*(x)=\frac{1}{2}R^{-1}\left[g(x)\right]^{\mathrm{T}}\left[\frac{\partial V}{\partial x}(x)\right]^{\mathrm{T}}$, equation (3) is rearranged to

$$\frac{\partial V}{\partial x}\left(f-gk^*\right)=-\left\{h^{\mathrm{T}}Qh+k^{*\mathrm{T}}Rk^*\right\}.$$

Moreover, we know that

$$\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)=-\left\{h^{\mathrm{T}}Qh+k_N^{\mathrm{T}}Rk_N\right\}$$

The above two PDEs can be rearranged as follows:

$$\frac{\partial V}{\partial x}\left(f-gk_N\right)+\frac{\partial V}{\partial x}g\left(k_N-k^*\right)=-\left\{h^{\mathrm{T}}Qh+k^{*\mathrm{T}}Rk^*\right\}$$

$$\frac{\partial v_{N+1}}{\partial x}\left(f-gk_N\right)=$$
$$-\left\{h^{\mathrm{T}}Qh+k^{*\mathrm{T}}Rk^*+\left(k_N-k^*\right)^{\mathrm{T}}R\left(k_N-k^*\right)\right\}-k^{*\mathrm{T}}2R\left(k_N-k^*\right)$$

Substituting the above by sides, we obtain

$$\left(\frac{\partial v_{N+1}}{\partial x}-\frac{\partial V}{\partial x}\right)\left(f-gk_N\right)-\frac{\partial V}{\partial x}g\left(k_N-k^*\right)=$$
$$-\left(k_N-k^*\right)^{\mathrm{T}}R\left(k_N-k^*\right)-k^{*\mathrm{T}}2R\left(k_N-k^*\right)$$

and since $k^{*\mathrm{T}}=\frac{1}{2}\frac{\partial V}{\partial x}gR^{-1}$, we obtain finally

$$\frac{\partial}{\partial x}\left(v_{N+1}-V\right)\cdot\left(f-gk_N\right)=-\left(k_N-k^*\right)^{\mathrm{T}}R\left(k_N-k^*\right)$$

Since, by Property 1, $\dot{x}=f(x)-g(x)k_N(x)$ is locally asymptotically stable, the solution $\left(v_{N+1}-V\right)$ to the above Zubov PDE is locally positive definite. □

Corollary from Properties 2 and 3:

$V(x) \leq \ldots \leq v_{N+1}(x) \leq v_N(x) \leq \ldots \leq v_2(x)$ *for every* x,

in a neighborhood of the origin.

Property 4: The sequence $v_N(x)$ *is convergent pointwise in* x*, in a neighborhood of the origin.*

Proof: By the above corollary, the sequence $v_N(x)$ is monotonic and bounded for every x. Therefore, it is convergent pointwise in x. □

Finally, denote by $v_\infty(x)$ the limit of the sequence $v_N(x)$, i.e. the limit of the recursion

$$\frac{\partial v_{N+1}}{\partial x}\left(f-\frac{1}{2}gR^{-1}g^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}\right]^{\mathrm{T}}\right)= =-\left\{h^{\mathrm{T}}Qh+\frac{1}{4}\frac{\partial v_N}{\partial x}gR^{-1}g^{\mathrm{T}}\left[\frac{\partial v_N}{\partial x}\right]^{\mathrm{T}}\right\}$$

As long as $v_\infty(x)$ is continuously differentiable, it follows that it will satisfy

$$\frac{\partial v_\infty}{\partial x}\left(f-\frac{1}{2}gR^{-1}g^{\mathrm{T}}\left[\frac{\partial v_\infty}{\partial x}\right]^{\mathrm{T}}\right)= =-\left\{h^{\mathrm{T}}Qh+\frac{1}{4}\frac{\partial v_\infty}{\partial x}gR^{-1}g^{\mathrm{T}}\left[\frac{\partial v_\infty}{\partial x}\right]^{\mathrm{T}}\right\}$$

or

$$\frac{\partial v_\infty}{\partial x}f-\frac{1}{4}\frac{\partial v_\infty}{\partial x}gR^{-1}g^{\mathrm{T}}\left[\frac{\partial v_\infty}{\partial x}\right]^{\mathrm{T}}+h^{\mathrm{T}}Qh=0$$

i.e. it will be a solution of the Hamilton-Jacobi-Bellman equation (3).

V. ILLUSTRATIVE EXAMPLE

Consider the dynamic system

$$\frac{dx_1}{dt}=\frac{x_1}{2}\left(1+x_1^2+x_2^2-\frac{1}{1+x_1^2+x_2^2}\right)+x_2+u$$
$$\frac{dx_2}{dt}=-x_1 \tag{16}$$
$$y=x_1$$

and the performance index

$$J=\int_0^\infty\left(x_1^2+u^2\right)dt \tag{17}$$

The associated Hamilton-Jacobi-Bellman equation (3) for this problem is

$$\frac{\partial V}{\partial x_1}(x_1,x_2)\left[\frac{x_1}{2}\left(1+x_1^2+x_2^2-\frac{1}{1+x_1^2+x_2^2}\right)+x_2\right]+ +\frac{\partial V}{\partial x_2}(x_1,x_2)\left(-x_1\right)-\frac{1}{4}\left[\frac{\partial V}{\partial x_1}(x_1,x_2)\right]^2+x_1^2=0 \tag{18}$$

The positive definite solution to the above equation can be expressed in closed form. It is

$$V(x_1,x_2)=x_1^2+x_2^2+\frac{1}{2}\left(x_1^2+x_2^2\right)^2 \tag{19}$$

The Newton-Kantorovich iteration developed in the previous section will now be applied to solve the above equation and the results will be compared to the exact solution.

Equations (14)-(15) take the form:

$$\frac{\partial v_{N+1}}{\partial x_1}(x_1,x_2)\left[\frac{x_1}{2}\left(1+x_1^2+x_2^2-\frac{1}{1+x_1^2+x_2^2}\right)+x_2-k_N(x_1,x_2)\right]- -\frac{\partial v_{N+1}}{\partial x_2}(x_1,x_2)\cdot x_1=-\left\{x_1^2+\left[k_N(x_1,x_2)\right]^2\right\} \tag{20}$$

where

$$k_N(x_1,x_2)=\frac{1}{2}\frac{\partial v_N}{\partial x_1}(x_1,x_2) \tag{21}$$

The Newton-Kantorovich iteration is initialized at $v_1(x_1,x_2)=x_1^2+x_2^2$, $k_1(x_1,x_2)=x_1$.
At each step of the iteration, the Zubov equation is solved via the power series algorithm of [5] up to 25th order.

Figure 1 depicts $v_2(x_1,x_2)$, $v_3(x_1,x_2)$, $v_4(x_1,x_2)$ together with the exact solution $V(x_1,x_2)$ of (18) over the rectangle $\Omega=\left\{-0.5\leq x_1\leq 0.5, -0.5\leq x_2\leq 0.5\right\}$. Table I provides their point values over a rectangular grid of mesh 0.25. From Table I, one can observe the sandwich property (Corollary from Properties 2 and 3) and also that $v_4(x_1,x_2)$ essentially coincides with $V(x_1,x_2)$, which indicates that convergence has been achieved after the 3rd iteration.

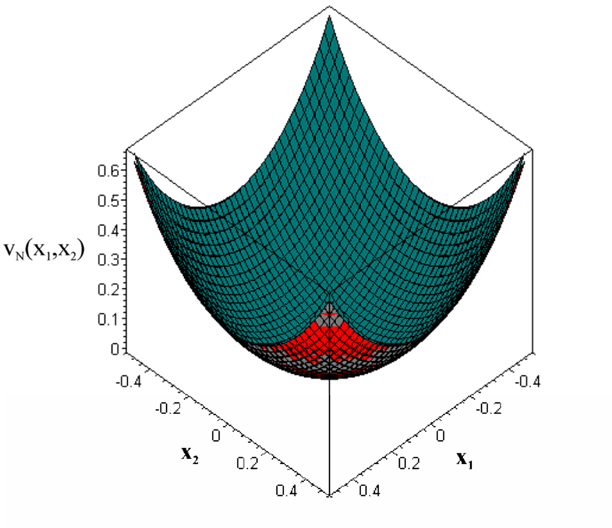

Fig. 1: Solution for $v_N(x_1,x_2)$ for different iteration numbers N = 2, 3, 4 together with the exact solution V.

TABLE I
REPRESENTATIVE VALUES OF V_N, N=1,2,3,4 AND THE EXACT SOLUTION V

	x_1 = -0.5	x_1 = -0.25	$x_1 = 0$	$x_1 = 0.25$	$x_1 = 0.5$
x_2 = -0.5	$v_1 = 0.5$	$v_1 = 0.3125$	$v_1 = 0.25$	$v_1 = 0.3125$	$v_1 = 0.5$
	$v_2 = 0.6561$	$v_2 = 0.3678$	$v_2 = 0.2844$	$v_2 = 0.3678$	$v_2 = 0.6561$
	$v_3 = 0.6266$	$v_3 = 0.3615$	$v_3 = 0.2813$	$v_3 = 0.3615$	$v_3 = 0.6266$
	$v_4 = 0.6250$	$v_4 = 0.3613$	$v_4 = 0.2813$	$v_4 = 0.3613$	$v_4 = 0.6250$
	$V = 0.6250$	$V = 0.3613$	$V = 0.2813$	$V = 0.3613$	$V = 0.6250$
x_2 = -0.25	$v_1 = 0.3125$	$v_1 = 0.1250$	$v_1 = 0.0625$	$v_1 = 0.1250$	$v_1 = 0.3125$
	$v_2 = 0.3678$	$v_2 = 0.1332$	$v_2 = 0.0645$	$v_2 = 0.1332$	$v_2 = 0.3678$
	$v_3 = 0.3615$	$v_3 = 0.1328$	$v_3 = 0.0645$	$v_3 = 0.1328$	$v_3 = 0.3615$
	$v_4 = 0.3613$	$v_4 = 0.1328$	$v_4 = 0.0645$	$v_4 = 0.1328$	$v_4 = 0.3613$
	$V = 0.3613$	$V = 0.1328$	$V = 0.0645$	$V = 0.1328$	$V = 0.3613$
$x_2 = 0$	$v_1 = 0.25$	$v_1 = 0.0625$	$v_1 = 0$	$v_1 = 0.0625$	$v_1 = 0.25$
	$v_2 = 0.2844$	$v_2 = 0.0645$	$v_2 = 0$	$v_2 = 0.0645$	$v_2 = 0.2844$
	$v_3 = 0.2813$	$v_3 = 0.0645$	$v_3 = 0$	$v_3 = 0.0645$	$v_3 = 0.2813$
	$v_4 = 0.2813$	$v_4 = 0.0645$	$v_4 = 0$	$v_4 = 0.0645$	$v_4 = 0.2813$
	$V = 0.2813$	$V = 0.0645$	$V = 0$	$V = 0.0645$	$V = 0.2813$
$x_2 = 0.25$	$v_1 = 0.3125$	$v_1 = 0.1250$	$v_1 = 0.0625$	$v_1 = 0.1250$	$v_1 = 0.3125$
	$v_2 = 0.3678$	$v_2 = 0.1332$	$v_2 = 0.0645$	$v_2 = 0.1332$	$v_2 = 0.3678$
	$v_3 = 0.3615$	$v_3 = 0.1328$	$v_3 = 0.0645$	$v_3 = 0.1328$	$v_3 = 0.3615$
	$v_4 = 0.3613$	$v_4 = 0.1328$	$v_4 = 0.0645$	$v_4 = 0.1328$	$v_4 = 0.3613$
	$V = 0.3613$	$V = 0.1328$	$V = 0.0645$	$V = 0.1328$	$V = 0.3613$
$x_2 = 0.5$	$v_1 = 0.5$	$v_1 = 0.3125$	$v_1 = 0.25$	$v_1 = 0.3125$	$v_1 = 0.5$
	$v_2 = 0.6561$	$v_2 = 0.3678$	$v_2 = 0.2844$	$v_2 = 0.3678$	$v_2 = 0.6561$
	$v_3 = 0.6266$	$v_3 = 0.3615$	$v_3 = 0.2813$	$v_3 = 0.3615$	$v_3 = 0.6266$
	$v_4 = 0.6250$	$v_4 = 0.3613$	$v_4 = 0.2813$	$v_4 = 0.3613$	$v_4 = 0.6250$
	$V = 0.6250$	$V = 0.3613$	$V = 0.2813$	$V = 0.3613$	$V = 0.6250$

Table II compares the $C^1(\Omega)$ norms $\| v_N(x_1,x_2) - V(x_1,x_2) \|$ for N = 1,2,3,4. The results indicate fast convergence of $v_N(x_1,x_2)$ to the exact solution $V(x_1,x_2)$.

TABLE II

N	$\| v_N - V \|$
1	0.5
2	0.2143
3	0.0177
4	0.0001

REFERENCES

[1] E. G. Al'Brekht, "On the optimal stabilization of nonlinear systems", *Journal of Applied Mathematics and Mechanics*, vol. 25, pp. 836-844, 1961.

[2] A. E. Bryson Jr. and Y. Ho, *Applied Optimal Control: Optimization, Estimation, and Control.* New York: Halsted Press, 1975.

[3] W. L. Garrard, and J. M. Jordan, "Design of Nonlinear Automatic Flight Control Systems", *Automatica*, vol. 13, pp. 497-505, 1977.

[4] L. V. Kantorovich, and , G. P. Akilov, *Functional Analysis in Normed Spaces*. New York: Macmillan, 1964.

[5] N. Kazantzis, C. Kravaris, C.Tseronis, and R. A. Wright "Optimal controller tuning for nonlinear processes," *Automatica*, vol. 41, no.1, pp. 79-86, Jan. 2005.

[6] H. K. Khalil, *Nonlinear Systems, 3rd edition*. New Jersey: Prentice Hall, Englewood Clifis, 2001.

[7] E. B. Lee, and L. Markus, *Foundations of Optimal Control Theory* New York: John Wiley, 1967.

[8] D. L. Lukes, "Optimal regulation of nonlinear dynamical systems" *SIAM J. Control*, vol. 7, no. 1, pp. 75–100, Feb. 1969.

[9] S. G. Margolis and W. G. Vogt, "Control engineering applications of V. I. Zubov's construction procedure for Lyapunov functions", *IEEE Trans. Autom. Contr*, vol. 8, pp. 104, 1963.

[10] A. Pappas and C. Kravaris, "A Newton-Kantorovich Iteration for the Solution of the Hamilton-Jacobi-Bellman Equation", in *Proc. 6th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2004)*, Stuttgart, Germany, Sept. 2004.

[11] A. J. van der Schaft, "L_2-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback H_∞ Control", *IEEE Transactions on Automatic Control*, vol. 37, pp. 770-783, 1992.

[12] V. I. Zubov, *Methods of A.M. Lyapunov and their Applications.* Groningen: Noordhoff LTD, 1964.