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Robustness Properties and Output Feedback of Optimization Based Sampled-data Open-loop Feedback

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Abstract— **This paper investigates the inherent robustness properties and the output-feedback problem of sampled-data open-loop feedback control. One typical sampled-data open-loop feedback scheme is nonlinear model predictive control. Specifically, the influence on the stability of the closed loop for model-plant mismatch, exogenous disturbances, unknown delays, numerical errors, and state estimation errors is examined. It is shown that if the decreasing function of the sampled-data open-loop feedback is continuous, then the closed-loop possesses inherent robustness properties. As one specific application of the derived result conditions for the semi-regional practical output-feedback stabilization via observer-based state-feedback are derived.**

I. Introduction

In this paper we investigate the inherent robustness properties and the output-feedback problem for sampled-data open-loop feedback control. Furthermore, we outline the implication of these results to nonlinear model predictive control.

Since the advent of microprocessors the control of continuous time systems using sampled-data inputs has become increasingly important. By now most practically employed controllers are implemented in discrete time using microprocessors. Typically, the interconnection between the discrete and continuous time is achieved using suitable A/D and D/A converters (often referred to as sampler and zero-order holds), compare Figure 1. Controlling a continuous

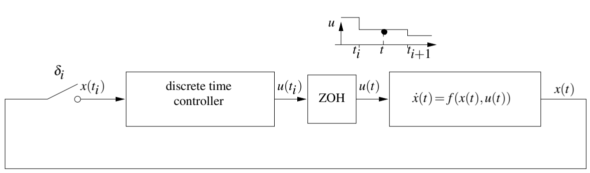

Fig. 1. Sampled-data state-feedback.

time systems using a discrete time feedback is classically referred to as sampled-data control [2, 6, 18]. Over the last decades many results concerning sampled-data control of continuous time systems have been derived, see [6, 20, 32, 33] and references therein.

In comparison to classical sampled-data control for continuous time systems we consider a slightly modified problem. We do not consider that the applied input is sampled/kept constant in between the recalculation instants, compare Figure 2. There are several reasons for not considering to sample-and-hold the input in between the recalculation times. Firstly microprocessors and A/D and D/A converters are becoming faster and faster. Frequently, the speed of the A/D and D/A converters/microprocessors are not the limiting factors for practical implementations anymore, at least for control problems typically encountered in the process industry. Rather than the speed of the A/D and D/A converters, typically slow state "measurements" are key limiting factors. Slow state measurements might for example be due to slow sensors such as concentration measurements, or due to the required extraction of the state information from secondary measurements involving for example computationally intense image processing. Furthermore, the recalculation time might be, as for example in the case of nonlinear model predictive control (NMPC), dictated by the time required to solve a computationally expensive optimal control problem. Typically, the sampling time of the process control system, at which the A/D and D/A converters operate, is in the order of milli- or even micro-seconds, whereas the recalculation time and availability of sensor measurements might be in the order of seconds. If in this case the input is kept constant in between recalculation instants, the achievable performance can degrade significantly. One possibility to overcome this problem is to open-loop apply an input signal obtained at the recalculation time t_i. Even so the D/A converters/sample-and-hold elements will lead to an approximation error of the open-loop input, these effects can often be neglected

Based on recent nominal stability results for sampled-data open-loop feedback control [14], we analyze the influence of model-plant mismatch, exogenous disturbances, unknown delays, numerical errors, and state estimation errors on the stability of the closed-loop. Is it still possible to achieve stability and good performance, at least in the case of small disturbances? Also, what type of performance and stability can be expected, if the disturbances are persistent? As shown, this question is closely connected to the continuity of the decreasing/Lyapunov function. The results are of practical interest as they underpin that small disturbances, for example due to model-plant mismatch or numerical errors, can be tolerated. Based on the inherent stability results we furthermore derive conditions for the semi-regional practical output-feedback stabilization via observer-based state-feedback.

The paper is structured as follows: In Section II we state

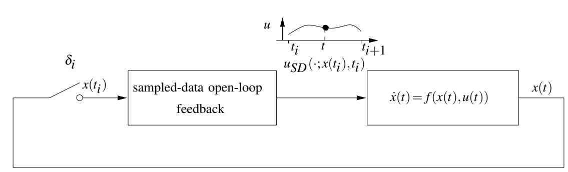

Fig. 2. Sampled-data open-loop feedback.

the nominal problem setup and review the nominal stability result obtained in [14]. The inherent robustness properties of sampled-data open-loop are examined in Section III. Section IV presents results related to the observer based output-feedback sampled-data state-feedback problem, before some final conclusions are drawn in Section V.

II. Setup and Nominal Stability

We consider time-invariant nonlinear systems given by

$$\dot{x}(t) = f(x(t),u(t)), \qquad t \geq 0,\ x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, and $u(t) \in \mathbb{R}^m$ denotes the input. With respect to the vector field f we assume that:

Assumption 1: The vector field $f : \mathscr{X} \times \mathscr{U} \to \mathbb{R}^n$ is continuous in u and locally Lipschitz in x. Furthermore, $f(0,0) = 0$. Here $\mathscr{X} \subseteq \mathbb{R}^n$ denotes the set of feasible states and the compact set $\mathscr{U}$ with $\mathscr{U} \subseteq \mathbb{R}^m$ denotes the set of feasible inputs. We assume, that

Assumption 2: $(0,0) \in \mathscr{X} \times \mathscr{U}$.

We consider the stabilization of (1) under sampled-data open-loop feedback of the form:

$$u(t) = u_{SD}(t;x(t_i),t_i). \tag{2}$$

Here u_{SD} denotes the open-loop input trajectory of the sampled-data feedback controller given by a so called admissible input generator as defined later. The input signal is based on the state information $x(t_i)$ at the recalculation instant t_i. The recalculation times are assumed to be given by a partition π of the time axis:

Definition 1: (Partition) A partition is a series $\pi = (t_i)$, $i \in \mathbb{N}$ of (finite) positive real numbers such that $t_0 = 0$, $t_i < t_{i+1}$ and $t_i \to \infty$ for $i \to \infty$.

Whenever t and t_i occur together, t_i should be taken as the closest previous sampling instant with $t_i < t$. We refer in the following to an admissible input generator as

Definition 2: (*Admissible input generator*) An input generator is called admissible with respect to the sets $\mathscr{X}_0 \subseteq \mathscr{X} \subseteq \mathbb{R}^n$, $\mathscr{U} \subseteq \mathbb{R}^m$, and a partition π, if for any $x \in \mathscr{X}_0$ and any $t_i \in \pi$:

1) $u_{SD}(\cdot;x,t_i) \in \mathscr{L}^\infty([t_i,t_{i+1}],\mathscr{U})$
2) the solution $x(\cdot;x(t_i),u_{SD}(\cdot;x(t_i),t_i))$ of (1) under the input u_{SD} starting from $x(t_i)$ is absolutely continuous on $[t_i,t_{i+1})$ with
   a) $x(\tau;x(t_i),u_{SD}(\cdot;x(t_i),t_i)) \in \mathscr{X}\ \ \forall \tau \in [t_i,t_{i+1})$
   b) $x(t_{i+1};x(t_i),u_{SD}(\cdot;x(t_i),t_i)) \in \mathscr{X}_0$.

Here $\mathscr{L}^\infty([a,b],\mathscr{U})$ denotes Lebesgue measurable and essentially bounded functions mapping from $[a,b]$ into the admissible input set $\mathscr{U}$ (a.e.). In other words, a feasible input generator maps from an initial state inside the set $\mathscr{X}_0$ and a sampling instant t_i to an input for $[t_i,t_{i+1})$ that is measurable, satisfies the input constraints almost everywhere (besides a number of points with measure zero), keeps the state inside of the allowed set of states $\mathscr{X}$, and (at least) renders the set $\mathscr{X}_0$ invariant at the recalculation instants.

Remark 1: (Admissible input generators) One question occurring is, whether there are any controller designs available that can provide for a single state measurement an input signal rather then a "fixed" input value? One classical example is optimal control. Further examples are sampled-data open-loop NMPC [16]or open-loop input generators as outlined in [1, 26], which might for example be based on differential flatness or other structural considerations. Furthermore, as shown in [14], any stabilizing instantaneous feedback can be used to obtain suitable open-loop input trajectories by feed forward simulation.

A. Nominal Stability and Sampled-data Open-loop Feedback

With respect to the feedback u_{SD} we assume that it stabilizes the origin of the nominal system with a region of attraction $\mathscr{R} \subseteq \mathscr{X}$, $0 \in \mathscr{R}$, and that a Lipschitz assumption on the corresponding decreasing function is satisfied. In the spirit of the nominal stability results presented in[14] this is guaranteed provided that the following assumptions hold:

Assumption 3: (Nominal stability)

1) The input generator u_{SD} is admissible with respect to a set $\mathscr{R}$, the input and state constraint sets $\mathscr{U}$, $\mathscr{X}$, and the partition π.
2) There exists a locally Lipschitz continuous positive definite function $\alpha : \mathscr{R} \to \mathbb{R}^+$ and a continuous positive definite function $\beta : \mathscr{R} \to \mathbb{R}^+$, such that for all $t_i \in \pi$, $x(t_i) \in \mathscr{R}$ and $\tau \in [0, t_{i+1} - t_i)$
   a) $$\alpha(x(t_i+\tau;x(t_i),u_{SD}(\cdot;x(t_i),t_i))) - \alpha(x(t_i)) \leq -\int_{t_i}^{t_i+\tau} \beta(x(s;x(t_i),u_{SD}(\cdot;x(t_i),t_i)))ds. \tag{3}$$
   b) For all compact strict subsets $\mathscr{S} \subset \mathscr{R}$ there is at least one compact sub-level set $\Omega_c = \{x \in \mathscr{R} | \alpha(x) \leq c\}$ s.t. $\mathscr{S} \subset \Omega_c$.

We denote α in the following as decreasing or Lyapunov like function. Given that Assumption 1- 3 hold, the following theorem can be established:

Theorem 1: Assume that Assumption 1- 3 hold. Then for all $x(0) \in \mathscr{X}_0$: 1.) The solution of (1) subject to (2) exists for all times. 2.) The input and state constraints are satisfied. 3.) $x(t_i) \in \mathscr{X}_0\ \forall t_i \in \pi$. 4.) $\|x(t)\| \to 0$ as $t \to \infty$.

For a proof of this result we refer to [13, 14]. Thus, under the given conditions the sampled-data open-loop feedback u_{SD} nominally stabilizes (in the sense of convergence) the origin. As shown in [14] the Lipschitz assumption on the decreasing function α and the existence of compact level sets Ω_c is actually not necessary to achieve the desired robustness properties. Rather we assume it here to derive the robustness results for sampled-data open-loop feedback. This requirement is in correspondence with recent results on the stability and robustness of discontinuous feedbacks with sample-and-hold [22, 23].

III. Inherent Robustness

The reviewed results only establish nominal stability. In reality, however, model-plant mismatch, exogenous disturbances, unknown delays, numerical errors, and state estimation errors are present. Analyzing the influence of such unknown disturbances is important since the state information is only fed back at the recalculation times, i.e. the controller cannot immediately react to disturbances.

Note that we do not consider the design of robustly stabilizing controllers. Rather we analyze the inherent robustness properties of sampled-data open-loop feedback. Especially, we show that sampled-data open-loop feedback possesses inherent robustness properties if the decreasing function is locally Lipschitz.

The derived results are related to robustness results for discrete time systems [35] as well as to results on sampled-data feedback considering sample-and-hold elements for the input [7, 23].

Remark 2: We consider persistent disturbances and the repeated application of open-loop inputs, i.e. we cannot react instantaneously to disturbances. Thus, asymptotic stability cannot be achieved, and the nominal region of attraction $\mathscr{R}$ can in general not be rendered invariant under disturbances. As a consequence, we desire only "ultimate boundedness" results, i.e. we desire that the norm of the state after some time becomes small. Furthermore, we show that the bound can be made arbitrarily small depending on the bound on the disturbance and the sampling time (*practical stability*), and that the region where this holds can be made an arbitrarily inner approximation with respect to $\mathscr{R}$ (*semi-regional*). In view of Assumption 3 and for simplicity of presentation, we parameterize these regions with level sets.

Specifically, we derive bounds for the maximum allowable disturbance and sampling time that ensure that the state converges from any arbitrary level set of initial conditions $\Omega_{c_0} \subset \mathscr{R}$ in finite time to an arbitrary small set Ω_γ around the origin without leaving a desired set $\Omega_c \subset \mathscr{R}$, compare Figure 3. Certainly, the maximum allowable disturbance

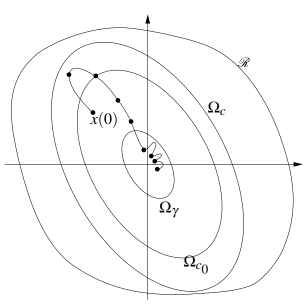

Fig. 3. Set of initial conditions Ω_{c_0}, maximum attainable set Ω_c, desired region of convergence Ω_γ and nominal region of attraction $\mathscr{R}$.

depends on the size of the region of convergence Ω_γ and on the "distance" between Ω_c and Ω_{c_0}.

The results are based on the observation that small disturbances and model uncertainties lead to a (small) difference between the nominal open-loop state and the real state. The influence of the disturbance on the decreasing function α can be bounded by

$$\alpha(x(t_{i+1}))-\alpha(x(t_i)) \leq -\int_{t_i}^{t_{i+1}} \beta(x(\tau;x(t_i),u_{SD}(\cdot;x(t_i),t_i)))d\tau + \varepsilon(t_i), \quad (4)$$

where ε corresponds to the "disturbance contribution". Thus, if the disturbance contribution ε "scales" with the size of the disturbance, one can achieve contraction of the level sets, at least at the recalculation instants. Since the integral contribution in (4) depends on the distance of the system state to the origin, while the disturbance contribution mainly depends on the size of the disturbances, the decrease cannot continue until reaching the origin, i.e. in general only practical stability can be achieved. For the results we need the function $\Delta\alpha_{\min}(c,\gamma)$ which is defined as:

Definition 3: ($\Delta\alpha_{\min}$) For any $c > \gamma > 0$ with $\Omega_c \subset \mathscr{R}$, the value of $\Delta\alpha_{\min}(c,\gamma)$ is defined as

$$\Delta\alpha_{\min}(c,\gamma) = \min_{\substack{x_0 \in \Omega_c/\Omega_\gamma \\ t_i \in \pi}} \int_{t_i}^{t_{i+1}} \beta(\bar{x}(s;x_0,u_{SD}(\cdot;x_0,t_i)))ds. \quad (5)$$

Here $\bar{x}$ is the state of the nominal system under the nominal sampled-data open-loop feedback, i.e.

$$\dot{\bar{x}}(s) = f(\bar{x}(s),u_{SD}(s;x(t_i),t_i)), \qquad s \in [t_i,t_{i+1}],\ \bar{x}(t_i) = x_0.$$

Note that for any $c > \gamma > 0$ with $\Omega_c \subset \mathscr{R}$, $\Delta\alpha_{\min}(c,\gamma)$ is nontrivial and finite. In general it is difficult to obtain an explicit expression or even a good lower bound for $\Delta\alpha_{\min}$.

A. Additive Disturbances

We first examine the robustness with respect to additive disturbances. Specifically, we consider additive disturbances of the form:

$$\dot{x}(t) = f(x(t),u_{SD}(t;x(t_i),t_i)) + p(t). \quad (6)$$

All appearing disturbances and model-plant uncertainties are lumped in the disturbance term p. With respect to the additive disturbance p we can derive the following result

Theorem 2: (Additive disturbances) Given arbitrary level sets $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathscr{R}$ and assume that Assumptions 1-3 hold. Then, there exists a constant $p_{\max} > 0$, such that for any disturbance satisfying for all $t_i \in \pi$

$$\left\| \int_{t_i}^{t_i+\tau} p(s)ds \right\| \leq p_{\max}\tau, \quad \tau \in [0, t_{i+1}-t_i], \quad (7)$$

the trajectories of the disturbed system for any $x_0 \in \Omega_{c_0}$

$$\dot{x}(t) = f(x(t),u_{SD}(t;x(t_i),t_i)) + p(t), \qquad x(0) = x_0, \quad (8)$$

exist for all times, will not leave the set Ω_c, $x(t_i) \in \Omega_{c_0}$ $\forall i \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$.

Due to space limitations we refer to [13] for the proof.

Remark 3: The bound (7) ensures existence of solutions and convergence to the set Ω_c. Examples are constant additive disturbances and time varying disturbances. Note that it is not necessary to require that the disturbance vanishes over time, since we do not desire to achieve asymptotic convergence. In general, the disturbances also depends on the state and input or sampling time. The derived result can be used in this case, if the integrability condition (7) on p holds.

Theorem 2 establishes robustness of sampled-data open-loop feedbacks with respect to small additive disturbances. The degree of robustness strongly depends on the dynamics of the system, the Lipschitz condition on the decreasing function α, and on the minimum and maximum recalculation time.

Remark 4: Calculating the robustness bound $p_{\max}$ is difficult, since it is necessary to at least know a lower bound on the minimum decrease $\Delta\alpha_{\min}(c,\gamma/4)$, see[13]. Nevertheless, the result is of value, since it underpins that small additive disturbances can be tolerated.

B. Input Disturbances

The derived results can be tailored to disturbances that directly act on the input. The consideration of disturbances acting directly on u is of interest, since this covers a series of practically important disturbances as discussed later.

To derive the results it is necessary to assume that f is locally Lipschitz in u over a compact set $\tilde{\mathscr{U}}$ which is slightly larger then $\mathscr{U}$ with $\mathscr{U} \subset \tilde{\mathscr{U}}$, since the nominal controller could use values on the boundary of $\mathscr{U}$:

Assumption 4:

The vector field $f : \mathscr{X} \times \tilde{\mathscr{U}} \to \mathbb{R}^n$ is locally Lipschitz in x and u. Furthermore, $f(0,0) = 0$.

We assume that the disturbed input is given by $u_{SD}(t;x(t_i),t_i) + v(t)$. Under the given assumptions the following theorem can be established:

Theorem 3: (Robustness with respect to input disturbances) Given arbitrary level sets $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathscr{R}$ and assume that Assumptions 2- 4 hold. Then, there exists a constant $v_{\max} > 0$ such that for any disturbance satisfying for all $t_i \in \pi$

$$\left\| \int_{t_i}^{t_i+\tau} v(s)ds \right\| \leq v_{\max}\tau, \qquad \tau \in [0, t_{i+1} - t_i], \tag{9}$$

and

$$\|v(t)\| \leq v_{\text{dist}\mathscr{U},\tilde{\mathscr{U}}}, \quad t \geq 0, \tag{10}$$

the trajectories of the disturbed system for any $x_0 \in \Omega_{c_0}$

$$\dot{x}(t) = f(x(t), u_{SD}(t;x(t_i),t_i) + v(t)), \qquad x(0) = x_0, \tag{11}$$

exist for all times, will not leave the set Ω_c, $x(t_i) \in \Omega_{c_0}$ $\forall i \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$.

For the proof we refer to [13].

Besides the important case of disturbances that directly act on the input, the derived result has a series of direct implications.

a) Numerical approximation errors: One direct implication of this result is that approximated solutions to the optimal control problem in NMPC can be tolerated, if the approximation error is sufficiently small. Such approximated solutions can for example result from the numerical integration of the differential equations, or errors due to the application of direct solution approaches for the optimal control problem in NMPC. Related arguments have been used in [9, 10] to establish stability of a NMPC scheme that employs approximated solutions of the optimal control problem.

b) Computational delays: The derived result underlines that sufficiently small computational delays can be tolerated. Since the state on which the input calculation is based on remains unchanged, it becomes immediately clear that condition (9) is satisfied if the delay is sufficiently small. In this case condition (10) vanishes, since the resulting input is only shifted in time. This result is of special interest for open-loop sampled-data NMPC, since delays will always be present even for fast calculations.

c) Neglected fast actuator dynamics: One further application of the derived result might be the question, if in the case of neglected, but fast actuator dynamics, practical stability can be guaranteed. In principle this is possible, following ideas presented in [22] for the case of sampled-data feedback with sample-and-hold elements.

C. Measurement and State Estimation Errors

In this section we consider the problem of measurement and state estimation errors. The derived result lays the basis for the output-feedback results given in Section IV. Instead of the real system state $x(t_i)$ we assume that at every sampling instant only a disturbed state $x(t_i) + e(t_i)$ is available. The disturbance $e(t_i)$ could for example be the result of measurement noise, small measurement delays, or state estimation errors. Instead of the nominal open-loop feedback the following "disturbed" feedback is applied:

$$u(t;\hat{x}(t_i)) = u_{SD}(t;x(t_i) + e(t_i), t_i), \quad t \in [t_i, t_{i+1}). \tag{12}$$

Note that only the state and disturbance $e(t_i)$ at the recalculation time is of interest for the robustness. The influence of disturbances in between recalculation times does not influence the feedback immediately.

Similar considerations as for the other disturbances lead to the following theorem:

Theorem 4: (Measurement/state estimation errors)

Given arbitrary level sets $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathscr{R}$ and assume that Assumptions 1- 3 hold. Then, there exists a constant $e_{\max} > 0$ such that for any measurement disturbance and state estimation error $e(t_i)$ satisfying for all $t_i \in \pi$

$$\|e(t_i)\| \leq e_{\max}, \tag{13}$$

the trajectories of the system for any $x_0 \in \Omega_{c_0}$

$$\dot{x}(t) = f(x(t), u_{SD}(t;x(t_i) + e(t_i), t_i)), \qquad x(0) = x_0, \tag{14}$$

exist for all times, will not leave the set Ω_c, $x(t_i) \in \Omega_{c_0}$ $\forall i \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$.

This result underpins that sufficiently small measurement/estimation errors can be tolerated to achieve stability in a practical sense. Thus, small measurement noise, but also state observation errors can be tolerated. One direct application of this result is the sampled-data open-loop feedback separation principle as outlined in the next section.

D. Robustness and NMPC

In the case of NMPC some inherent robustness results already exist [5, 25, 27, 35]. However, these results are either only valid for instantaneous NMPC [5, 25, 27], or discrete time NMPC [35], or they consider special NMPC implementations, such as dual-mode predictive control [30] or contractive predictive control formulations [8, 38].

The inherent robustness results derived in this section are also applicable to NMPC. Many NMPC approaches that guarantee stability already satisfy the decrease condition (3). However, it is in general not possible to answer the question if a given sampled-data open-loop NMPC schemes satisfies the Lipschitz condition on the decreasing function, which in the case of NMPC is the value function. This problem stems from the fact that the applied input is based on the solution of an optimal control problem, which can be discontinuous as a function of the considered state [16, 17, 28]. There are only a few NMPC schemes that guarantee that the value function is locally Lipschitz [19, 21]. Most of these do not consider constraints.

IV. SAMPLED-DATA OPEN-LOOP OUTPUT-FEEDBACK

The results on nominal stability and on inherent robustness of sampled-data open-loop feedback are based on the assumption that the full state information is available. In practical applications it is, however, often not possible to measure all states. In practice the problem of output-feedback is often "solved" according to the certainty equivalence principle, i.e. instead of the true, but unknown, system state, a state estimate provided by a state observer is used for feedback. However, since no general separation principle for nonlinear systems exists, the stability of the closed-loop cannot be deduced from the stability of the observer and the state-feedback separately. In this Section we outline conditions ensuring semi-regional practical stability of the closed-loop for a combination of of sampled-data open-loop state-feedback controllers and state observers.

The results are inspired by special nonlinear separation principles for instantaneous feedbacks employing high-gain observers, see e.g. [3, 36]. The state estimation error is basically considered as a disturbance acting on the nominal closed-loop. It is shown that if the sampled-data feedback possesses inherent robustness properties and if the observer error converges sufficiently fast, it is possible to achieve stability stability of the closed loop.

A. Setup and Stability

In addition to the state equations (1) we consider an output/measurement equation of the form

$$y(t) = h(x(t), u(t)) \tag{15}$$

where $y(t) \in \mathbb{R}^p$ are the measured outputs. The input to the system is given by a sampled-data open-loop feedback controller

$$u(t) = u_{SD}(t; \hat{x}(t_i), t_i). \tag{16}$$

Here $\hat{x}(t_i)$ is the estimated state provided by the used state observer. In the following we denote the state estimation error by $e = x - \hat{x}$.

In addition to Assumption 3 we assume that:

Assumption 5: For all $x(t_i) \notin \mathscr{R}$, $t_i \in \pi$ the sampled-data open-loop feedback u_{SD} is defined as $u_{SD}(\tau; x(t_i), t_i) = u_c, \quad \tau \in [t_i, t_{i+1}]$, where $u_c \in \mathscr{U}$ is constant.

This is necessary, since the state estimate of the observer can be outside of $\mathscr{R}$, at least in some initial phase.

On the observer we require that:

Assumption 6: (*Observer error convergence)* For any desired maximum state estimation error $e_{\max} > 0$ there exist observer parameters such that $\|x(t_i) - \hat{x}(t_i)\| \leq e_{\max}, \quad \forall t_i \geq t_{k_{\text{conv}}}$. Here $k_{\text{conv}} > 0$ is a freely chosen, but fixed number of recalculation instants after which the observer error has to satisfy the error bound.

Remark 5: Depending on the observer, further conditions on the system might be necessary (e.g. observability assumptions). Also, note that the observer does not have to operate continuously since the state information is only required at the recalculation instants t_i. Thus, it is in principle possible to apply a discrete time observer for the continuous time system, or a state estimator utilizing a certain piece of the output trajectory at once, such as moving horizon state estimation [31, 34].

In principle we follow the ideas used for inherent robustness with respect to measurement errors in the previous section, i.e. we show that if $e_{\max}$ is sufficiently small, then a decrease of the disturbed decreasing function α from recalculation time to recalculation time can be retained. However, in comparison to the previous results we must take into account that the observer requires a certain convergence time to achieve the desired maximum observer error $e_{\max}$. To avoid that the system state leaves the set Ω_c during this time it might thus be necessary to sufficiently decrease the maximum recalculation time $\bar{\pi}$. Under the given setup the following theorem holds

Theorem 5: (Semi-regional practical stability) Given level sets Ω_γ, Ω_c, and Ω_{c_0} with $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathscr{R}$. Then, under the Assumptions 2-6 there exists a maximum allowable observer error $e_{\max}$ and a maximum recalculation time $\bar{\pi}$ such that for all initial conditions $x_0 \in \Omega_{c_0}$ the state trajectories of the closed-loop satisfy $x(\tau) \in \Omega_c \ \tau \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma \ \forall \tau \geq T_\gamma$.

The proof can be found in [13], or for the special case of sampled-data NMPC in[15].

The most critical conditions for the application of the derived semi-regional practical stability result is the requirement that the observer satisfies Assumption 6. Even so this assumption is rather strong, a series of observer designs exist achieving the desired properties. Examples are high-gain observers [37], optimization based moving horizon observers with contraction constraint [31], observers possessing a linear error dynamics where the poles can be chosen arbitrarily (e.g. based on normal form considerations and output injection [4, 24]), and observers achieving finite convergence time such as sliding mode observers [11] or the approach presented in [12, 29].

V. CONCLUSIONS

Based on nominal stability results for sampled-data open-loop feedback we investigated the question of the inherent robustness with respect to external disturbances, model-plant mismatch, and measurement/state estimation errors. Specifically, we showed that that under certain continuity assumptions, sampled-data open-loop feedbacks possesses inherent robustness properties. Of practical importance are the robustness to small input uncertainties such as numerical optimization errors, the robustness to small input delays, the robustness to measurement and state estimation errors, and the robustness to neglected fast actuator and sensor dynamics. The results have direct implications to nonlinear model predictive control, since they underline that under certain conditions optimization errors, state estimation errors, and model plant mismatch can be tolerated. Analyzing the influence of such unknown disturbances is important since the state information is only fed back at the recalculation times, i.e. the controller cannot immediately react to disturbances. As a direct application of the presented results we outlined a separation principle like stability theorem for observer based state-feedback sampled-data open-loop control.

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