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H^∞ control of systems with multiple I/O delays. Part II: simplifications

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Abstract— The paper provides a solution to the standard H_∞ control problem with multiple i/o delays. The derivation is considerably simpler than previous solutions.

I. INTRODUCTION

Input/output time delays arise naturally in numerous control application, both from physical delays in processes and control interfaces and from the use of delays to model complicated high-frequency dynamics. Optimal control of time-delay systems has been an active research area since the late 60's, first in the H^2 (LQG) [1], [2] and then in the H^∞ [3], [4] settings.

Time-delay systems can in principle be treated in the framework of a general theory of infinite-dimensional systems, both in the time [5] and in the frequency [3] domains. These approaches, however, result in rather abstract results (i.e., in terms of operator Riccati equations), from which it may not be clear what the structures of solvability conditions and controllers are and how (if) they can be computed and implemented. This motivated researchers to seek for more problem-oriented approaches that exploit the special structure of the delay operator, see the review paper [4] and the references therein.

Although substantial progress has been made in this direction during the last two decades, the vast majority of the results (in both H^2 and H^∞ settings) is still limited to systems with a *single* delay. On the other hand, in MIMO systems different input/output channels can have different delays, so that multiple delay results are of great importance. Earlier treatments of multiple-delay systems either produced quite complicated solutions [2], [3] or were heavily based on the simplifying assumption that the delay operator commutes with the plant [7]. An exception to this is a recent work by Kojima and Ishijima [8], who derive explicit H^∞ solution for the case when the disturbance and/or control inputs are delayed. Yet in [8] only input delays are considered and it is assumed that the controller has access to the full plant state.

The general problem was recently solved in [6], but the machinery needed in that paper is still rather involved. The purpose of the present paper is to setup a mathematical theory in which most of the technicalities disappear. This brings about a simplification of the theory and a short-cut in the derivation of the solution presented in [6].

a) Notation: Borrowing from [11] we define the *completion* operator π_h, which completes the impulse response of an h-delay system to a delay-free system. Informally:
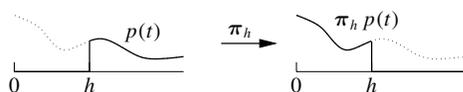
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The completion operator for delayed systems of the form $\mathrm{e}^{-sh}P = \mathrm{e}^{-sh}C(sI - A)^{-1}B$ is defined as

$$\pi_h(\mathrm{e}^{-sh}P) = \left[\begin{array}{c|c} A & B \\ \hline C\mathrm{e}^{-Ah} & 0 \end{array}\right] - \mathrm{e}^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

(for $h > 0$). This way the sum of $\mathrm{e}^{-sh}P$ and its completion $\pi_h(\mathrm{e}^{-sh}P)$ is finite dimensional.

A mapping $Q \in H^\infty$ is *contractive* if $\|Q\|_\infty < 1$. A transfer matrix Q is *bistable* if $Q, Q^{-1} \in H^\infty$.
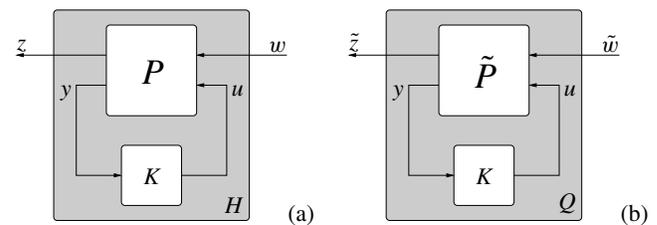
Fig. 1. Reduction of standard problem

II. PROBLEM FORMULATION

The standard H^∞ problem *without* delay, as we all know it, is to determine for a given plant P in Fig. 1(a) a stabilizing causal controller K that renders the closed loop system mapping H from w to z is contractive (or to show that none such K exists). Under the usual assumptions the problem is converted — using two Riccati equations and a coupling condition — to a simpler problem with different external signals $\tilde{w}$ and $\tilde{z}$ but of the same structure — see Fig. 1(b) — where now

- $\begin{bmatrix} \tilde{z} \\ y \end{bmatrix} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ u \end{bmatrix}$ with $\tilde{P}_{12}$ and $\tilde{P}_{21}$ invertible and biproper,
- The closed loop is stable iff the closed loop system mapping Q is stable.
- $\|Q\|_\infty < 1 \iff \|H\|_\infty < 1$.

This then solves the problem in the delay-free case because the mapping from K to Q is invertible, so *any* Q may be achieved by appropriate choice of K, and the assumptions on the plant P are typically such that $Q = 0$ is achieved for some proper or even strictly proper rational K.

This conversion of the H_∞ problem to a simpler H_∞ problem also has some bearing on the case where there are input and output delays, see Fig. 2(a). In this configuration Λ_u and Λ_y denote multiple delay operators

$$\Lambda_u(s) = \begin{bmatrix} \mathrm{e}^{-h_{u,1}s} & & & \\ & \mathrm{e}^{-h_{u,2}s} & & \\ & & \ddots & \\ & & & \mathrm{e}^{-h_{u,n_u}s} \end{bmatrix},$$

$$\Lambda_y(s) = \begin{bmatrix} \mathrm{e}^{-h_{y,1}s} & & & \\ & \mathrm{e}^{-h_{y,2}s} & & \\ & & \ddots & \\ & & & \mathrm{e}^{-h_{y,n_y}s} \end{bmatrix}.$$
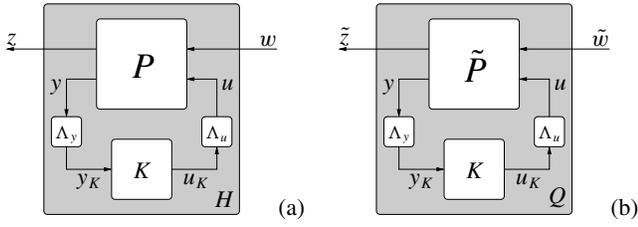
Fig. 2. Reduction of standard problem with delays

Fig. 4. Chain scattering representation

Since delay operators Λ_u and Λ_y impose constraints on the mapping $\Lambda_u K \Lambda_y$ it is clear that the H_∞ problem with delays is solvable *only if* so is the delay-free case. Hence when considering the H^∞ problem for systems with delays we may without loss of generality begin our analysis with the converted system with plant $\tilde{P}$ as in Fig. 2(b). It turns out to be useful to perform yet another, quite standard, conversion at this point: since $\tilde{P}_{12}$ and $\tilde{P}_{21}$ are invertible we may describe the closed loop of Fig. 2(b) also as in Fig. 3 where now G is the mapping from $\begin{bmatrix} u \\ y \end{bmatrix}$ to $\begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix}$. It is well known that this mapping G when derived this way from $\tilde{P}$ is actually bistable [15].
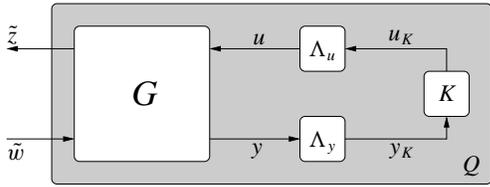
Fig. 3. A chain scattering description

From now on we will assume that $G(s)$ is bistable and that it has realization

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & I \end{array} \right]. \qquad (1)$$

Essentially the only assumption here is that the direct feedthrough matrix $G(\infty)$ equals identity, but also this can be relaxed (see [6, page 208]).

A final conversion that turns out to be useful is to combine the two delay operators Λ_u and Λ_y into a single joint delay operator Λ that maps $\begin{bmatrix} u_K \\ y_K \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$, see Fig. 4. Clearly this is achieved if we take

$$\Lambda = \begin{bmatrix} \Lambda_u & 0 \\ 0 & \Lambda_y^{-1} \end{bmatrix}$$

(mind the inverse). Due to the inverse the joint delay operator may have advance elements. However, by advancing $\begin{bmatrix} u_K \\ y_K \end{bmatrix}$ (which does not affect the controller K) the mapping Λ from $\begin{bmatrix} u_K \\ y_K \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ may be made causal, or, to put it differently, Q does not change if the joint delay operator $\Lambda(s)$ is replaced with $e^{-sT}\Lambda(s)$. Hence we may without loss of generality assume that Λ is causal, if so desired:

$$\Lambda(s) = \begin{bmatrix} e^{-h_0 s} & & & \\ & e^{-h_1 s} & & \\ & & \ddots & \\ & & & e^{-h_m s} \end{bmatrix}, \qquad h_k \ge 0. \qquad (2)$$

(It turns out to be convenient to begin with zero indexed delay h_0.)

So then we finally arrive at the H_∞ problem that we shall address in this paper:

Consider Fig. 4 and assume G is bistable with realization (1) and that Λ is as in (2). Determine all causal controllers K that render Q stable and contractive or show that no such K exists.
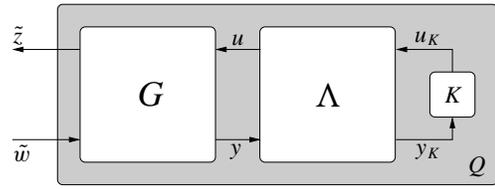
III. INTERMEZZO: DELAY OPERATORS

The main point of this paper is to set up the mathematical language in such a way that technicalities are reduced. Such an approach is particularly important for multiple-delay problems as the results tend to be rather technical. In this section we recap some easy but handy rules of calculus for multiple delay operators on some finite horizon signal space $L^2[0, T]$.

Consider the multiple delay operator $\Lambda : L^2[0, T] \to L^2[0, T]$ defined as

$$\Lambda u(t) = \begin{bmatrix} u_1(t - h_0)1_{[h_0, T]}(t) \\ u_2(t - h_1)1_{[h_1, T]}(t) \\ \vdots \\ u_m(t - h_m)1_{[h_m, T]}(t) \end{bmatrix} \qquad \text{(with } 0 \le h_j \le T\text{)}$$

where u_k denotes the kth entry of u and $1_{[a,b]}$ denotes the indicator function on $[a, b]$. The indicator function is added to avoid confusion about the extend of the domain on which u is defined.

The dual of Λ is readily seen to satisfy

$$\Lambda^* z(t) = \begin{bmatrix} z_1(t + h_0)1_{[0, T - h_0]}(t) \\ z_2(t + h_1)1_{[0, T - h_1]}(t) \\ \vdots \\ z_m(t + h_m)1_{[0, T - h_m]}(t) \end{bmatrix}. \qquad (3)$$

Now the mapping $\Lambda^* \Lambda u$ shifts u forward and then backward where each time the support of the result is clipped to $[0, T]$, so

$$\Lambda^* \Lambda u(t) = \begin{bmatrix} u_1(t)1_{[0, T - h_0]}(t) \\ u_2(t)1_{[0, T - h_1]}(t) \\ \vdots \\ u_m(t)1_{[0, T - h_m]}(t) \end{bmatrix}. \qquad (4)$$

Now something less standard. Even though $\Lambda^* \Lambda$ is not invertible, the equation

$$\Lambda^* \Lambda u = \Lambda^* z$$

for any $z \in L^2[0, T]$ always *does* have a solution $u \in L^2[0, T]$ (though not unique) which is evident from the support of $\Lambda^* \Lambda u$ and $\Lambda^* z$, cf. (3) and (4). We will denote this mapping as

$$u = (\Lambda^* \Lambda)^{-1} \Lambda^* z$$

and we want to stress that this mapping only identifies u on

$$\begin{bmatrix} [0, T - h_0] \\ \vdots \\ [0, T - h_m] \end{bmatrix}. \qquad (5)$$

Loosely speaking the nonuniqueness is that the "tail" of u is completely left unspecified by the equation $u = (\Lambda^* \Lambda)^{-1} \Lambda^* z$. If however we delay the unspecified "tail" of u enough $y := \Lambda u$ then $y \in L^2[0, T]$ *is* uniquely determined by z. That is, the mapping from z to y defined as

$$y = \Lambda(\Lambda^* \Lambda)^{-1} \Lambda^* z$$

is unique. It is such that $y_j(t) = z_j(t)$ on $[h_{j-1}, T]$ and zero elsewhere, i.e., $\Lambda(\Lambda^*\Lambda)^{-1}\Lambda^*$ is a multiplication operator

$$\Lambda(\Lambda^*\Lambda)^{-1}\Lambda^* = \begin{bmatrix} 1_{[h_0,T]} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1_{[h_m,T]} \end{bmatrix}. \quad (6)$$

IV. THE MAIN RESULT

Consider the H_∞ problem as defined at the end of Section II and depicted in Fig. 4. For the moment we restrict attention to some *finite* horizon $[0, T]$ and wonder whether there exists a causal controller K such that

$$\int_0^T \|\tilde{z}(t)\|^2 - \|\tilde{w}(t)\|^2 \, dt \le 0, \quad \forall \tilde{w}.$$

Now clearly we have from $\begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix} = G\Lambda \begin{bmatrix} u_K \\ y_K \end{bmatrix}$ that

$$\int_0^T \|\tilde{z}(t)\|^2 - \|\tilde{w}(t)\|^2 \, dt = \int_0^T \begin{bmatrix} u_K(t) \\ y_K(t) \end{bmatrix}^* \Lambda^* G^* J G \Lambda \begin{bmatrix} u_K(t) \\ y_K(t) \end{bmatrix} dt$$

where * denote adjoints for mappings on $L^2[0, T]$ and

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \quad (7)$$

Lemma IV.1. *Suppose G satisfies* (1). *Then the operator*

$$\Lambda^* G^* J G \Lambda \quad : \quad L^2[0, T] \mapsto L^2[0, T]$$

is singular in the sense that $\exists v \in L^2[0, T]$ *such that*

$$\Lambda v \ne 0, \quad \Lambda^* G^* J G \Lambda v = 0$$

iff $\det \Sigma_{22}(T) = 0$, *where J is as in* (7), *and* $\Sigma_{22}(t)$ *is the lower-right block of* $\Sigma(t)$ *defined as*

$$\dot{\Sigma}(t) = H(t)\Sigma(t), \quad \Sigma(0) = I$$

with

$$H(t) = \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix}$$
$$- \begin{bmatrix} B \\ -C'J \end{bmatrix} \Lambda(\Lambda^*J\Lambda)^{-1}\Lambda^* \begin{bmatrix} JC & B' \end{bmatrix}. \quad (8)$$

Proof. Standard duality theory shows that $\Lambda^* G^* J G \Lambda v = 0$ iff

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B \\ -C'J \end{bmatrix} \Lambda v \\ 0 = \Lambda^* \begin{bmatrix} JC & B' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \Lambda^* J \Lambda v \end{cases} \quad (9)$$

with the boundary conditions $x(0) = p(T) = 0$. The second of the two equations partly determines v on $[0, T]$ as $v = -(\Lambda^* J \Lambda)^{-1} \Lambda^* \begin{bmatrix} JC & B' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$, which when inserted in the first equation shows that state and costate x an p satisfy $\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = H(t) \begin{bmatrix} x \\ p \end{bmatrix}$. The mapping $H(t)$ is uniquely defined (even though $\Lambda^* J \Lambda$ is not invertible over $[0, T]$, see Section III). Now

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Sigma(t) \begin{bmatrix} x(0) \\ p(0) \end{bmatrix}.$$

So a nontrivial solution $\begin{bmatrix} x \\ p \end{bmatrix}$ exists for which $x(0) = p(T) = 0$ iff $\Sigma_{22}(T)$ is singular. Finally if $\begin{bmatrix} x \\ p \end{bmatrix}$ is trivial then Λv is trivial, and conversely if Λv is trivial then $\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$ which with the given boundary conditions means $\begin{bmatrix} x \\ p \end{bmatrix}$ is trivial. $\square$

Now we are in a position to formulate our main result. We want to point out here that the result in a way is available in the literature [6] but the formulation and proof below is more transparent.

Theorem IV.2. *Consider Fig. 4 and assume G satisfies* (1) *and that* Λ *is as in* (2). *Then there exists a causal K such that* $\|Q\|_\infty < 1$ *iff* $\det \Sigma_{22}(t) \ne 0$ *for all* $t \in [0, T]$, *with* $T := \max h_k$ *and* Σ *as in Lemma IV.1.*

For the moment we only prove one direction.

Proof (only if). Assume to the contrary that $\Sigma_{22}(t)$ is singular for some $t_* \in [0, T]$. In what follows all mappings and inner products are with respect to $L^2[0, t_*]$. By Lemma IV.1 this means that a nonzero $v \in L^2[0, t_*]$ exists such that

$$\Lambda^* G^* J G \Lambda v = 0, \quad \Lambda v \ne 0.$$

Now for any such v define the "worst" signals

$$\begin{bmatrix} z^\circ \\ w^\circ \end{bmatrix} \doteq G\Lambda v.$$

(Notice that $w^\circ \ne 0$ because $G(\infty) = I$ has full column rank, $\Lambda v \ne 0$ and by construction, $\|z^\circ\|_{L^2[0,t_*]} = \|w^\circ\|_{L^2[0,t_*]}$). Take $\tilde{w} := w^\circ$ as input to the system of Fig. 4. Then given any causal K the resulting closed loop signals $\begin{bmatrix} u_K \\ y_K \end{bmatrix}$ are unique and they are such that

$$\begin{bmatrix} Qw^\circ \\ w^\circ \end{bmatrix} = G\Lambda \begin{bmatrix} u_K \\ y_K \end{bmatrix}.$$

Hence

$$\langle Qw^\circ, z^\circ \rangle - \langle w^\circ, w^\circ \rangle = \langle \begin{bmatrix} Q \\ I \end{bmatrix} w^\circ, J \begin{bmatrix} z^\circ \\ w^\circ \end{bmatrix} \rangle$$
$$= \langle G\Lambda \begin{bmatrix} u_K \\ y_K \end{bmatrix}, JG\Lambda v \rangle = 0.$$

This together with the fact that $\langle z^\circ, z^\circ \rangle = \langle w^\circ, w^\circ \rangle$ shows that

$$\langle Qw^\circ, z^\circ \rangle = \langle w^\circ, w^\circ \rangle = \langle z^\circ, z^\circ \rangle.$$

Cauchy-Schwartz inequality yields then that the induced norm $\|Q\|_{L^2[0,t_*]} \ge 1$ (and equality holds only if $Qw^\circ = z^\circ$, in which case $\|Qw^\circ\|_2 = \|w^\circ\|_2$, hence the name "worst disturbance" for w°). The proof of the necessary part is complete on noting that $\|Q\|_{L^2[0,t_*]} \le \|Q\|_{L^2[0,T]}, \forall t_* \le T$. $\square$

Theorem IV.2 does not specify a controller K. For actual computation of K it is necessary to introduce more structure. First of all we shall assume that the delays are ordered ascendingly, with the first delay equal to zero (which is without loss of generality), and that equal delays are grouped,

$$\Lambda(s) = \begin{bmatrix} e^{-h_0 s} I_{n_0} & & & \\ & e^{-h_1 s} I_{n_1} & & \\ & & \ddots & \\ & & & e^{-sh_m} I_{n_m} \end{bmatrix}, \quad 0 = h_0 < h_1 < \cdots \quad (10)$$

Then we partition G compatibly with Λ as

$$G(s) = \begin{bmatrix} A & B \\ \hline C & I \end{bmatrix} = \begin{bmatrix} A & B_0 & \dots & B_m \\ \hline C_0 & I_{n_0} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_m & 0 & 0 & I_{n_m} \end{bmatrix} \quad (11)$$

The Hamiltonian $H(t)$ of (8) is piecewise constant and switches only at the delays $t = h_k$. This is because $\Lambda(\Lambda^*J\Lambda)^{-1}\Lambda^*$ is the piecewise constant multiplication operator (see Section III)

$$\Lambda(\Lambda^*J\Lambda)^{-1}\Lambda^* = J \begin{bmatrix} 1_{[h_0=0,T]} I_{n_0} & 0 & 0 & 0 \\ 0 & 1_{[h_1,T]} I_{n_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1_{[h_m,T]} I_{n_m} \end{bmatrix}. \quad (12)$$

Because of the ordering of the delays, see (10), this matrix (12) is in fact a matrix whose "J-block" grows in dimension with time:

$$\Lambda(\Lambda^*J\Lambda)^{-1}\Lambda^*(t) = \begin{cases} \begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t \in [0, h_1) \\[6pt] \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } t \in [0, h_2) \\[8pt] \vdots \quad\quad \vdots & \\ J & t > h_m \end{cases}$$

where J_i denote the corresponding $n_i \times n_i$ blocks of J. (In most cases J_i is either $+I$ or $-I$ but for one i it can happen that J_i is of the form $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.) If $t \in [0, h_1)$ then $H(t)$ equals the constant matrix

$$H_1 := \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix} - \begin{bmatrix} B_0 \\ -C_0'J_0 \end{bmatrix} J_0 \begin{bmatrix} J_0C_0 & B_0' \end{bmatrix}.$$

On the following interval $t \in [h_1, h_2)$ the Hamiltonian $H(t)$ of (8) becomes

$$H_2 := \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix} - \begin{bmatrix} B_0 \\ -C_0'J_0 \end{bmatrix} J_0 \begin{bmatrix} J_0C_0 & B_0' \end{bmatrix}$$
$$- \begin{bmatrix} B_1 \\ -C_1'J_1 \end{bmatrix} J_1 \begin{bmatrix} J_1C_1 & B_1' \end{bmatrix}$$

etcetera. Continuing in this way, with H_i the Hamiltonian on $[h_{i-1}, h_i)$, we can express the Hamiltonians H_i recursively as

$$H_0 := \begin{bmatrix} A & 0 \\ -C'JC & -A' \end{bmatrix}, \tag{13a}$$

$$H_{i+1} = H_i - \begin{bmatrix} B_i \\ -C_i'J_i \end{bmatrix} J_i \begin{bmatrix} J_iC_i & B_i' \end{bmatrix}. \tag{13b}$$

Since $H(t)$ is piecewise constant, the transition matrix $\Sigma(t)$ defined as $\dot{\Sigma}(t) = H(t)\Sigma(t)$ is a finite product of symplectic matrix exponentials. For instance

$$\Sigma(t) = e^{H_i(t-h_{i-1})} e^{H_{i-1}(h_{i-1}-h_{i-2})} \cdots e^{H_1 h_1} \text{ if } t \in [h_{i-1}, h_i] \tag{14}$$

and in particular we have for the largest delay $t = h_m$ that

$$\Sigma(h_m) = e^{H_m(h_m-h_{m-1})} \cdots e^{H_2(h_2-h_1)} e^{H_1 h_1}. \tag{15}$$

In summary, we specialized Theorem IV.2 to:

Theorem IV.3. *Consider Fig. 4 and assume G has realization (11) and that Λ is as in (10). Then there exists a causal K such that $\|Q\|_\infty < 1$ iff $\det \Sigma_{22}(t) \neq 0$ for all $t \in [0, h_m]$, with Σ_{22} the lower-right block of Σ as defined in (14).*

A final note about the choice of $T = h_m$: Theorems IV.2 and IV.3 claim that Q can be made contractive over infinite horizon iff it can be done so over the finite horizon $[0, h_m]$. Then clearly this should also be equivalent to solvability over *any* finite horizon $[0, T]$ with $T \geq h_m$. Therefore it should be that nonsingularity of $\Sigma_{22}(t)$ for all $t \in [0, h_m]$ should imply nonsingularity of $\Sigma_{22}(t)$ for all $t > h_m$. Indeed that is the case because for $t > h_m$ we have that $\Lambda(\Lambda^*J\Lambda)^{-1}\Lambda^* = J$ and as a result that the $H(t)$ is block-upper triangular

$$H(t) = H_\infty := \begin{bmatrix} A - BC & -BJB' \\ 0 & -(A-BC)' \end{bmatrix}.$$

Therefore

$$\Sigma(t) = e^{H_\infty(t-h_m)}\Sigma(h_m) \quad \text{for all } t > h_m$$

resulting in $\Sigma_{22}(t) = e^{-(A-BC)'(t-h_m)}\Sigma_{22}(h_m)$.

V. CONSTRUCTION OF CONTROLLER

So far we only proved necessity of the nonsingularity of $\Sigma_{22}(t)$ for all $t \in [0, T]$. In this section we prove sufficiency by constructing a controller K that solves the problem if this nonsingularity condition is met. In the remainder of this paper systems are assumed to operate over all time, not just the finite horizon of previous sections. We shall assume the notation of the previous section, to be precise we assume that Λ is as in (10) which together with realization (11) renders $\Sigma(t)$ as in (14) with H_i defined by (13).

The construction of the controller hinges on J-spectral factorization of

$$\Phi_0 := \Lambda^\sim G^\sim JG\Lambda.$$

Because of the delay operator Λ this Φ_0 has delay and advance elements. These elements we will sequentially peel off by applying $\mathcal{S}_k$-transformation, which is the self-inverse transformation that corresponds to i/o-swapping of the kth signal block. To define $\mathcal{S}_k$-transformation associate with Φ_0 the equation

$$\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_m \end{bmatrix} = \Phi_0 \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}, \quad \eta_k(t), \zeta_k(t) \in \mathbb{R}^{n_k}$$

where the partitioning is compatible with that of the delay operator. Now $\mathcal{S}_0(\Phi_0)$ is defined by the property that it corresponds to swapping the first block of inputs and outputs, ζ_0 and η_0:

$$\begin{bmatrix} \eta_0 \\ \zeta_1 \\ \vdots \\ \zeta_m \end{bmatrix} = \mathcal{S}_0(\Phi_0) \begin{bmatrix} \zeta_0 \\ \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}.$$

Similarly $\mathcal{S}_k(\Phi_0)$ corresponds to swapping ζ_k and η_k. A useful property of $\mathcal{S}$-transformation is that it transforms a certain type of multiplication into addition: If Π_k has $r_k := n_0 + \cdots + n_{k-1}$ rows and $c_k := n_k + \cdots + n_m$ columns then

$$\mathcal{S}_0\mathcal{S}_1 \cdots \mathcal{S}_{k-1}\left(\begin{bmatrix} I & 0 \\ \Pi_k^\sim & I \end{bmatrix} \Omega \begin{bmatrix} I & \Pi_k \\ 0 & I \end{bmatrix}\right)$$
$$= \mathcal{S}_0\mathcal{S}_1 \cdots \mathcal{S}_{k-1}(\Omega) + \begin{bmatrix} 0 & -\Pi_k \\ \Pi_k^\sim & 0 \end{bmatrix}. \tag{16}$$

Note that $\mathcal{S}_0\mathcal{S}_1 \cdots \mathcal{S}_{k-1}$ means swapping all first k signal *blocks*, i.e., it means swapping all first $r_k := n_0 + \cdots + n_{k-1}$ signal *entries*. This turns out be a handy rule and allows to reduce factorization of Φ_0 to that of a rational matrix:

Lemma V.1. *Let $r_k = n_0 + \cdots + n_{k-1}$ and $c_k = n_k + \cdots + n_m$ and define the $c_k \times c_k$ delay operators Λ_k as*

$$\Lambda_0 := \Lambda, \quad \begin{bmatrix} I_{n_{k-1}} & 0 \\ 0 & e^{-s(h_k-h_{k-1})}\Lambda_k(s) \end{bmatrix} := \Lambda_{k-1}(s).$$

For Φ_k defined as

$$\Phi_0 = \Lambda^\sim G^\sim JG\Lambda, \quad \Phi_{k+1} = \mathcal{S}_k(\Phi_k) + \begin{bmatrix} 0 & -\Pi_k \\ \Pi_k^\sim & 0 \end{bmatrix}$$

with Π_k the $r_k \times c_k$ stable FIR systems

$$\Pi_k = -\pi_h\left(\begin{bmatrix} I_{r_k} & 0 \end{bmatrix} \mathcal{S}_{k-1}(\Phi_{k-1}) \begin{bmatrix} 0 \\ I_{c_k} \end{bmatrix} \Lambda_k^{-1}\right) \Lambda_k$$

we have that Φ_m is rational and that

$$\Phi_0 = Z^\sim (\mathcal{S}_m(\Phi_m))^{-1}Z, \tag{17}$$

where Z is the bistable

$$Z = \begin{bmatrix} I_{r_m} & -\Pi_m \\ 0 & I_{c_m} \end{bmatrix}\begin{bmatrix} I_{r_{m-1}} & -\Pi_{m-1} \\ 0 & I_{c_{m-1}} \end{bmatrix} \cdots \begin{bmatrix} I_{r_1} & -\Pi_1 \\ 0 & I_{c_1} \end{bmatrix} = I - \sum_{k=1}^m \begin{bmatrix} 0_{r_k} & \Pi_k \\ 0 & 0_{c_k} \end{bmatrix}.$$

Proof. This is a condensed version of an idea originating from [10] and further elaborated upon in [6]. We first prove that Φ_m is rational. Because $\Lambda = \begin{bmatrix} I & 0 \\ 0 & e^{-sh_1}\Lambda_1 \end{bmatrix}$ we have that

$$\mathcal{S}_0(\Phi_0) = \begin{bmatrix} \Omega_{11} & e^{-sh_1}\Omega_{12}\Lambda_1 \\ e^{sh_1}\Lambda_1^\sim\Omega_{12}^\sim & \Lambda_1^\sim\Omega_{22}\Lambda_1 \end{bmatrix}$$

for rational $\Omega := \mathcal{S}_0(G^\sim JG)$. Given that $\Pi_1 = -\pi_{h_1}(e^{-sh_1}\Omega_{12})$ we arrive at

$$\Phi_1 = \mathcal{S}_0(\Phi_0) + \begin{bmatrix} 0 & -\Pi_k \\ \Pi_k^\sim & 0 \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12}\Lambda_1 \\ \Lambda_1^\sim\Omega_{12}^\sim & \Lambda_1^\sim\Omega_{22}\Lambda_1 \end{bmatrix}$$

$$=: \begin{bmatrix} \Omega_{11} & \Omega_{12a} & e^{-s(h_2-h_1)}\Omega_{12b}\Lambda_2 \\ \Omega_{12a}^\sim & \Omega_{22aa} & e^{-s(h_2-h_1)}\Omega_{22ab}\Lambda_2 \\ e^{s(h_2-h_1)}\Lambda_2^\sim\Omega_{12b}^\sim & e^{s(h_2-h_1)}\Lambda_2^\sim\Omega_{22ab}^\sim & \Lambda_2^\sim\Omega_{22bb}\Lambda_2 \end{bmatrix}.$$

The delay operator Λ_2 in Φ_1 has $m-1$ delay blocks (i.e., one less than Λ_1 has in Φ_0). Continuing this process m times results in a delay free Φ_m. Remains to establish (17) and the following formula for Z.

Using (16) we get that

$$\mathcal{S}_0\left(\begin{bmatrix} I_{n_0} & \Pi_1 \\ 0 & I \end{bmatrix}^\sim \Phi_0 \begin{bmatrix} I_{n_0} & \Pi_1 \\ 0 & I \end{bmatrix}\right) = \mathcal{S}_0(\Phi_0) + \begin{bmatrix} 0 & -\Pi_1 \\ \Pi_1^\sim & 0 \end{bmatrix} = \Phi_1.$$

Now as $\mathcal{S}_k$ is a self inverse transformation we have that $\begin{bmatrix} I_{n_0} & \Pi_1 \\ 0 & I \end{bmatrix}^\sim \Phi_0 \begin{bmatrix} I_{n_0} & \Pi_1 \\ 0 & I \end{bmatrix} = \mathcal{S}_0(\Phi_1)$, which is the same as

$$\Phi_0 = \begin{bmatrix} I_{n_0} & -\Pi_1 \\ 0 & I \end{bmatrix}^\sim \mathcal{S}_0(\Phi_1) \begin{bmatrix} I_{n_0} & -\Pi_1 \\ 0 & I \end{bmatrix}.$$

Repeating this procedure a couple of times similarly gives

$$\Phi_0 = \begin{bmatrix} I & -\Pi_1 \\ 0 & I \end{bmatrix}^\sim \cdots \begin{bmatrix} I & -\Pi_m \\ 0 & I \end{bmatrix}^\sim \mathcal{S}_0 \cdots \mathcal{S}_{m-1}(\Phi_m) \begin{bmatrix} I & -\Pi_m \\ 0 & I \end{bmatrix} \cdots \begin{bmatrix} I & -\Pi_1 \\ 0 & I \end{bmatrix}.$$

Note that $\mathcal{S}_0 \cdots \mathcal{S}_m$ is the same inversion so the above is equivalent to (17). That Z defined as a *product* can also be written as a sum $Z = I - \sum_{k=1}^m \begin{bmatrix} 0 & \Pi_k \\ 0 & 0 \end{bmatrix}$ follows from the dimensions of Π_k. $\qquad\square$

The state space formula for Φ_k and Π_k actually follow quite elegantly. Bring in the "state-space realization" of Φ_0,

$$\Phi_0 := \Lambda^\sim G^\sim JG\Lambda = \Lambda^\sim \left[\begin{array}{c|c} H_0 & \begin{bmatrix} B \\ -C'J \end{bmatrix} \\ \hline \begin{bmatrix} JC & B' \end{bmatrix} & J \end{array}\right] \Lambda, \qquad (18)$$

where H_0 is the Hamiltonian matrix defined in (13). Partition this realization as

$$\Phi_0 = \Lambda^\sim \left[\begin{array}{c|cccc} H_0 & \hat{B}_0 & \hat{B}_1 & \ldots & \hat{B}_m \\ \hline \hat{C}_0 & J_0 & 0 & \ldots & 0 \\ \hat{C}_1 & 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{C}_m & 0 & 0 & \ldots & J_m \end{array}\right] \Lambda \qquad (19)$$

with the partitioning compatible with that of the delay operator Λ. (Note that $J_i^{-1} = J_i$ for all i.) Since $\mathcal{S}_k(\Lambda^\sim\Omega\Lambda) = \mathcal{S}_k(\Omega)$ we have that swapping the first i/o signal block Φ_0 amounts to that of $G^\sim JG$:

$$\mathcal{S}_0(\Phi_0) = \begin{bmatrix} I & 0 \\ 0 & \Lambda_1^\sim \end{bmatrix} \left[\begin{array}{c|cccc} H_1 & \hat{B}_0 J_0 & \hat{B}_1 & \ldots & \hat{B}_m \\ \hline -J_0\hat{C}_0 & J_0 & 0 & \ldots & 0 \\ \hat{C}_1 & 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{C}_m & 0 & 0 & \ldots & J_m \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}$$

where $H_1 = H_0 - \hat{B}_0 J_0\hat{C}_0$. From this realization we read that

$$\Pi_1 = \pi_{h_1}\left(e^{-sh_1}\left[\begin{array}{c|cccc} H_1 & \hat{B}_1 & \hat{B}_2 & \ldots & \hat{B}_m \\ \hline J_0\hat{C}_0 & 0 & 0 & \ldots & 0 \end{array}\right]\right)\Lambda_1$$

and hence

$$\Phi_1 := \mathcal{S}_0(\Phi_0) + \begin{bmatrix} 0 & -\Pi_1 \\ \Pi_1^\sim & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & \Lambda_1^\sim \end{bmatrix} \left[\begin{array}{c|cccc} H_1 & \Sigma_1\hat{B}_0 J_0 & \hat{B}_1 & \ldots & \hat{B}_m \\ \hline -J_0\hat{C}_0\Sigma_1^{-1} & J_0 & 0 & \ldots & 0 \\ \hat{C}_1 & 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_m & 0 & 0 & \ldots & J_m \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix},$$

where $\Sigma_1 := e^{H_1 h_1}$. Swapping the second input-output block of Φ_1 results in

$$\mathcal{S}_1(\Phi_1) = \begin{bmatrix} I & 0 \\ 0 & \Lambda_1^\sim \end{bmatrix} \left[\begin{array}{c|cccc} H_2 & \Sigma_1\hat{B}_0 J_0 & \hat{B}_1 J_1 & \ldots & \hat{B}_m \\ \hline -J_0\hat{C}_0\Sigma_1^{-1} & J_0 & 0 & \ldots & 0 \\ -J_1\hat{C}_1 & 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_m & 0 & 0 & \ldots & J_m \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}$$

with $H_2 := H_1 - \hat{B}_1 J_1\hat{C}_1$. Then, similarly, we can read of immediately that

$$\Pi_2 = \pi_{h_2-h_1}\left(e^{-s(h_2-h_1)}\left[\begin{array}{c|ccc} H_2 & \hat{B}_2 & \ldots & \hat{B}_m \\ \hline J_0\hat{C}_0\Sigma_1^{-1} & 0 & \ldots & 0 \\ J_1\hat{C}_1 & 0 & \ldots & 0 \end{array}\right]\right)\Lambda_2$$

yielding

$$\Phi_2 = \mathcal{S}_1(\Phi_1) + \begin{bmatrix} 0 & -\Pi_2 \\ \Pi_2^\sim & 0 \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 \\ 0 & \Lambda_2^\sim \end{bmatrix} \left[\begin{array}{c|cccc} H_2 & \Sigma_2\Sigma_1\hat{B}_0 J_0 & \Sigma_2\hat{B}_1 J_1 & \ldots & \hat{B}_m \\ \hline -J_0\hat{C}_0\Sigma_1^{-1}\Sigma_2^{-1} & J_0 & 0 & \ldots & 0 \\ -J_1\hat{C}_1\Sigma_2^{-1} & 0 & J_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_m & 0 & 0 & \ldots & J_m \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

where $\Sigma_2 \doteq e^{H_2(h_2-h_1)}$. We may continue in this fashion and each time the dimension of the number of delay blocks in the delay operator Λ_k decreases and in the end vanishes resulting in the rational

$$\Phi_m =$$

$$\left[\begin{array}{c|cccc} H_m & \Sigma_m\cdots\Sigma_1\hat{B}_0 J_0 & \ldots & \Sigma_m\hat{B}_{m-1}J_{m-1} & \hat{B}_m \\ \hline -J_0\hat{C}_0\Sigma_1^{-1}\cdots\Sigma_m^{-1} & J_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -J_{m-1}\hat{C}_{m-1}\Sigma_m^{-1} & 0 & \ldots & J_{m-1} & 0 \\ \hat{C}_m & 0 & \ldots & 0 & J_m \end{array}\right],$$

where $H_k = H_0 - \sum_{i=0}^{k-1}\hat{B}_i J_i\hat{C}_i$ and $\Sigma_k = e^{H_k(h_k-h_{k-1})}$. In (17) we need

$$\mathcal{S}_m(\Phi_m) =$$

$$\left[\begin{array}{c|cccc} H_\infty & \Sigma_m\cdots\Sigma_1\hat{B}_0 J_0 & \ldots & \Sigma_m\hat{B}_{m-1}J_{m-1} & \hat{B}_m J_m \\ \hline -J_0\hat{C}_0\Sigma_1^{-1}\cdots\Sigma_m^{-1} & J_0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -J_{m-1}\hat{C}_{m-1}\Sigma_m^{-1} & 0 & \ldots & J_{m-1} & 0 \\ -J_m\hat{C}_m & 0 & \ldots & 0 & J_m \end{array}\right],$$

and its inverse. Here $H_\infty := H_0 - \sum_{i=0}^m\hat{B}_i J_i\hat{C}_i$. It is readily verified (as $H_i = \Sigma_i H_i\Sigma_i^{-1}$) that the "$A$" matrix of $(\mathcal{S}_m(\Phi_m))^{-1}$ is $\Sigma_m\cdots\Sigma_2\Sigma_1 H_0 \Sigma_1^{-1}\Sigma_2^{-1}\cdots\Sigma_m^{-1}$, i.e., it is similar to the "A" matrix

H_0 of Φ_0. This, in turn, leads to

$$(\mathcal{S}_m(\Phi_m))^{-1} = \left[\begin{array}{c|cccc} H_0 & \hat{B}_0 & \Sigma_1^{-1}\hat{B}_1 & \dots & \Sigma_1^{-1}\cdots\Sigma_m^{-1}\hat{B}_m \\ \hline \hat{C}_0 & J_0 & 0 & \dots & 0 \\ \hat{C}_1\Sigma_1 & 0 & J_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_m\Sigma_m\cdots\Sigma_1 & 0 & 0 & \dots & J_m \end{array}\right].$$

Application of standard factorization techniques (see e.g. [6]) says that

$$(\mathcal{S}_m(\Phi_m))^{-1} = \tilde{G}^\sim J \tilde{G}$$

for the bistable $\tilde{G}$ equal to

$$\tilde{G} = \left[\begin{array}{c|c} A & \tilde{B} \\ \hline C & I \end{array}\right] \tag{20a}$$

in which

$$\tilde{B} = \begin{bmatrix} I_n & 0 \end{bmatrix}\begin{bmatrix} \hat{B}_0 & \Sigma_1^{-1}\hat{B}_1 & \cdots & \Sigma_1^{-1}\cdots\Sigma_m^{-1}\hat{B}_m \end{bmatrix} \tag{20b}$$

$$\tilde{C} = J\begin{bmatrix} \hat{C}_0 \\ \hat{C}_1\Sigma_1 \\ \vdots \\ \hat{C}_m\Sigma_m\cdots\Sigma_1 \end{bmatrix}\begin{bmatrix} I_n \\ M \end{bmatrix} \tag{20c}$$

$$M = -\Sigma_{22}^{-1}(h_m)\Sigma_{21}(h_m). \tag{20d}$$
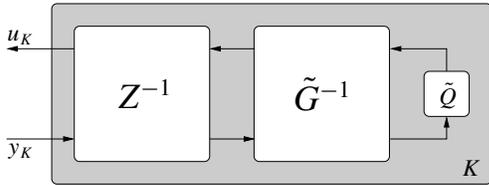
Fig. 5. All solutions K

Theorem V.2. *Consider Fig. 4 and assume that Λ is as in (10) and that G is bistable with realization (11). Then there exists a causal K such that $\|Q\|_\infty < 1$ iff $\det\Sigma_{22}(t) \neq 0$ for all $t \in [0, h_m]$, with Σ as in (14), (13). In that case, using the short hands of (18) and (19), K is a solution iff it is of the form shown in Fig. 5 in which $\|\tilde{Q}\|_\infty < 1$ and $\tilde{G}$ is as in (20) and Z is as in Lemma V.1 where*

$$\Pi_i = \pi_{h_i - h_{i-1}}\left(\mathrm{e}^{-s(h_i - h_{i-1})}\left[\begin{array}{c|ccc} H_i & \hat{B}_i & \dots & \hat{B}_m \\ \hline J_0\hat{C}_0\Sigma_1^{-1}\cdots\Sigma_{i-1}^{-1} & 0 & \dots & 0 \\ J_1\hat{C}_1\Sigma_2^{-1}\cdots\Sigma_{i-1}^{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ J_{i-1}\hat{C}_{i-1} & 0 & \dots & 0 \end{array}\right]\right)\Lambda_i$$

with Λ_i as in Lemma V.1 and $\Sigma_i = \mathrm{e}^{H_i(h_i - h_{i-1})}$.

Proof. These follow well documented arguments: By construction $G\Lambda Z^{-1}\tilde{G}^{-1}$ is J-unitary. In fact it is J-lossless, which follows from a continuity argument (see [6]). Consequently the mapping Q in Fig. 4 is contractive if-and-only-if so is $\tilde{Q}$ (see for instance [6]). The proof is complete on noting that K is causal iff $\tilde{Q}$ is causal. This is consequence of the fact that $\lim_{s\to\infty}\tilde{G}(s)Z(s) = I$. □

VI. Concluding remarks

There are a number of issues not addressed in this paper due to space limitations: (a) the controller has interesting interpretations owing to its specific structure (for example, the FIR systems Π_i from which Z is defined have non-overlapping support), (b) the

H^2 problem for systems with multiple delays can be handled quite easily as well using the machinery presented in this paper, and (c) J-spectral factorization can also be done through completion of squares much like the way it is often done for delay-free systems. This last approach will further simplify the derivation.

References

[1] D. L. Kleinman, "Optimal control of linear systems with time-delay and observation noise," *IEEE Trans. Automat. Control*, vol. 14, no. 5, pp. 524–527, 1969.

[2] M. A. Soliman and W. H. Ray, "Optimal feedback control for linear-quadratic systems having time delays," *Int. J. Control*, vol. 15, no. 4, pp. 609–627, 1972.

[3] C. Foias, H. Özbay, and A. Tannenbaum, *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods*, ser. Lecture Notes in Control and Information Sciences. London: Springer-Verlag, 1996, vol. 209.

[4] L. Mirkin and G. Tadmor, "H^∞ control of system with I/O delay: A review of some problem-oriented methods," *IMA J. Math. Control & Information*, vol. 19, pp. 185–199, 2002.

[5] B. van Keulen, H^∞ *Control for Distributed Parameter Systems: A State Space Approach*. Boston: Birkhäuser, 1993.

[6] G. Meinsma and L. Mirkin, " H^∞ Control of Systems with Multiple I/O Delays via Decomposition to Adobe Problems," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 199–211, 2005.

[7] M. J. Grimble and G. Hearns, "LQG controllers for state-space systems with pure transport delays: Application to hot strip mills," *Automatica*, vol. 34, no. 10, pp. 1169–1184, 1998.

[8] A. Kojima and S. Ishijima, "H^∞ control for preview and delayed strategies," in *Proc. 40th IEEE Conf. Decision and Control*, Orlando, FL, 2001, pp. 991–996.

[9] G. Meinsma, L. Mirkin, and Q.-C. Zhong, "Control of systems with I/O delay via reduction to a one-block problem," *IEEE Trans. Automat. Control*, vol. 47, no. 11, pp. 1890–1895, 2002.

[10] G. Meinsma and H. Zwart, "On H^∞ control for dead-time systems," *IEEE Trans. Automat. Control*, vol. 45, no. 2, pp. 272–285, 2000.

[11] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parametrizations," *IEEE Trans. Automat. Control*, vol. 48, no. 4, pp. 543–553, 2003.

[12] N. F. Jerome and W. H. Ray, "High-performance multivariable control strategies for systems having time delays," *AIChE Journal*, vol. 32, no. 6, pp. 914–931, 1986.

[13] G. Weiss, "Transfer functions of regular linear systems. Part I: Characterizations of regularity," *Trans. Amer. Math. Soc.*, vol. 342, no. 2, pp. 827–854, 1994.

[14] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[15] H. Kimura, *Chain-Scattering Approach to H^∞ Control*. Boston: Birkhäuser, 1996.

[16] I. Gohberg and M. A. Kaashoek, "Time varying linear systems with boundary conditions and integral operators, I. The transfer operator and its properties," *Integral Equations and Operator Theory*, vol. 7, pp. 325–391, 1984.

[17] G. Meinsma and L. Mirkin, "H^∞ control of systems with multiple I/O delays via decomposition to adobe problems. Part II: simplifications and interpretations," 2004, in preparation.

[18] M. Kristalny and L. Mirkin, "On the multi-channel H^∞ fixed-lag smoothing," in *Proc. 5th IFAC Workshop on Time Delay Systems*, K.U. Leuven, Belgium, 2004.

[19] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[20] H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, ser. Operator Theory. Basel: Birkhäuser, 1979, vol. 1.

[21] E. G. F. Thomas, "Vector-valued integration with applications to the operator valued H^∞ space," *IMA J. Math. Control & Information*, vol. 14, pp. 109–136, 1997.

[22] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. NY: Springer-Verlag, 1995.