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Pseudospectra for analytic matrix functions and application to time-delay systems

Wim Michiels, Kirk Green, Thomas Wagenknecht and Silviu-Iulian Niculescu

***Abstract*— Definitions for pseudospectra of an analytic matrix function are given, where the structure of the function is exploited. Various perturbation measures are considered and computationally tractable formulae are derived. The results are applied to a class of retarded delay differential equations. Special properties of the pseudospectra of such equations are determined and illustrated.**

I. Introduction

Closeness to instability and transient response are key issues in understanding the behaviour of physical systems subject to perturbation. The computation of pseudospectra has become an established tool in analysing and gaining insights for both phenomena (see, for instance, Trefethen [1], and the references therein). More explicitly, pseudospectra of a system are sets in the complex plane to which its eigenvalues can be shifted under a perturbation of a given size. In the simplest case of a matrix (or linear operator) A, the ϵ-pseudospectrum Λ_ϵ is defined as

$$\Lambda_\epsilon(A) := \{\lambda \in \mathbb{C} : \lambda \in \Lambda(A+P), \quad \text{for some } P \text{ with } \|P\| \leq \epsilon\}, \quad (1)$$

where Λ denotes the spectrum and $\|\cdot\|$ denotes an arbitrary matrix (or operator) norm. Equation (1) is known to be equivalent to the following

$$\Lambda_\epsilon(A) = \{\lambda \in \mathbb{C} : \|R(\lambda, A)\| \geq 1/\epsilon\},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the corresponding resolvent operator.

Although most systems can be written in a first-order form, it is often advantageous to exploit the underlying structure of an equation in its analysis, for example, one may wish to compute pseudospectra of higher-order or delay differential equations (DDEs). In particular, this can be of importance in sensitivity investigations that respect the structure of the governing system. For example, many physical problems involving vibration of structural systems and vibro-acoustics are modelled by second-order differential equations of the form $A_2\ddot{x} + A_1\dot{x} + A_0 x = 0$, where A_2, A_1, and A_0 represent mass, damping and stiffness matrices, respectively. Stability is inferred from the eigenvalues, found as solutions of $\det(A_2\lambda^2 + A_1\lambda + A_0) = 0$. To understand the sensitivity of these eigenvalues with respect to perturbations with weights α_i applied to A_i, $i = 0, 1, 2$, the ϵ-pseudospectrum of the matrix polynomial $P(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ can be defined as

$$\begin{aligned}\Lambda_\epsilon(P) :=& \\ \{\lambda \in \mathbb{C} \ :& \ (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ & \text{and } \Delta P(\lambda) = \delta A_2\lambda^2 + \delta A_1\lambda + \delta A_0 \\ & \text{with } ||\delta A_i|| \leq \epsilon\alpha_i,\ i = 0, 1, 2\},\end{aligned}$$

More recently, pseudospectra for matrix functions that arise as characteristic equations in DDEs have been defined and analysed [2]. In its simplest form of one, fixed, discrete delay $\tau \in \mathbb{R}^+$, the delayed characteristic is of the form $Q(\lambda) = \lambda I - A_0 - A_1 \exp(-\lambda\tau)$. Similar to (2) the associated pseudospectra is defined in [2] as

$$\begin{aligned}\Lambda_\epsilon(Q)& \\ := \{\lambda \in \mathbb{C} \ :& \ (Q(\lambda) + \Delta Q(\lambda))x = 0 \text{ for some } x \neq 0 \\ & \text{and } \Delta Q(\lambda) = \delta A_0 + \delta A_1 \exp(-\lambda\tau) \\ & \text{with } ||\delta A_i|| \leq \epsilon\alpha_i,\ i = 0, 1\}.\end{aligned}$$

The aim of this paper is twofold: first, to present a *unified* theory for the definition and computation of pseudospectra of *general* matrix functions of the form

$$\det\left\{\sum_{i=0}^{m} A_i p_i(\lambda)\right\} = 0, \quad (2)$$

where p_i is an entire function. It is easy to see that the cases described above are in this class of matrix functions. Also various perturbation measures are discussed, of which the above is only a particular case (Section II). The second aim is to emphasize some nice, and special properties in the case of delay systems (Section III). In this sense, we discuss the effects of weighting factors on the sensitivity of the eigenvalues in $\mathbb{C}^+$, and $\mathbb{C}^-$, respectively. Next, special attention is devoted to the asymptotic behaviour, and to the relationship between the decay of the pseudospectrum function to zero and root chains coming from infinity. To the best of the authors' knowledge, there does not exist any similar analysis in the literature.

It is important to mention that one of the practical applications of our results concerns the *stability radius* $r_\mathbb{C}$ of (2), that is, a measure of the distance of the matrix function to instability (see also [3], [4], [5]). Specifically, if we decompose $\mathbb{C}$ into two disjoint regions, a desired region $\mathbb{C}_d$ and an undesired region $\mathbb{C}_u$, the complex stability radius is defined as

$$r_\mathbb{C}(\mathbb{C}_d, \|\cdot\|_{\text{glob}}) := \inf_{\lambda \in \mathbb{C}_u} \inf_{\Delta} \left\{ \|\Delta\|_{\text{glob}} :\ \det\left(\sum_{i=0}^{m}(A_i + \delta A_i)p_i(\lambda)\right) = 0 \right\}, \quad (3)$$

where $\|\Delta\|_{\text{glob}}$ is a global measure of the perturbation Δ, a combination of δA_i; this is discussed in detail in Section II. In other words, $r_\mathbb{C}$ defines the norm of the smallest perturbation that destroys the $\mathbb{C}_d$-stability. Furthermore, $r_\mathbb{C}$ corresponds to the smallest ϵ value at which the ϵ-pseudospectrum has a non-empty intersection with $\mathbb{C}_u$.

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II. PSEUDOSPECTRA FOR GENERAL MATRIX FUNCTIONS

We study the roots of the generalised matrix function given by Eq. (2), where $A_i \in \mathbb{C}^{n\times n},\ i = 0,\dots,m$ and the functions $p_i : \mathbb{C} \to \mathbb{C}, i = 0,\dots,m$ are entire. In particular, we are interested in the effect of bounded perturbations of the matrices A_i on the position of the roots. For this, we analyze the perturbed equation,

$$\det\left\{\sum_{i=0}^{m}(A_i+\delta A_i)p_i(\lambda)\right\}=0. \tag{4}$$

The first step in our robustness analysis is to define the class of perturbations under consideration, as well as a measure of the combined perturbation

$$\Delta := (\delta A_0,\dots,\delta A_m).$$

In this work we assume that the allowable perturbations $\delta A_i,\ i = 0,\dots,m$, are *complex* matrices, that is, $\Delta \in \mathbb{C}^{n\times n\times(m+1)}$.

Introducing weights $w_i \in \bar{\mathbb{R}}_0^+, i = 0,\dots,m$, where $\bar{\mathbb{R}}_0^+ = \mathbb{R}^+ \setminus \{0\} \cup \{\infty\}$, we define three global measures of the perturbations:

$$\|\Delta\|_{\text{glob}} := \|[w_0\ \delta A_0 \dots w_m\ \delta A_m]\|_p, \tag{5}$$

or

$$\|\Delta\|_{\text{glob}} := \left\|\begin{bmatrix} w_0\ \delta A_0 \\ \vdots \\ w_m\ \delta A_m\end{bmatrix}\right\|_p, \tag{6}$$

where $\|M\|_p$ is the induced matrix norm given by $\sup_{\|x\|_p=1}\|Mx\|_p,\ p\in\bar{\mathbb{N}}$. Notice that $w_j=\infty$ for some j means that no perturbation on A_j is allowed when the combined perturbation Δ is required to be bounded, that is $w_j=\infty \implies \delta A_j = 0$, for some j. Finally, we also consider a measure of mixed-type:

$$\|\Delta\|_{\text{glob}} := \left\|\begin{bmatrix} w_0\|\delta A_0\|_{p_1} \\ \vdots \\ w_m\|\delta A_m\|_{p_1}\end{bmatrix}\right\|_{p_2},\ p_1,p_2\in\bar{\mathbb{N}}_0. \tag{7}$$

For instance, when $p_2=\infty$ and all weights are equal to one, the condition $\|\Delta\|_{\text{glob}}<\epsilon$ corresponds to the natural assumptions of taking perturbations satisfying $\|\delta A_i\|_{p_1}<\epsilon,\ i=0,\dots,m$. In this special case (7) is also equal to the p_1-norm of the *block diagonal* perturbation matrix $\text{diag}(\delta A_0,\dots,\delta A_m)$, considered in [4], [5] for polynomial matrices.

Notice that, if all weights are finite, then the measures given by (5)-(7) are norms.

For any of the above definitions of $\|\Delta\|_{\text{glob}}$, we define the ϵ-pseudospectrum of (2) as the set

$$\Lambda_\epsilon := \left\{\lambda\in\mathbb{C}:\ \det\left(\sum_{i=0}^{m}(A_i+\delta A_i)p_i(\lambda)\right)=0 \right.$$
$$\left. \text{for some } \Delta \text{ with } \|\Delta\|_{\text{glob}}\le\epsilon\right\}. \tag{8}$$

We define $f:\ \mathbb{C}\to\bar{\mathbb{R}}^+$ as the inverse of the size of the smallest perturbation which shifts a root to λ if such perturbations exist, and zero otherwise, more precisely,

$$f(\lambda)= \begin{cases} 0, & \text{when } \det\left(\sum_{i=0}^m(A_i+\delta A_i)p_i(\lambda)\right)\neq 0,\ \forall\Delta \\ +\infty, & \text{when } \det\sum_{i=0}^m A_ip_i(\lambda)=0 \\ \left(\inf\left\{\|\Delta\|_{\text{glob}}:\ \det\left(\sum_{i=0}^m(A_i+\delta A_i)p_i(\lambda)=0\right)\right\}\right)^{-1}, & \text{otherwise.}\end{cases} \tag{9}$$

Therefore, we can also define the ϵ-pseudospectra as

$$\Lambda_\epsilon=\left\{\lambda\in\mathbb{C}:\ f(\lambda)\ge\frac{1}{\epsilon}\right\}.$$

The boundary of pseudospectra is thus formed by the level sets of the function f, which can be computed as follows:

Theorem 2.1: For the perturbation measures (5)-(7) the function (9) satisfies

$$f(\lambda)=\begin{cases}\left\|\left(\sum_{i=0}^m A_ip_i(\lambda)\right)^{-1}\right\|_\alpha\cdot\|w(\lambda)\|_\beta, & \det\sum_{i=0}^m A_ip_i(\lambda)\neq0, \\ +\infty, & \det\sum_{i=0}^m A_ip_i(\lambda)=0,\end{cases}$$

where

$$w(\lambda)=\begin{bmatrix}\frac{p_0(\lambda)}{w_0} & \cdots & \frac{p_m(\lambda)}{w_m}\end{bmatrix}^T \tag{10}$$

and

$$\begin{array}{lll}\alpha=p,\ \beta=p, & & \text{perturb. measure (5)},\\ \alpha=p,\ \beta=q, & \frac{1}{p}+\frac{1}{q}=1, & \text{perturb. measure (6)},\\ \alpha=p_1,\ \beta=q_2, & \frac{1}{p_2}+\frac{1}{q_2}=1 & \text{perturb. measure (7)}.\end{array}$$

Proof: Based on the *small gain* theorem and exploiting the structure of Δ and (2). See the full version of the paper for details. ■

A. Connection with stability radii

Pseudospectra are closely related to the concept of the stability radius given by (3). Note that an eigenvalue can leave the desired region $\mathbb{C}_d$ of $\mathbb{C}$, due to a perturbation of the system matrices, in two ways:

1) The perturbation shift roots from $\mathbb{C}_d$ to $\mathbb{C}_u$;
2) The perturbation causes roots at infinity to appear in $\mathbb{C}_u$.

From Rouché's theorem one derives that the individual roots of (2) are *continuous* at each value of the system matrices A_i. If the second case can be excluded, a loss of stability is associated with roots on the boundary of $\mathbb{C}_d$ and it becomes sufficient to scan this boundary in the outer optimization of (3). In other words, the stability radius is the smallest value of ϵ for which an ϵ-pseudospectrum contour reaches the boundary of $\mathbb{C}_d$. Formally, using (9) one has,

Corollary 2.2:

$$r_{\mathbb{C}}(\mathbb{C}_d,\|\cdot\|_{\text{glob}})=\inf_{\lambda\in\Gamma_{\mathbb{C}_d}}\frac{1}{f(\lambda)}=\frac{1}{\sup_{\lambda\in\Gamma_{\mathbb{C}_d}}f(\lambda)},$$

where $\Gamma_{\mathbb{C}_d}$ is the boundary of the set $\mathbb{C}_d$.

The following example demonstrates that Corollary 2.2 does not hold if perturbations create roots at infinity in $\mathbb{C}_u$.

Example 2.3: The equation $p(\lambda)=0$, with

$$p(\lambda)=\lambda+1+\delta a\ e^{\lambda},$$

is $\mathbb{C}^-$-stable for $\delta a=0$. With $\|\Delta\|_{\mathrm{glob}}=|\delta a|$, we have

$$\inf_{\lambda\in\Gamma_{\mathbb{C}^-}}\frac{1}{f(\lambda)}=\inf_{\omega\geq 0}\frac{|1+j\omega|}{|e^{j\omega}|}=1,$$

that is, shifting roots to the imaginary axis requires $|\delta a|\geq 1$. However, the stability radius is zero because for any real $\delta a\neq 0$, there are infinitely many roots in the open right half plane, whose real parts move off to plus infinity as $|\delta a|\to 0+$. To see this, note that $p(-\lambda)$ can be interpreted as the characteristic function of the DDE $\dot{x}(t)=x(t)+\delta a\ x(t-1)$, which has infinitely many eigenvalues located in logarithmic sections of the left half plane [6].

To conclude this section we give in Table I an overview of publications, where results from Theorem 2.1 or Corollary 2.2 were obtained for special cases.

III. Pseudospectra of delay differential equations

We apply the results of Section II to linear DDEs of the form

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{m}A_ix(t-\tau_i), \tag{11}$$

(under appropriate initial conditions) where we assume that $0<\tau_1<\ldots<\tau_m$ and that the system matrices $A_i\in\mathbb{R}^{n\times n},\ i=0,\ldots,m$ are uncertain. Following (8) and (3), we have:

Proposition 3.1: For perturbations $\delta A_i\in\mathbb{C}^{n\times n},\ i=0,\ldots,m$, measured by (5)-(7), the pseudospectrum Λ_ϵ satisfies

$$\Lambda_\epsilon=\left\{\lambda\in\mathbb{C}:\ \left\|\left(\lambda I-A_0-\sum_{i=1}^{m}A_ie^{-\lambda\tau_i}\right)^{-1}\right\|_\alpha \times\|w(\lambda)\|_\beta\geq\frac{1}{\epsilon}\right\} \tag{12}$$

and the associated stability radius of (11) satisfies

$$r_{\mathbb{C}}(\mathbb{C}^-,\|\cdot\|_{\mathrm{glob}})=\frac{1}{\left(\sum_{i=0}^{m}w_i^{-\beta}\right)^{\frac{1}{\beta}}\ \sup_{\omega\geq 0}\left\|\left(j\omega I-A_0-\sum_{i=1}^{m}A_ie^{-j\omega\tau_i}\right)^{-1}\right\|_\alpha}, \tag{13}$$

where $w(\lambda)=\left[\frac{1}{w_0}\ \frac{e^{-\lambda\tau_1}}{w_1}\ldots\frac{e^{-\lambda\tau_m}}{w_m}\right]^T$ and α and β are defined as in Theorem 2.1.

Remark 3.2: For the system

$$\dot{x}(t)=(A+\delta A)x(t), \tag{14}$$

with $\|\Delta\|_{\mathrm{glob}}=\|\delta A\|_p$, expression (12) simplifies to

$$\Lambda_\epsilon=\left\{\lambda\in\mathbb{C}:\|\mathcal{R}(\lambda,A)\|_p\geq\frac{1}{\epsilon}\right\}, \tag{15}$$

where $\mathcal{R}(\lambda,A)=(\lambda I-A)^{-1}$ is the *resolvent* of A. The right-hand side of (15) can also be considered as a definition for the ϵ-pseudospectrum of (14).

In general, one can formulate (11) as an abstract evolution equation over the space $X:=\mathcal{C}([-\tau_m,\ 0],\mathbb{R}^n)$, equipped with the supremum norm, $\|\phi\|_s=\sup_{\theta\in[-\tau_m,\ 0]}\|\phi(\theta)\|_2,\ \ \phi\in X$, namely: $\frac{d}{dt}\ x_t=\mathcal{A}x_t$, where $\mathcal{A}$ is the infinitesimal generator of (11) Now, one can alternatively define the ϵ-pseudospectrum as the set

$$\left\{\lambda\in\mathbb{C}:\ \|\mathcal{R}(\lambda,\mathcal{A})\|\geq\frac{1}{\epsilon}\right\}. \tag{16}$$

Definition (16) is related with the effect of *unstructured* perturbations of the operator $\mathcal{A}$ on stability. In this paper we have chosen a more practical definition, by directly relating pseudospectra to concrete perturbations on the system matrices.

A. Effect of weighting

Applying different weights to the system matrices A_i of (11), $i=1,\ldots,m$, leads to changes in the pseudospectra. This can be understood by investigating the weighting function $w(\lambda)=w(\sigma+j\omega)$, where

$$\|w(\sigma+j\omega)\|_\beta=\left\|\left[\frac{1}{w_0},\frac{e^{-\sigma\tau_1}}{w_1},\ldots,\frac{e^{-\sigma\tau_m}}{w_m}\right]^T\right\|_\beta,\ \ \forall\sigma,\omega\in\mathbb{R}. \tag{17}$$

Note that $w(\lambda)$ only depends on the real part σ, that is, $w(\lambda)\equiv w(\sigma)$. From (17) the following conclusions can be drawn:

1) Eigenvalues in the right half plane are more sensitive to perturbations of the non-delayed term A_0;
2) Eigenvalues in the left half plane are more sensitive to perturbations of delayed terms A_i, $i=1,\ldots,m$;
3) Furthermore, the intersection of an ϵ-pseudospectrum contour with the imaginary axis is independent of the weights, provided that the β-norm of $w(\lambda)=w(0)$ is constant.

B. Asymptotic properties.

We investigate the behaviour of

$$f(\lambda):=\left\|\left(\lambda I-A_0-\sum_{i=1}^{m}A_ie^{-\lambda\tau_i}\right)^{-1}\right\|_\alpha\|w(\lambda)\|_\beta$$

as $|\lambda|\to\infty$, in order to characterize boundedness properties of pseudospectra. We have the following results, whose proof is omitted due to space limitations:

Proposition 3.3: For all $\mu\in\mathbb{R}$,

$$\lim_{R\to\infty}\inf\left\{f(\lambda)^{-1}:\ \Re(\lambda)>\mu,\ |\lambda|>R\right\}=\infty. \tag{18}$$

As a consequence the cross-section between any pseudospectrum $\Lambda_\epsilon,\ \epsilon>0$, and any right-half plane is bounded.

Proposition 3.4: Assume that w_m is finite. For all $\gamma\in\mathbb{R}^+$, let the set $\Psi_\gamma\subseteq\mathbb{C}$ be defined as

$$\Psi_\gamma:=\left\{\lambda\in\mathbb{C}:\ \Re(\lambda)<-\gamma,\ |\lambda|<e^{-(\Re(\lambda)+\gamma)\tau_m}\right\}. \tag{19}$$

Furthermore, let

$$l=\begin{cases}\frac{w_m}{\|A_m^{-1}\|}, & A_m\ \text{regular},\\ 0, & A_m\ \text{singular}.\end{cases}$$

reference	problem	perturbation measure	weights
[7]	matrix pencil	(5)	/
[5]	polynomial matrices	(5),(6),(7) with $p_2=\infty$	/
[4]	polynomial matrices	(5),(7) with $p_2=\infty$	yes
[8]	polynomial matrices	(5),(7) with $p_2=\infty$	yes
[2]	delay systems	(7) with $p_1=1$ and $p_2=\infty$	yes

TABLE I

SPECIAL CASES OF THEOREM 2.1/ COROLLARY 2.2, TREATED IN THE LITERATURE.

Then the following convergence property holds:

$$\forall \epsilon>0,\ \exists \gamma>0 \text{ such that } |f(\lambda)^{-1}-l|<\epsilon,\ \forall \lambda \in \Psi_\gamma. \quad (20)$$

Notice that for any $\gamma>0$ the set Ψ_γ is a *logarithmic sector* stretching out into the left half plane. Furthermore, the collection $\{\Psi_\gamma\}_{\gamma\geq 0}$ is nested in the sense

$$\gamma_1 \leq \gamma_2 \Rightarrow \Psi_{\gamma_2} \subseteq \Psi_{\gamma_1}.$$

Restating the proposition in terms of pseudospectra yields:

Corollary 3.5: Let Ψ_γ be defined as in Proposition 3.4. If A_m is regular, then

$$\begin{array}{l} \forall \epsilon \in \left(0,\ \frac{w_m}{\|A_m^{-1}\|_\alpha}\right),\ \exists \gamma>0 \text{ such that } \Psi_\gamma \cap \Lambda_\epsilon = \phi, \\ \forall \epsilon > \frac{w_m}{\|A_m^{-1}\|_\alpha},\ \exists \gamma>0 \text{ such that } \Psi_\gamma \subset \Lambda_\epsilon. \end{array} \quad (21)$$

If A_m is singular, then

$$\forall \epsilon>0,\ \exists \gamma>0 \text{ such that } \Psi_\gamma \subset \Lambda_\epsilon. \quad (22)$$

In case of singular A_m, the pseudospectrum Λ_ϵ thus stretches out along the negative real axis, for *any* value of $\epsilon>0$, unlike the case of regular A_m, where this only happens for $\epsilon > w_m/\|A_m^{-1}\|_\alpha$. As a consequence, *infinitesimal* perturbations may result in the introduction of eigenvalues with *small* imaginary parts (but large negative real parts).

IV. ILLUSTRATIVE EXAMPLES

To demonstrate the above results we first consider the following DDE,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) \quad (23)$$

where

$$A_0 = \begin{bmatrix} -5 & 1 \\ 2 & -6 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}. \quad (24)$$

Figure 1(a) shows the spectrum of (23), where we have used DDE-BIFTOOL, a Matlab package for the bifurcation analyses of DDEs [9]. The system is shown to be stable with all eigenvalues confined to the left-half plane. To investigate how this stability may change under perturbations of the matrices A_0 and A_1 we need to compute the corresponding pseudospectra.

To this end, we consider perturbations of A_0 and A_1 using the global measure (7) with $p_1=2$ and $p_2=\infty$. Pseudospectra can then be *computed using Theorem 2.1* with $\alpha=2$ and $\beta=1$. Specifically,

$$f(\lambda) = \left\| \left(\lambda I - A_0 - A_1 e^{-\lambda}\right)^{-1} \right\|_2 \left(\frac{1}{w_0} + \frac{e^{-\lambda}}{w_1} \right). \quad (25)$$

The first term on the right-hand side of (25) can be computed as the minimum singular value of $\lambda I - A_0 - A_1 e^{-\lambda}$ [2]. Thus, by evaluating $f(\lambda)$ for λ on a grid over a region of the complex plane, and by using a contour plotter to view the results, the boundaries of ϵ-pseudospectra are identified.

Figures 1(b)–(d) show the ϵ-pseudospectra of (23) where different weights have been applied to A_0 and A_1. Specifically, $(w_0,w_1)=(\infty,1)$ (b), $(w_0,w_1)=(2,2)$ (c), and $(w_0,w_1)=(1,\infty)$ (d). In each panel, from outermost to innermost (or rightmost to leftmost if the curve is not closed), the curves correspond to boundaries of ϵ-pseudospectra with $\epsilon=10^{1.25}$, $10^{1.0}$, $10^{0.75}$, $10^{0.5}$, $10^{0.25}$, 10^{0}, and $10^{-0.5}$. It can be seen that the conclusions drawn in Section III-A hold, that is, perturbations of A_0 stretch pseudospectra lying in the right-half plane (d). While perturbations applied to A_1 stretch the pseudospectra lying in the left-half plane (b). Furthermore, Fig. 2 shows the intersection of ϵ-pseudospectrum curves with the imaginary axis. In each panel, the darkest curve corresponds to an ϵ-pseudospectrum curve of Fig. 1(a), the next to a curve of Fig. 1(b), and the lightest curve corresponds to an ϵ-pseudospectrum curve of Fig. 1(c). Specifically, Fig. 2(a) shows the intersection of the three curves for $\epsilon=10^{1.25}$, Fig. 2(b) for $\epsilon=10^{1.0}$, and Fig. 2(c) for $\epsilon=10^{0.75}$. For a given ϵ, these curves are seen to intersect the imaginary axis at the same point, independent of the weighting applied to the system matrices. Thus, demonstrating the third conclusion of Section III-A.

Figure 3 shows for each $\omega \in [-50,\ 50]$ which ϵ-pseudospectrum curve intersects the imaginary axis at $\lambda=j\omega$, that is $f^{-1}(j\omega)$. The minimum of this curve represents the stability radius of the system,

$$r_{\mathbb{C}}(\mathbb{C}^-, \|\cdot\|_{\text{glob}}) \approx 3.28011.$$

Since the minimum is reached for $\omega=0$ the smallest destabilizing perturbations shift an eigenvalue to the origin.

Proposition 3.4 applies to this problem with

$$l = \frac{w_1}{\|A_1^{-1}\|_2} \approx 0.4282\ w_1. \quad (26)$$

In Figure 4(a) we show ϵ-pseudospectra for the weights $(w_0,w_1)=(\infty,1)$ and $\epsilon=0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$. In Figure 4(b) we take $(w_0,w_1)=(2,2)$ and $\epsilon=0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$. In both cases only the ϵ-pseudospectrum for the largest value of ϵ stretches out infinitely far along the negative real axis, as follows from (26).

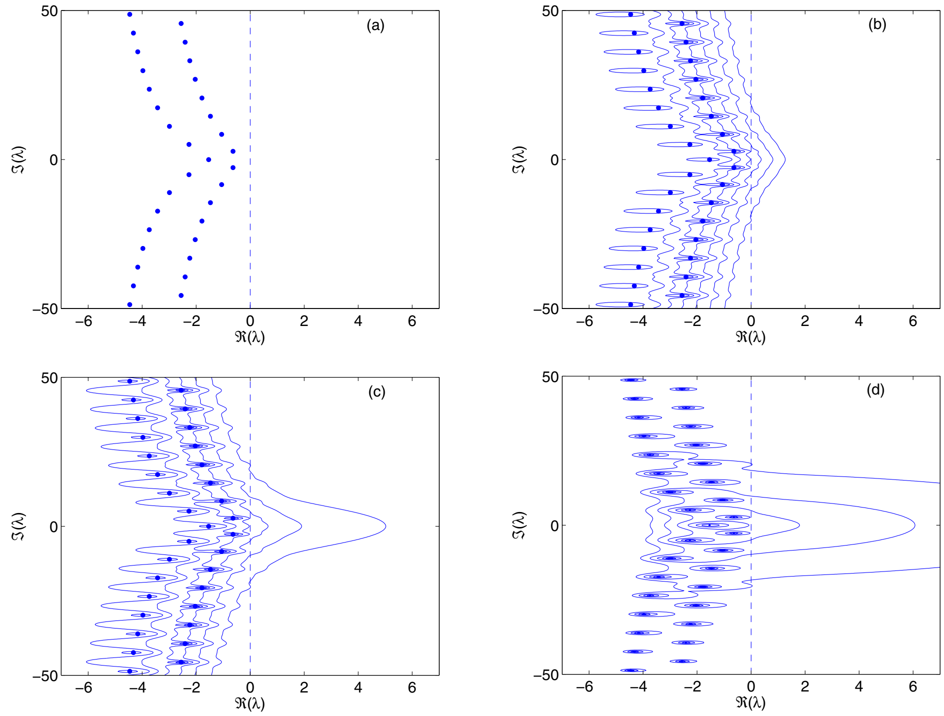

Fig. 1. Weighted pseudospectra of the DDE (23). Panel (a) shows the spectrum of the unperturbed problem. In all other panels, from rightmost to leftmost, the contours correspond to $\epsilon = 10^{1.25}$, $10^{1.0}$, $10^{0.75}$, $10^{0.5}$, $10^{0.25}$, 10^{0}, and $10^{-0.5}$. From (b) to (d), the weights w_0 and w_1 applied to the A_0 and A_1 matrices were $(w_0, w_1) = (\infty, 1)$, $(w_0, w_1) = (2, 2)$, and $(w_0, w_1) = (1, \infty)$, respectively.

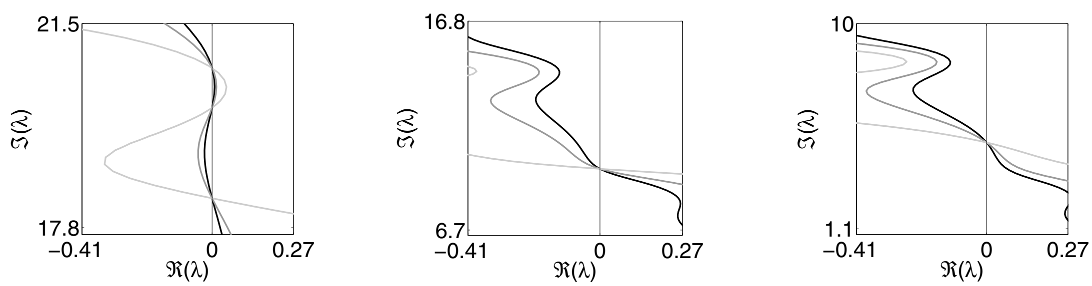

Fig. 2. Crossings of ϵ-pseudospectrum curves with the imaginary axis, for $\epsilon = 10^{1.25}$ (left), $\epsilon = 10$ (middle) and $\epsilon = 10^{0.75}$ (right). In the three cases the darkest contour contour corresponds to the weights $(w_0, w_1) = (\infty, 1)$, the middle curve to $(2, 2)$ and the lightest curve to $(1, \infty)$.

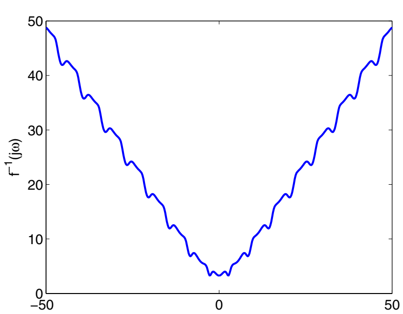

Fig. 3. The function $\omega \rightarrow f^{-1}(j\omega)$ for the system (23). The minimum is the complex stability radius.

To illustrate the effects of a singular matrix corresponding to the largest delay, we consider the system:

$$\dot{x}(t) = A_1 x(t-1) + A_2 x(t-2) \qquad (27)$$

where

$$A_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_4 \end{pmatrix}.$$

Figure 5(a) shows the spectrum of (27) for $\delta_1 = 10$, $\delta_2 = 2$, $\delta_4 = 1$, and varying $\delta_3 = 0$ ('$\circ$'), 0.1 ('$\times$') and 0.5 ('$+$'). When the matrix A_2, corresponding to the largest delay, is singular, that is, $\delta_3 = 0$, the system has three tails of eigenvalues. As δ_3 is increased, an *additional* tail of

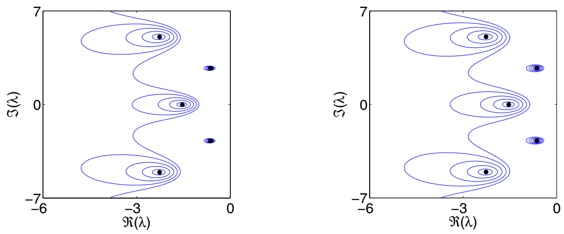

Fig. 4. (left) - ϵ-pseudospectrum curves for $(w_0, w_1) = (\infty, 1)$ and $\epsilon = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$. (right) - ϵ-pseudospectrum curves for $(w_0, w_1) = (2, 2)$ and $\epsilon = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$.

eigenvalues enters our region of interest from $\Re(\lambda) = -\infty$. Consequently, as predicted also by Corollary **3.1**, for $\delta_3 = 0$ we expect ϵ-pseudospectra to be unbounded to the left, even for arbitrarily small ϵ. This is confirmed in Fig. 5(b) where for $\delta_3 = 0$ we show contours representing pseudospectra for $\epsilon = 10^{-1}$, $10^{-0.5}$, 10^0 and $10^{0.25}$, with the weights set to $w_1 = 2$ and $w_2 = 2$. Importantly, the two contours on the left of Fig. 5(b) correspond to pseudospectrum contours with $\epsilon = 10^{-0.5}$ (leftmost) and 10^0. These contours are associated with the eigenvalues accumulated at $\Re(\lambda) = -\infty$.

Next, if δ_3 is fixed to one, and δ_2 is brought to zero such that A_1 becomes singuar, then a different mechanism can be observed[1]: two of the four tails of eigenvalues collapse to one tail, instead of having one tail moving off towards $\Re(\lambda) = -\infty$. Note that the latter is not possible by Corollary **3.1**, since A_2 is regular.

V. Conclusions

A unifying treatment of pseudospectra of *analytic matrix functions* was presented. For a class of retarded time-delay systems special properties of pseudospectra were derived and related to the behaviour of eigenvalues, emphasizing the effect of the weights in the perturbation measures and the asymptotic behaviour of pseudospectra.

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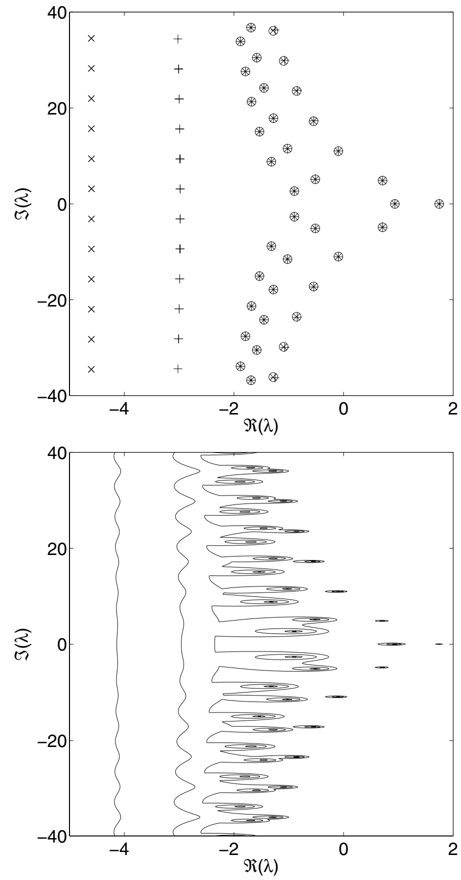

Fig. 5. The top panel shows the spectrum of (27) for $\delta_1 = 10$, $\delta_2 = 2$, $\delta_4 = 1$, and varying $\delta_3 = 0$ ('$\circ$'), 0.1 ('$\times$') and 0.5 ('$+$'). The panel below (b), from rightmost to leftmost (or form outermost to innermost for closed curves), the corresponding pseudospectra contours for $\epsilon = 10^{0.25}$, 10^0, $10^{-0.5}$, and 10^{-1}. The weights were set to $(w_1, w_2) = (2, 2)$.

References

[1] L. Trefethen, "Pseudospectra of linear operators," *SIAM Review*, vol. 39, no. 3, pp. 383–406, 1997.

[2] K. Green and T. Wagenknecht, "Pseudospectra of delay differential equations," Bristol Centre for Applied Nonlinear Mathematics, University of Bristol, Tech. Rep., 2004, available at www.enm.bris.ac.uk/anm/publications-2004.html.

[3] W. Michiels and D. Roose, "An eigenvalue based approach for the robust stabilization of linear time-delay systems," *International Journal of Control*, vol. 76, no. 7, pp. 678–686, 2003.

[4] G. Pappas and D. Hinrichsen, "Robust stability of linear systems described by higher order dynamic equations," *IEEE Transactions on Automatic Control*, vol. 38, pp. 1430–1435, 1993.

[5] Y. Genin, R. Stefan, and P. Van Dooren, "Real and complex stability radii of polynomial matrices," *Linear Algebra and its Applications*, vol. 351-352, pp. 381–410, 2002.

[6] J. Hale and S. Verduyn Lunel, *Introduction to functional differential equations*, ser. Applied mathematical sciences. Springer Verlag, 1993, vol. 99.

[7] P. Van Dooren and V. Vermaut, "On stability radii of generalized eigenvalue problems," in *Proceedings of the 1997 European Control Conference (ECC'97)*, Brussels, Belgium, 1997.

[8] F. Tisseur and N. Higham, "Structured pseudospectra for polynomial eigenvalue problems with applications," *SIAM J. Matrix Analysis and Applications*, vol. 23, no. 1, pp. 187–208, 2001.

[9] K. Engelborghs, T. Luzyanina, and G. Samaey, "DDE-BIFTOOL v. 2.00: a Matlab package for bifurcation analysis of delay differential equations," Department of Computer Science, Katholieke Universiteit Leuven, Belgium, TW Report 330, October 2001.

[1] the eigenvalue plots are omitted due to space limitations.