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Uniform Disturbance Attenuation for Markovian Jump Linear Systems in Discrete Time

Ji-Woong Lee and Geir E. Dullerud

Abstract— **A complete characterization of almost surely uniformly stable and contractive Markovian jump linear systems is given in the discrete-time domain via the union of an increasing family of linear matrix inequality conditions. This characterization draws on the facts that the Riccati difference inequality associated with a stable and contractive linear time-varying system admits a solution which has finite memory of past parameters, and that each Markovian jump linear system can be treated as a switched linear system where the underlying Markov chain defines the switching path constraint. The result leads to a semidefinite programming–based controller synthesis technique, from which optimal finite-path dependent dynamic output feedback controllers arise naturally.**

I. Introduction

The Markovian jump linear system is a multi-modal linear system whose mode changes within a finite set according to the state transitions of an underlying Markov chain. We consider the problem of almost sure uniform disturbance attenuation for Markovian jump linear systems in the discrete-time domain. The problem is not deterministic because the Markov chain determines the switching sequence (i.e., the sequence of modes); yet our technique is non-stochastic due to the uniformity requirement on the almost sure stability and performance. We formulate the problem in the framework of switched linear systems with switching path constraints, and provide a complete solution without any assumption on the parameters or the Markov chain.

A key contribution of this work to the control of Markovian jump linear systems is that an exact "control-oriented" condition for almost sure (uniform) stabilization and disturbance attenuation is provided for the first time. This condition is control-oriented in the sense that it leads to semidefinite programming–based techniques that render optimal controllers very efficiently. The usual approach in the literature to disturbance attenuation of Markovian jump linear systems has been based on the notion of stochastic stability in both continuous time [1], [2], [3] and discrete time [4], [5], [6]. These results, however, are either partly satisfactory or exact but not well-suited for efficient controller synthesis; moreover, although stochastically stable Markovian jump linear systems are almost surely stable [7], the usual approach does not guarantee almost sure disturbance attenuation.

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J.-W. Lee is with the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801, USA jiwoong@uiuc.edu

G. E. Dullerud is with the Department of Mechanical and Industrial Engineering, University of Illinois, Urbana, IL 61801, USA dullerud@uiuc.edu

Our approach is quite different and based on a fundamental property of linear time-varying systems. The performance analysis relies on the extended Kalman-Yacubovitch-Popov (KYP) inequality [8] for stable and contractive linear time-varying systems, from which we derive a uniformly stabilizing solution to the associated Riccati difference inequality that has finite memory of past parameters. On the other hand, the contoller synthesis is done via the linear matrix inequality (LMI) embedding technique originally developed for H^∞ control of linear time-invariant systems [9], [10]. As a result, we obtain an increasing family of LMI conditions whose "union" is necessary and sufficient for the Markovian jump linear system to have a finite-path dependent controller (i.e., a controller that has finite memory of past modes) that yields an almost surely uniformly stable and contractive closed-loop system, where the uniformity is in time and over all realizations of the underlying Markov chain.

The reason why our result involves a family of LMI conditions is that, under our notion of stability and performance, each Markovian jump linear system is equivalent to a switched linear system (i.e., a collection of linear time-varying systems whose parameters vary within a single finite set). The basic problem in switched linear systems is to determine the stability of every admissible switching sequence [11], [12]. This problem is considered semidecidable and involves a countable but increasing family of LMIs to check [13]. Our result inherits the same limitations due to the problem nature, even though this limitation is not likely to manifest in practice. (The number of LMIs whose feasibility needs to be checked is often very small.) Nevertheless, there has been little work on the input-output performances of switched linear systems, and we provide the first complete solution to the uniform stabilization and disturbance attenuation of discrete-time switched linear systems as well.

Due to the space constraint, lemmas and theorems are presented without proof. For more detailed discussion of our results, complete with proofs and several examples, the reader is referred to [14], [15].

Notation

If $\mathbf{X} \in \mathbb{R}^{m\times n}$, the range (or image) of $\mathbf{X}$ is denoted by $\operatorname{Im}\mathbf{X}$, the null space (or kernel) of $\mathbf{X}$ by $\operatorname{Ker}\mathbf{X}$, and the rank of $\mathbf{X}$ by $\operatorname{rank}\mathbf{X}$; denoted by $\mathbf{N}(\mathbf{X})$ is any particular full-rank matrix such that $\operatorname{Im}\mathbf{N}(\mathbf{X}) = \operatorname{Ker}\mathbf{X}$. If $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times n}$ are symmetric and $\mathbf{X}-\mathbf{Y}$ is positive definite (resp. nonnegative definite), we write $\mathbf{X} > \mathbf{Y}$ (resp. $\mathbf{X} \geq \mathbf{Y}$). The identity matrix is denoted by $\mathbf{I}$ with n understood.

For $x \in \mathbb{R}^n$, denoted by $\|x\|$ is the Euclidean norm of x. If $\mathbf{X} \in \mathbb{R}^{m\times n}$, the Euclidean norm induces the spectral norm $\|\mathbf{X}\|$ of $\mathbf{X}$. If $\mathbf{x} = (x(0), x(1), \dots)$ is a sequence in $\mathbb{R}^n$, then we write $\mathbf{x} \in \ell^2(\mathbb{R}^n)$ whenever the ℓ^2 norm of $\mathbf{x}$ is finite.

II. Performance of Linear Time-Varying Systems

Let a subset $\boldsymbol{\mathcal{G}}$ of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times m}$ and a sequence $\boldsymbol{\theta} = (\theta(0), \theta(1), \dots)$ in $\{0, 1, \dots\}$ be as follows:

$$\boldsymbol{\mathcal{G}} = \bigcup\nolimits_{i=0}^{\infty} \{(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)\}, \quad \theta(t) = t. \tag{1}$$

Then $(\mathbf{A}_{\theta(t)}, \mathbf{B}_{\theta(t)}, \mathbf{C}_{\theta(t)}, \mathbf{D}_{\theta(t)}) = (\mathbf{A}_t, \mathbf{B}_t, \mathbf{C}_t, \mathbf{D}_t)$, so the pair $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ defines the linear time-varying system that has the state-space representation

$$\begin{aligned} x(t+1) &= \mathbf{A}_t x(t) + \mathbf{B}_t w(t), \\ z(t) &= \mathbf{C}_t x(t) + \mathbf{D}_t w(t), \end{aligned} \tag{2}$$

where $\boldsymbol{\mathcal{G}}$ defines an indexed family of parameter quadruples, and the sequence $\boldsymbol{\theta}$ chooses one quadruple among $\boldsymbol{\mathcal{G}}$ for each $t \geq 0$. Given the initial state $x(0)$ and disturbance sequence $\mathbf{w} = (w(0), w(1), \dots)$, system (2) determines the state sequence $\mathbf{x} = (x(0), x(1), \dots)$ and output sequence $\mathbf{z} = (z(0), z(1), \dots)$. If $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$ are all zero matrices, we write $(\boldsymbol{\mathcal{A}}, \boldsymbol{\theta})$ for $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ where

$$\boldsymbol{\mathcal{A}} = \{\mathbf{A}_0, \mathbf{A}_1, \dots\}.$$

Definition 1: The system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$, and hence $(\boldsymbol{\mathcal{A}}, \boldsymbol{\theta})$, is said to be *uniformly (exponentially) stable* if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that, whenever $\mathbf{w} = 0$,

$$\|x(t)\| \leq c\, \lambda^{t-t_0} \, \|x(t_0)\| \tag{3}$$

for $t \geq t_0 \geq 0$ and $x(t_0) \in \mathbb{R}^n$.

Definition 2: The system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ is said to be *uniformly (strictly) contractive* if there exists a $\gamma \in (0, 1)$ such that, whenever $x(t_0) = 0$,

$$\sum\nolimits_{s=t_0}^{t} \|z(s)\|^2 \leq \gamma^2 \sum\nolimits_{s=t_0}^{t} \|w(s)\|^2 \tag{4}$$

for $t \geq t_0 \geq 0$ and $\mathbf{w} \in \ell^2(\mathbb{R}^m)$.

Lemma 1: Let $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\theta}$ be as in (1); let $\boldsymbol{\mathcal{G}}$ be bounded. The following are equivalent:

(a) The system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ is uniformly exponentially stable and uniformly strictly contractive.

(b) There exist symmetric positive definite matrices $\mathbf{X}_t \in \mathbb{R}^{n\times n}$, uniformly bounded above and below, such that

$$\begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_{t+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \end{bmatrix} - \begin{bmatrix} \mathbf{X}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} < \mathbf{0} \tag{5}$$

holds uniformly for all $t \geq 0$.

(c) There exist symmetric positive definite matrices $\mathbf{Y}_t \in \mathbb{R}^{n\times n}$, uniformly bounded above and below, such that

$$\begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \end{bmatrix} \begin{bmatrix} \mathbf{Y}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} \mathbf{Y}_{t+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} < \mathbf{0} \tag{6}$$

holds uniformly for all $t \geq 0$.

Moreover, if either (b) or (c) holds, then one may take $\mathbf{X}_t = \mathbf{Y}_t^{-1}$ for $t \geq 0$.

Lemma 1 is a time-varying version of the classical KYP lemma—the equivalence of (a) and (b) is proved in [8]. Inequality (5) is called the extended KYP inequality, and (6) is its dual form. Solutions of these inequalities are obtained by solving the associated Riccati difference equations. Let $\mathbb{S}$ be the set of all symmetric matrices in $\mathbb{R}^{n\times n}$. For each i, let $\mathbb{Y}_i$ be the set of symmetric matrices $\mathbf{Y} \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{V}_i(\mathbf{Y}) = \mathbf{I} - \mathbf{D}_i\mathbf{D}_i^{\mathrm{T}} - \mathbf{C}_i\mathbf{Y}\mathbf{C}_i^{\mathrm{T}}$$

is invertible. Define $\mathcal{R}_i \colon \mathbb{Y}_i \to \mathbb{S}$, $i = 0, 1, \dots$, by

$$\begin{aligned} \mathcal{R}_i(\mathbf{Y}) &= \mathbf{A}_i\mathbf{Y}\mathbf{A}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{B}_i^{\mathrm{T}} \\ &\quad + (\mathbf{A}_i\mathbf{Y}\mathbf{C}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{D}_i^{\mathrm{T}})\mathcal{V}_i(\mathbf{Y})^{-1}(\mathbf{C}_i\mathbf{Y}\mathbf{A}_i^{\mathrm{T}} + \mathbf{D}_i\mathbf{B}_i^{\mathrm{T}}) \end{aligned}$$

for $\mathbf{Y} \in \mathbb{Y}_i$.

Lemma 2: Let $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\theta}$ be as in (1); let $\boldsymbol{\mathcal{G}}$ be bounded. The following are equivalent:

(a) The system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ is uniformly exponentially stable and uniformly strictly contractive.

(b) There exists a constant $\varepsilon_0 > 0$ such that, whenever $\varepsilon \in [0, \varepsilon_0]$, the Riccati difference equation

$$\mathbf{Y}_{t_0}^{(\varepsilon,t_0)} = \varepsilon\mathbf{I}, \quad \mathbf{Y}_{t+1}^{(\varepsilon,t_0)} = \mathcal{R}_t\big(\mathbf{Y}_t^{(\varepsilon,t_0)}\big) + \varepsilon\mathbf{I} \tag{7}$$

yields symmetric positive definite matrices $\mathbf{Y}_t^{(\varepsilon,t_0)} \in \mathbb{R}^{n\times n}$, uniformly bounded above and below, such that $\mathcal{V}_t\big(\mathbf{Y}_t^{(\varepsilon,t_0)}\big) > \mathbf{0}$ holds uniformly for $t \geq t_0 \geq 0$.

Moreover, if (b) holds, one may take $\mathbf{Y}_t = \mathbf{Y}_t^{(\varepsilon,0)}$ in (6).

The proof of Lemma 2 is based on the following standard lemma. For $\mathbf{Y} \in \mathbb{Y}_i$, let

$$\mathcal{A}_i(\mathbf{Y}) = \mathbf{A}_i + (\mathbf{A}_i\mathbf{Y}\mathbf{C}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{D}^{\mathrm{T}})\mathcal{V}_i(\mathbf{Y})^{-1}\mathbf{C}_i.$$

Lemma 3: Let $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)} \in \mathbb{Y}_i$. Then

$$\begin{aligned} \mathcal{R}_i\big(\mathbf{Y}^{(1)}\big) - \mathcal{R}_i\big(\mathbf{Y}^{(2)}\big) &= \mathcal{A}_i\big(\mathbf{Y}^{(2)}\big)\boldsymbol{\Delta}^{(12)}\mathcal{A}_i\big(\mathbf{Y}^{(2)}\big)^{\mathrm{T}} \\ &\quad + \mathcal{A}_i\big(\mathbf{Y}^{(2)}\big)\boldsymbol{\Delta}^{(12)}\mathbf{C}_i^{\mathrm{T}}\mathcal{V}_i\big(\mathbf{Y}^{(1)}\big)^{-1}\mathbf{C}_i\boldsymbol{\Delta}^{(12)}\mathcal{A}_i\big(\mathbf{Y}^{(2)}\big)^{\mathrm{T}} \end{aligned}$$

where $\boldsymbol{\Delta}^{(12)} = \mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}$.

Lemma 3 is useful in proving asymptotic properties of the Riccati difference equation. In particular, an immediate consequence of Lemma 3 is that $\mathcal{R}_i\big(\mathbf{Y}^{(1)}\big) \geq \mathcal{R}_i\big(\mathbf{Y}^{(2)}\big)$ if $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)} \in \mathbb{Y}_i$ and $\mathbf{Y}^{(1)} \geq \mathbf{Y}^{(2)}$; moreover, it is not hard to see from this lemma that

$$\mathcal{R}_i\big(\mathbf{Y}^{(1)}\big) - \mathcal{R}_i\big(\mathbf{Y}^{(2)}\big) = \mathcal{A}_i\big(\mathbf{Y}^{(1)}\big)\boldsymbol{\Delta}^{(12)}\mathcal{A}_i\big(\mathbf{Y}^{(2)}\big)^{\mathrm{T}} \tag{8}$$

whenever $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)} \in \mathbb{Y}_i$, and $\boldsymbol{\Delta}^{(12)} = \mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}$.

Lemmas 1–3 and (8) lead to the following result:

Theorem 1: Let $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\theta}$ be as in (1); let $\boldsymbol{\mathcal{G}}$ be bounded. Suppose that the system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ is uniformly exponentially stable and uniformly strictly contractive, so that condition (b) of Lemma 2 holds. Then the following hold:

(a) For $\varepsilon \in (0, \varepsilon_0)$ and $t_0 \geq 0$, let

$$\begin{aligned} \boldsymbol{\mathcal{A}}^{(\varepsilon,t_0)} &= \big\{\mathcal{A}_{t_0}\big(\mathbf{Y}_{t_0}^{(\varepsilon,t_0)}\big), \mathcal{A}_{t_0+1}\big(\mathbf{Y}_{t_0+1}^{(\varepsilon,t_0)}\big), \dots\big\}, \\ \boldsymbol{\theta}^{(t_0)} &= (t_0, t_0+1, \dots). \end{aligned}$$

Then each system $\left(\boldsymbol{\mathcal{A}}^{(\varepsilon,t_0)}, \boldsymbol{\theta}^{(t_0)}\right)$ is uniformly exponentially stable. Moreover, for each $\varepsilon \in (0, \varepsilon_0)$, there exist $c_\varepsilon \geq 1$ and $\lambda_\varepsilon \in (0,1)$ such that, for $t > t_0 \geq 0$,

$$\left\|\mathcal{A}_{t-1}\big(\mathbf{Y}_{t-1}^{(\varepsilon,t_0)}\big) \cdots \mathcal{A}_{t_0}\big(\mathbf{Y}_{t_0}^{(\varepsilon,t_0)}\big)\right\| \leq c_\varepsilon\, \lambda_\varepsilon^{t-t_0}.$$

(b) For each $\varepsilon \in (0, \varepsilon_0)$, there exists a nonnegative integer M such that the symmetric positive definite matrices

$$\mathbf{Y}_t = \begin{cases} \mathbf{Y}_t^{(\varepsilon,t-M)}, & t \geq M; \\ \mathbf{Y}_t^{(\varepsilon,0)}, & t < M, \end{cases}$$

are uniformly bounded above and below, and satisfy (6) uniformly for $t \geq 0$.

If $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ is uniformly stable, it follows from (8) and the results in [16], [17] that, for each $t_0 \geq 0$, $\mathbf{Y}_t^{(\varepsilon,t_0)}$ converges to the unique "moving equilibrium" (i.e., the maximal solution) of the Riccati difference equation (7) as $t - t_0 \to \infty$. However, part (a) of Theorem 1 says that this convergence is uniform in (t, t_0) because the uniform stability of $(\boldsymbol{\mathcal{A}}^{(\varepsilon,t_0)}, \boldsymbol{\theta}^{(t_0)})$ is again uniform in t_0. Part (b) of Theorem 1 says that the KYP inequality (or equivalently, the Riccati inequality) associated with a uniformly stable and contractive linear time-varying system has a solution that has finite memory of past parameters.

III. Control of Switched Linear Systems

Markovian jump linear systems will be treated in the next section as if they are switched linear systems. Fix a positive integer N, and let $\mathbf{\Omega}$ be the set of all infinite sequences in $\{1, \ldots, N\}$. Let $\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $\mathbf{B}_i \in \mathbb{R}^{n\times m}$, $\mathbf{C}_i \in \mathbb{R}^{l\times n}$, $\mathbf{D}_i \in \mathbb{R}^{l\times m}$ for $i = 1, \ldots, N$. If

$$\boldsymbol{\mathcal{G}} = \{(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1), \ldots, (\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, \mathbf{D}_N)\}, \quad (9)$$

and if $\mathbf{\Theta}$ is a nonempty subset of $\mathbf{\Omega}$, then the *switched linear system*, identified with the pair $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$, is the family of linear time-varying systems $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$, $\boldsymbol{\theta} = (\theta(0), \theta(1), \ldots) \in \mathbf{\Theta}$, with state-space representations

$$\begin{aligned} x(t+1) &= \mathbf{A}_{\theta(t)} x(t) + \mathbf{B}_{\theta(t)} w(t), \\ z(t) &= \mathbf{C}_{\theta(t)} x(t) + \mathbf{D}_{\theta(t)} w(t). \end{aligned} \quad (10)$$

If $\theta(t) = i$, then the system is said to be in *mode* i at time t and its parameters at time t are given by $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$. Each $\boldsymbol{\theta} \in \mathbf{\Theta}$ is called a *switching sequence*.

In particular, the pair $(\boldsymbol{\mathcal{G}}, \mathbf{\Omega})$ is the usual *discrete linear inclusion* without any constraints on the switching sequence; on the other hand, if $\mathbf{\Theta}$ is a singleton $\{\boldsymbol{\theta}\}$, then $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$ is the linear time-varying system $(\boldsymbol{\mathcal{G}}, \boldsymbol{\theta})$ that has a finite parameter set $\boldsymbol{\mathcal{G}}$. We require that the stability and contractiveness of the system $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$ be uniform not only in time, but also over all switching sequences in $\mathbf{\Theta}$.

Definition 3: The system $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$ is said to be *uniformly (exponentially) stable* if there exist $c \geq 1$ and $\lambda \in (0,1)$ such that, whenever $\mathbf{w} = 0$, inequality (3) holds for $t \geq t_0 \geq 0$, $x(t_0) \in \mathbb{R}^n$ and $\theta \in \mathbf{\Theta}$.

Definition 4: The system $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$ is said to be *uniformly (strictly) contractive* if there exists a $\gamma \in (0,1)$ such that, whenever $x(t_0) = 0$, inequality (4) holds for $t \geq t_0 \geq 0$, $\mathbf{w} \in \ell^2(\mathbb{R}^m)$ and $\theta \in \mathbf{\Theta}$.

Introduce a dummy mode 0 and think of each $\boldsymbol{\theta} \in \mathbf{\Theta}$ as a two-sided sequence $(\ldots, \theta(-1), \theta(0), \theta(1), \ldots)$ by putting $\theta(t) = 0$ for $t < 0$. Finite sequences in $\{0, \ldots, N\}$ will be called *(finite) switching paths*; in particular, for each nonnegative integer L, elements of $\{0, \ldots, N\}^{L+1}$ are switching paths of length L, and called L*-paths*. Given $\boldsymbol{\theta} \in \mathbf{\Theta}$, let

$$\theta_L(t) = (\theta(t-L), \ldots, \theta(t))$$

for all t and L; the set of L-paths occurring in $\mathbf{\Theta}$ is

$$\boldsymbol{\mathcal{L}}_L(\mathbf{\Theta}) = \{\, \theta_L(t) \colon \boldsymbol{\theta} \in \mathbf{\Theta}, t \geq 0 \,\}.$$

If $(i_0, \ldots, i_L) \in \boldsymbol{\mathcal{L}}_L(\mathbf{\Theta})$, then write $(i_0, \ldots, i_L)_- = (i_0, \ldots, i_{L-1})$ and $(i_0, \ldots, i_L)_+ = (i_1, \ldots, i_L)$ for $L > 0$; write $(i_0, \ldots, i_L)_- = (i_0, \ldots, i_L)_+ = 0$ for $L = 0$. Let $\boldsymbol{\mathcal{M}}_0(\mathbf{\Theta}) = \boldsymbol{\mathcal{L}}_0(\mathbf{\Theta})$, and for $L > 0$ define $\boldsymbol{\mathcal{M}}_L(\mathbf{\Theta})$ to be the smallest subset of $\boldsymbol{\mathcal{L}}_L(\mathbf{\Theta})$ such that the following hold: for each $j \in \boldsymbol{\mathcal{M}}_0(\mathbf{\Theta})$, there is a switching path $(i_0^j, \ldots, i_{L-1}^j) \in \{0, \ldots, N\}^L$ such that, for every $\boldsymbol{\theta} \in \mathbf{\Theta}$ with $\theta(0) = j$, we have $(i_0^j, \ldots, i_{L-1}^j, \theta(0))$, $(i_1^j, \ldots, i_{L-1}^j, \theta(0), \theta(1))$, …, $(i_{L-1}^j, \theta(0), \ldots, \theta(L-1)) \in \boldsymbol{\mathcal{M}}_L(\mathbf{\Theta})$; and $\theta_L(t) \in \boldsymbol{\mathcal{M}}_L(\mathbf{\Theta})$ for all $t \geq L$ and for all $\boldsymbol{\theta} \in \mathbf{\Theta}$. The sets $\boldsymbol{\mathcal{M}}_L(\mathbf{\Theta})$, $L = 0$, 1, …, are unique, so well-defined. Let

$$\boldsymbol{\mathcal{M}}_L^-(\mathbf{\Theta}) = \{\, \mathcal{I}_- \colon \mathcal{I} \in \boldsymbol{\mathcal{M}}_L(\mathbf{\Theta}) \,\}.$$

Now we are ready to state our characterization of uniformly stable and contractive switched linear systems.

Theorem 2: Let $\boldsymbol{\mathcal{G}}$ be as in (9); let $\mathbf{\Theta} \subset \mathbf{\Omega}$ be nonempty. The system $(\boldsymbol{\mathcal{G}}, \mathbf{\Theta})$ is uniformly exponentially stable and uniformly strictly contractive if and only if there exist a nonnegative integer M and an indexed family $\bigcup_{\mathcal{I} \in \boldsymbol{\mathcal{M}}_M^-(\mathbf{\Theta})} \{\mathbf{X}_\mathcal{I}\}$ of symmetric positive definite matrices $\mathbf{X}_\mathcal{I} \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} \mathbf{A}_{i_M} & \mathbf{B}_{i_M} \\ \mathbf{C}_{i_M} & \mathbf{D}_{i_M} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_{(i_0,\ldots,i_M)_+} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{i_M} & \mathbf{B}_{i_M} \\ \mathbf{C}_{i_M} & \mathbf{D}_{i_M} \end{bmatrix} - \begin{bmatrix} \mathbf{X}_{(i_0,\ldots,i_M)_-} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} < \mathbf{0} \quad (11)$$

for all M-paths $(i_0, \ldots, i_M) \in \boldsymbol{\mathcal{M}}_M(\mathbf{\Theta})$.

Theorem 2 characterizes the uniform stability and disturbance attenuation performance of the switched linear system via the countably infinite union of an increasing family of systems of linear matrix inequalities. For uniform stability and contractiveness, not only each member of this family is sufficient, but also the union of the family is necessary.

Now, consider the set

$$\boldsymbol{\mathcal{T}} = \bigcup_1^N \{(\mathbf{A}_i, \mathbf{B}_{1,i}, \mathbf{B}_{2,i}, \mathbf{C}_{1,i}, \mathbf{C}_{2,i}, \mathbf{D}_{11,i}, \mathbf{D}_{12,i}, \mathbf{D}_{21,i})\} \quad (12)$$

with $\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $\mathbf{B}_{1,i} \in \mathbb{R}^{n\times m_1}$, $\mathbf{B}_{2,i} \in \mathbb{R}^{n\times m_2}$, $\mathbf{C}_{1,i} \in \mathbb{R}^{l_1\times n}$, $\mathbf{C}_{2,i} \in \mathbb{R}^{l_2\times n}$, $\mathbf{D}_{11,i} \in \mathbb{R}^{l_1\times m_1}$, $\mathbf{D}_{12,i} \in \mathbb{R}^{l_1\times m_2}$, $\mathbf{D}_{21,i} \in \mathbb{R}^{l_2\times m_1}$ for $i = 1, \ldots, N$. If $\mathbf{\Theta} \subset \mathbf{\Omega}$ is nonempty, then the pair $(\boldsymbol{\mathcal{T}}, \mathbf{\Theta})$ defines the *controlled switched linear* *system* represented by

$$\begin{aligned} x(t+1) &= \mathbf{A}_{\theta(t)}x(t) + \mathbf{B}_{1,\theta(t)}w(t) + \mathbf{B}_{2,\theta(t)}u(t),\\ z(t) &= \mathbf{C}_{1,\theta(t)}x(t) + \mathbf{D}_{11,\theta(t)}w(t) + \mathbf{D}_{12,\theta(t)}u(t),\\ y(t) &= \mathbf{C}_{2,\theta(t)}x(t) + \mathbf{D}_{21,\theta(t)}w(t) \end{aligned} \tag{13}$$

for $t \geq 0$ and $\boldsymbol{\theta} = (\theta(0), \theta(1), \dots) \in \boldsymbol{\Theta}$. Given the initial state $x(0)$, disturbance sequence $\mathbf{w} = (w(t))$ and control sequence $\mathbf{u} = (u(t))$, this system of equations defines the evolution of the state $x(t)$, controlled output $z(t)$, and measured output $y(t)$ for $t \geq 0$.

We make the usual assumption that *the mode $\theta(t)$ is perfectly observed at each time instant t*; however, relaxing the standard restriction to mode dependent controllers (i.e., controllers that do not recall past modes), we consider all the finite-path dependent controllers (i.e., controllers with finite memory of past modes). For each nonnegative integer L, let

$$\boldsymbol{\Theta}_L = \{\, (\theta_L(0), \theta_L(1), \dots) \colon (\theta(0), \theta(1), \dots) \in \boldsymbol{\Theta} \,\};$$

with $\mathbf{A}_{K,\mathcal{I}} \in \mathbb{R}^{n_K \times n_K}$, $\mathbf{B}_{K,\mathcal{I}} \in \mathbb{R}^{n_K \times l_2}$, $\mathbf{C}_{K,\mathcal{I}} \in \mathbb{R}^{m_2 \times n_K}$, $\mathbf{D}_{K,\mathcal{I}} \in \mathbb{R}^{m_2 \times l_2}$ for $\mathcal{I} = (i_0, \dots, i_L) \in \boldsymbol{\mathcal{L}}_L(\boldsymbol{\Theta})$, let

$$\boldsymbol{\mathcal{K}} = \bigcup\nolimits_{\mathcal{I} \in \boldsymbol{\mathcal{L}}_L(\boldsymbol{\Theta})} \{(\mathbf{A}_{K,\mathcal{I}}, \mathbf{B}_{K,\mathcal{I}}, \mathbf{C}_{K,\mathcal{I}}, \mathbf{D}_{K,\mathcal{I}})\}. \tag{14}$$

Then the pair $(\boldsymbol{\mathcal{K}}, \boldsymbol{\Theta}_L)$ defines the *L-path dependent (linear output feedback) controller (of order n_K)*, which determines the control sequence $\mathbf{u}$ according to

$$\begin{aligned} x_K(t+1) &= \mathbf{A}_{K,\theta_L(t)}x_K(t) + \mathbf{B}_{K,\theta_L(t)}y(t),\\ u(t) &= \mathbf{C}_{K,\theta_L(t)}x_K(t) + \mathbf{D}_{K,\theta_L(t)}y(t), \end{aligned} \tag{15}$$

given the initial controller state $x_K(0)$ and a switching sequence $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. An L-path dependent controller is called *finite-path dependent*, and a zero-path dependent controller is said to be *mode dependent*.

Let

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{K}}} = \bigcup\nolimits_{\mathcal{I} \in \boldsymbol{\mathcal{L}}_L(\boldsymbol{\Theta})} \{(\widetilde{\mathbf{A}}_{\mathcal{I}}, \widetilde{\mathbf{B}}_{\mathcal{I}}, \widetilde{\mathbf{C}}_{\mathcal{I}}, \widetilde{\mathbf{D}}_{\mathcal{I}})\}, \tag{16}$$

with

$$\begin{aligned} \widetilde{\mathbf{A}}_{(i_0,\dots,i_L)} &= \widehat{\mathbf{A}}_{i_L} + \widehat{\mathbf{B}}_{2,i_L}\mathbf{K}_{(i_0,\dots,i_L)}\widehat{\mathbf{C}}_{2,i_L},\\ \widetilde{\mathbf{B}}_{(i_0,\dots,i_L)} &= \widehat{\mathbf{B}}_{1,i_L} + \widehat{\mathbf{B}}_{2,i_L}\mathbf{K}_{(i_0,\dots,i_L)}\widehat{\mathbf{D}}_{21,i_L},\\ \widetilde{\mathbf{C}}_{(i_0,\dots,i_L)} &= \widehat{\mathbf{C}}_{1,i_L} + \widehat{\mathbf{D}}_{12,i_L}\mathbf{K}_{(i_0,\dots,i_L)}\widehat{\mathbf{C}}_{2,i_L},\\ \widetilde{\mathbf{D}}_{(i_0,\dots,i_L)} &= \mathbf{D}_{11,i_L} + \widehat{\mathbf{D}}_{12,i_L}\mathbf{K}_{(i_0,\dots,i_L)}\widehat{\mathbf{D}}_{21,i_L} \end{aligned} \tag{17}$$

where

$$\widehat{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{B}}_{1,i} = \begin{bmatrix} \mathbf{B}_{1,i} \\ \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{C}}_{1,i} = \begin{bmatrix} \mathbf{C}_{1,i} & \mathbf{0} \end{bmatrix},$$

$$\widehat{\mathbf{B}}_{2,i} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,i} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{C}}_{2,i} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}_{2,i} & \mathbf{0} \end{bmatrix},$$

$$\widehat{\mathbf{D}}_{12,i} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{12,i} \end{bmatrix}, \quad \widehat{\mathbf{D}}_{21,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{21,i} \end{bmatrix},$$

$$\mathbf{K}_{(i_0,\dots,i_L)} = \begin{bmatrix} \mathbf{A}_{K,(i_0,\dots,i_L)} & \mathbf{B}_{K,(i_0,\dots,i_L)} \\ \mathbf{C}_{K,(i_0,\dots,i_L)} & \mathbf{D}_{K,(i_0,\dots,i_L)} \end{bmatrix}.$$

If we define the closed-loop state by

$$\tilde{x}(t) = \left[\, x(t)^{\mathrm{T}}\ x_K(t)^{\mathrm{T}} \,\right]^{\mathrm{T}} \in \mathbb{R}^{n+n_K},$$

then the *closed-loop system* $(\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{K}}}, \boldsymbol{\Theta}_L)$ is represented by

$$\begin{aligned} \tilde{x}(t+1) &= \widetilde{\mathbf{A}}_{\theta_L(t)}\tilde{x}(t) + \widetilde{\mathbf{B}}_{\theta_L(t)}w(t),\\ z(t) &= \widetilde{\mathbf{C}}_{\theta_L(t)}\tilde{x}(t) + \widetilde{\mathbf{D}}_{\theta_L(t)}w(t) \end{aligned} \tag{18}$$

for $t \geq 0$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. As long as finite-path dependent controllers are concerned, the closed-loop system is still a switched linear system where the closed-loop modes are the L-paths in $\boldsymbol{\mathcal{L}}_L(\boldsymbol{\Theta})$.

The following lemma is a consequence of Theorem 2, and can be proved as in [10], where the decompositions (17), along with a Schur complement argument and an appropriate congruence transformation, are used.

Lemma 4: Let $\boldsymbol{\mathcal{T}}$ be as in (12); let $\boldsymbol{\Theta} \subset \boldsymbol{\Omega}$ be nonempty. Suppose that $\boldsymbol{\mathcal{K}}$ is as in (14) with some nonnegative integer L. Then the closed-loop system $(\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{K}}}, \boldsymbol{\Theta}_L)$ is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer $M \geq L$ and an indexed family $\bigcup_{\mathcal{I} \in \boldsymbol{\mathcal{L}}_M^-(\boldsymbol{\Theta})}\{\mathbf{X}_{\mathcal{I}}\}$ of symmetric positive definite matrices $\mathbf{X}_{\mathcal{I}} \in \mathbb{R}^{(n+n_K)\times(n+n_K)}$ such that

$$\begin{aligned} \mathbf{H}_{(i_0,\dots,i_M)} &+ \mathbf{G}_{i_M}^{\mathrm{T}}\mathbf{K}_{(i_{M-L},\dots,i_M)}^{\mathrm{T}}\mathbf{F}_{i_M}\\ &+ \mathbf{F}_{i_M}^{\mathrm{T}}\mathbf{K}_{(i_{M-L},\dots,i_M)}\mathbf{G}_{i_M} < \mathbf{0} \end{aligned} \tag{19}$$

for all M-paths $(i_0, \dots, i_M) \in \boldsymbol{\mathcal{L}}_M(\boldsymbol{\Theta})$, where

$$\mathbf{H}_{(i_0,\dots,i_M)} = \begin{bmatrix} -\mathbf{X}_{(i_0,\dots,i_M)_+}^{-1} & \widehat{\mathbf{A}}_{i_M} & \widehat{\mathbf{B}}_{1,i_M} & \mathbf{0} \\ \widehat{\mathbf{A}}_{i_M}^{\mathrm{T}} & -\mathbf{X}_{(i_0,\dots,i_M)_-} & \mathbf{0} & \widehat{\mathbf{C}}_{1,i_M}^{\mathrm{T}} \\ \widehat{\mathbf{B}}_{1,i_M}^{\mathrm{T}} & \mathbf{0} & -\mathbf{I} & \widehat{\mathbf{D}}_{11,i_M}^{\mathrm{T}} \\ \mathbf{0} & \widehat{\mathbf{C}}_{1,i_M} & \widehat{\mathbf{D}}_{11,i_M} & -\mathbf{I} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{F}_{i_M} \\ \mathbf{G}_{i_M} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{B}}_{2,i_M}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{D}}_{12,i_M}^{\mathrm{T}} \\ \mathbf{0} & \widehat{\mathbf{C}}_{2,i_M} & \widehat{\mathbf{D}}_{21,i_M} & \mathbf{0} \end{bmatrix}$$

for $(i_0, \dots, i_M) \in \boldsymbol{\mathcal{L}}_M(\boldsymbol{\theta})$.

Inequality (19) is amenable to the standard linear matrix inequality embedding technique, originally developed for linear time-invariant systems [9], [10]. Finite-path dependent controllers arise naturally from this technique.

Definition 5: The controller $(\boldsymbol{\mathcal{K}}, \boldsymbol{\Theta}_L)$ is said to be an *admissible (L-path dependent) synthesis (of order n_K)* for $(\boldsymbol{\mathcal{T}}, \boldsymbol{\Theta})$ if the closed-loop system $(\boldsymbol{\mathcal{T}}_{\boldsymbol{\mathcal{K}}}, \boldsymbol{\Theta}_L)$ is uniformly exponentially stable and uniformly strictly contractive.

Theorem 3: Let $\boldsymbol{\mathcal{T}}$ be as in (12); let $\boldsymbol{\Theta} \in \boldsymbol{\Omega}$ be nonempty. Suppose that $n_K \geq n$. There exists an admissible finite-path dependent synthesis of order n_K for the system $(\boldsymbol{\mathcal{T}}, \boldsymbol{\Theta})$ if and only if there exist a nonnegative integer M and an indexed family $\bigcup_{\mathcal{I} \in \boldsymbol{\mathcal{M}}_M^-(\boldsymbol{\Theta})}\{(\mathbf{R}_{\mathcal{I}}, \mathbf{S}_{\mathcal{I}})\}$ of pairs of symmetric positive definite matrices $\mathbf{R}_{\mathcal{I}}$, $\mathbf{S}_{\mathcal{I}} \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned} &\mathbf{N}_{F,i_M}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{A}_{i_M} & \mathbf{B}_{1,i_M} \\ \mathbf{C}_{1,i_M} & \mathbf{D}_{11,i_M} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{(i_0,\dots,i_M)_-} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right.\\ &\times \left. \begin{bmatrix} \mathbf{A}_{i_M} & \mathbf{B}_{1,i_M} \\ \mathbf{C}_{1,i_M} & \mathbf{D}_{11,i_M} \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} \mathbf{R}_{(i_0,\dots,i_M)_+} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right) \mathbf{N}_{F,i_M} < \mathbf{0}, \end{aligned} \tag{20a}$$

$$\mathbf{N}_{G,i_M}^{\mathrm{T}}\left(\begin{bmatrix}\mathbf{A}_{i_M} & \mathbf{B}_{1,i_M}\\ \mathbf{C}_{1,i_M} & \mathbf{D}_{11,i_M}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{S}_{(i_0,\ldots,i_M)+} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\right.$$
$$\left.\times\begin{bmatrix}\mathbf{A}_{i_M} & \mathbf{B}_{1,i_M}\\ \mathbf{C}_{1,i_M} & \mathbf{D}_{11,i_M}\end{bmatrix}-\begin{bmatrix}\mathbf{S}_{(i_0,\ldots,i_M)-} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\right)\mathbf{N}_{G,i_M}<\mathbf{0}, \tag{20b}$$

$$\begin{bmatrix}\mathbf{R}_{(i_0,\ldots,i_M)-} & \mathbf{I}\\ \mathbf{I} & \mathbf{S}_{(i_0,\ldots,i_M)-}\end{bmatrix}\geq\mathbf{0} \tag{20c}$$

hold for all M-paths $(i_0,\ldots,i_M)\in\boldsymbol{\mathcal{M}}_M(\boldsymbol{\Theta})$, where

$$\mathbf{N}_{F,i_M}=\mathbf{N}\big(\big[\,\mathbf{B}_{2,i_M}^{\mathrm{T}}\;\mathbf{D}_{12,i_M}^{\mathrm{T}}\,\big]\big),$$
$$\mathbf{N}_{G,i_M}=\mathbf{N}\big(\big[\,\mathbf{C}_{2,i_M}\;\mathbf{D}_{21,i_M}\,\big]\big).$$

Moreover, if (20) holds for all $(i_0,\ldots,i_M)\in\boldsymbol{\mathcal{M}}_M(\boldsymbol{\Theta})$, then there exist a nonnegative integer $L\leq M$ and controller gain matrices $\mathbf{K}_{(i_{M-L},\ldots,i_M)}\in\mathbb{R}^{(n_K+m_2)\times(n_K+l_2)}$ such that (19) holds for $(i_0,\ldots,i_M)\in\boldsymbol{\mathcal{M}}_M(\boldsymbol{\Theta})$ with

$$\mathbf{X}_{\mathcal{I}}=\begin{bmatrix}\mathbf{S}_{\mathcal{I}} & \mathbf{U}_{\mathcal{I}}\mathbf{V}_{\mathcal{I}}^{\frac{1}{2}}\\ \mathbf{V}_{\mathcal{I}}^{\frac{1}{2}}\mathbf{U}_{\mathcal{I}}^{\mathrm{T}} & \mathbf{V}_{\mathcal{I}}\end{bmatrix}>0,$$
$$\mathbf{X}_{\mathcal{I}}^{-1}=\begin{bmatrix}\mathbf{R}_{\mathcal{I}} & -\mathbf{R}_{\mathcal{I}}\mathbf{U}_{\mathcal{I}}\mathbf{V}_{\mathcal{I}}^{-\frac{1}{2}}\\ -\mathbf{V}_{\mathcal{I}}^{-\frac{1}{2}}\mathbf{U}_{\mathcal{I}}^{\mathrm{T}}\mathbf{R}_{\mathcal{I}} & \mathbf{V}_{\mathcal{I}}^{-\frac{1}{2}}(\mathbf{I}+\mathbf{U}_{\mathcal{I}}^{\mathrm{T}}\mathbf{R}_{\mathcal{I}}\mathbf{U}_{\mathcal{I}})\mathbf{V}_{\mathcal{I}}^{-\frac{1}{2}}\end{bmatrix} \tag{21}$$

for $\mathcal{I}\in\boldsymbol{\mathcal{M}}_M^-(\boldsymbol{\Theta})$, where $\mathbf{U}_{\mathcal{I}}\in\mathbb{R}^{n\times n_K}$, $\mathbf{V}_{\mathcal{I}}\in\mathbb{R}^{n_K\times n_K}$ are any matrices such that $\mathbf{U}_{\mathcal{I}}\mathbf{U}_{\mathcal{I}}^{\mathrm{T}}=\mathbf{S}_{\mathcal{I}}-\mathbf{R}_{\mathcal{I}}^{-1}$ and $\mathbf{V}_{\mathcal{I}}>\mathbf{0}$.

Given a nonnegative integer M, the number of systems of linear matrix inequalities (20) to solve simultaneously is equal to the cardinality of $\boldsymbol{\mathcal{M}}_M(\boldsymbol{\Theta})$, and is bounded above by $N^{M+1}+M$. Given a nonnegative integer L, the feasibility of (20) for some $M\leq L$ is sufficient but not necessary for the existence of an admissible L-path dependent synthesis.

Suppose that a set of controller gain matrices $\mathbf{K}_{\mathcal{I}}$, $\mathcal{I}\in\boldsymbol{\mathcal{M}}_L(\boldsymbol{\Theta})$, is obtained by solving (19) for some nonnegative integer $L\leq M$. If $L=0$, then it follows from $\boldsymbol{\mathcal{M}}_0(\boldsymbol{\Theta})=\boldsymbol{\mathcal{L}}_0(\boldsymbol{\Theta})$ that we have all the gain matrices $\mathbf{K}_i$, $i\in\boldsymbol{\mathcal{L}}_0(\boldsymbol{\Theta})$, of an admissible zero-path dependent controller synthesis. If $L>0$, on the other hand, then obtain $\mathbf{K}_{\mathcal{I}}$, $\mathcal{I}\in\boldsymbol{\mathcal{L}}_L(\boldsymbol{\Theta})$, as follows: for each $j\in\boldsymbol{\mathcal{M}}_0(\boldsymbol{\Theta})$, choose a switching path $(i_0^j,\ldots,i_{L-1}^j)\in\{0,\ldots,N\}^L$ such that $(i_0^j,\ldots,i_{L-1}^j,\theta(0))$, $(i_1^j,\ldots,i_{L-1}^j,\theta(0),\theta(1))$, $\ldots$, $(i_{L-1}^j,\theta(0),\ldots,\theta(L-1))\in\boldsymbol{\mathcal{M}}_L(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ with $\theta(0)=j$; and put

$$\mathbf{K}_{(\theta(t-L),\ldots,\theta(t))}=\mathbf{K}_{(i_t^{\theta(0)},\ldots,i_{L-1}^{\theta(0)},\theta(0),\ldots,\theta(t))}$$

for all $t<L$. Then the resulting set of gain matrices defines an admissible L-path dependent controller synthesis.

IV. Control of Markovian Jump Linear Systems

Let $\boldsymbol{\mathcal{G}}$ be as in (9). Let $p=(p_i)\in\mathbb{R}^{1\times N}$ be a row vector whose entries are nonnegative and sum to one; let $\mathbf{P}=(p_{ij})\in\mathbb{R}^{N\times N}$ be a (row) stochastic matrix, so that each row of $\mathbf{P}$ has nonnegative entries that sum to one. Then the discrete-time *Markovian jump linear system*, defined by the triple $(\boldsymbol{\mathcal{G}},\mathbf{P},p)$, has the representation (10) where the switching sequence $\boldsymbol{\theta}$ is a realization of the Markov chain defined by the pair $(\mathbf{P},p)$ with transition probability matrix $\mathbf{P}$ and initial distribution p. The state $\theta(t)$ of the chain $(\mathbf{P},p)$ at time t defines the mode of $(\boldsymbol{\mathcal{G}},\mathbf{P},p)$ at time t. Let $\boldsymbol{\Omega}$ be the space of all infinite sequences in $\{1,\ldots,N\}$. Let P be the unique consistent probability measure on $\boldsymbol{\Omega}$ such that

$$P\{\,\theta(t+1)=j\mid\theta(t)=i\,\}=p_{ij},$$
$$P\{\,\theta(0)=i\,\}=p_i$$

for all i, j and t.

Definition 6: The system $(\boldsymbol{\mathcal{G}},\mathbf{P},p)$ is said to be *almost surely uniformly (exponentially) stable* if there exists a set $\boldsymbol{\Theta}\subset\boldsymbol{\Omega}$ with $P(\boldsymbol{\Theta})=1$ such that the system $(\boldsymbol{\mathcal{G}},\boldsymbol{\Theta})$ is uniformly exponentially stable.

Definition 7: The system $(\boldsymbol{\mathcal{G}},\mathbf{P},p)$ is said to be *almost surely uniformly (strictly) contractive* if there exists a set $\boldsymbol{\Theta}\subset\boldsymbol{\Omega}$ with $P(\boldsymbol{\Theta})=1$ such that the system $(\boldsymbol{\mathcal{G}},\boldsymbol{\Theta})$ is uniformly strictly contractive.

A switching sequence $\boldsymbol{\theta}$ in $\{1,\ldots,N\}$ is said to be *admissible (with respect to* $(\mathbf{P},p)$*)* if $p_{\theta(0)}>0$ and $p_{\theta(t)\theta(t+1)}>0$ for $t\geq0$. If we define

$$\boldsymbol{\Theta}(\mathbf{P},p)=\{\,\boldsymbol{\theta}\colon\boldsymbol{\theta}\text{ is admissible with respect to }(\mathbf{P},p)\,\},$$

and let

$$\boldsymbol{\mathcal{M}}_L(\mathbf{P},p)=\boldsymbol{\mathcal{M}}_L(\boldsymbol{\Theta}(\mathbf{P},p)),$$
$$\boldsymbol{\mathcal{M}}_L^-(\mathbf{P},p)=\boldsymbol{\mathcal{M}}_L^-(\boldsymbol{\Theta}(\mathbf{P},p))$$

for nonnegative integers L, then we have $P(\Theta(\mathbf{P},p))=1$; on the other hand, whenever $(i_0,\ldots,i_L)\in\boldsymbol{\mathcal{M}}_L(\mathbf{P},p)$, we have that $P\{\,\boldsymbol{\theta}\in\boldsymbol{\Omega}\colon(i_0,\ldots,i_L)\in\boldsymbol{\mathcal{L}}_L(\{\boldsymbol{\theta}\})\,\}>0$, so that $\boldsymbol{\mathcal{M}}_L(\mathbf{P},p)\subset\boldsymbol{\mathcal{M}}_L(\boldsymbol{\Theta})$ whenever $\boldsymbol{\Theta}\subset\boldsymbol{\Omega}$ and $P(\boldsymbol{\Theta})=1$. The following result is immediate from this observation and Theorem 2.

Theorem 4: Let $\boldsymbol{\mathcal{G}}$ be as in (9); let $(\mathbf{P},p)$ be a Markov chain. The system $(\boldsymbol{\mathcal{G}},\mathbf{P},p)$ is almost surely uniformly exponentially stable and almost surely uniformly strictly contractive if and only if there exist a nonnegative integer M and an indexed family $\bigcup_{\mathcal{I}\in\boldsymbol{\mathcal{M}}_M^-(\mathbf{P},p)}\{\mathbf{X}_{\mathcal{I}}\}$ of symmetric positive definite matrices $\mathbf{X}_{\mathcal{I}}\in\mathbb{R}^{n\times n}$ such that (11) holds for all M-paths $(i_0,\ldots,i_M)\in\boldsymbol{\mathcal{M}}_M(\mathbf{P},p)$.

Remark 1: Define

$$n_i(0)=1,\quad n_i(L+1)=\sum\nolimits_{\{\,j\colon p_{ij}>0\,\}}n_j(L)$$

for $i\in\{1,\ldots,N\}$ and for $L=0,1,\ldots$. Given a Markov chain $(\mathbf{P},p)$, let

$$S(p)=\{\,j\colon p_j>0\,\},$$
$$T(\mathbf{P},p)=\{\,j\colon p_ip_{ij}^{(k)}>0\text{ for some }(i,k)\,\}$$

where $\mathbf{P}^k=\big(p_{ij}^{(k)}\big)$ is the k-step transition probability matrix [18]. Then, for each M in Theorem 4, the number of linear matrix inequalities (11) to solve simultaneously is equal to the cardinality of $\boldsymbol{\mathcal{M}}_M(\mathbf{P},p)$, which is precisely given by

$$\sum\nolimits_{j\in S(p)\setminus T(\mathbf{P},p)}\sum\nolimits_{k=0}^{M-1}n_j(k)+\sum\nolimits_{j\in S(p)\cup T(\mathbf{P},p)}n_j(M).$$

In particular, if $\mathbf{P}$ is irreducible (i.e., if the directed graph of $\mathbf{P}$ is strongly connected), then the cardinality of $\boldsymbol{\mathcal{M}}_M(\mathbf{P},p)$ is equal to $\sum_{j=1}^N n_j(M)$.

Remark 2: Theorem 4 implies that the Markovian jump linear system $(\mathcal{G}, \mathbf{P}, p)$ is almost surely uniformly stable and contractive if and only if the switched linear system $(\mathcal{G}, \mathbf{\Theta}(\mathbf{P}, p))$ is uniformly stable and contractive. Therefore, Markovian jump linear systems can be treated as if they are switched linear systems. Moreover, the almost sure uniform stability and contractiveness of $(\mathcal{G}, \mathbf{P}, p)$ is robust against sparsity pattern–preserving deviations from $\mathbf{P}$ and p.

Let $\mathcal{T}$ be as in (12), and let $(\mathbf{P}, p)$ be a Markov chain. Then the triple $(\mathcal{T}, \mathbf{P}, p)$ defines the *controlled Markovian jump linear system* represented by (13). Here, $\boldsymbol{\theta}$ is a realization of $(\mathbf{P}, p)$. As in the previous section, we make the standard assumption that *the state* $\theta(t)$ *of the chain* $(\mathbf{P}, p)$ *is perfectly observed at each time instant* t; we consider all finite-path dependent controllers.

Given a nonnegative integer L, let $\mathcal{K}$ be as in (14) with $\mathbf{\Theta}$ replaced by $\mathbf{\Theta}(\mathbf{P}, p)$. Then the pair $(\mathcal{K}, \mathbf{\Theta}(\mathbf{P}, p)_L)$ defines an L-path dependent controller, whose representation is given by (15). Label the L-paths in $\mathcal{L}_L(\mathbf{\Theta}(\mathbf{P}, p))$ in dictionary order from 1 to N_L, where N_L is the cardinality of $\mathcal{L}_L(\mathbf{\Theta}(\mathbf{P}, p))$. Let $\mathbf{P}^{(0)} = \mathbf{P}$, and define $\mathbf{P}^{(L)} = (q_{ij}) \in \mathbb{R}^{N_L \times N_L}$ for each $L > 0$ as follows: whenever $(i_0, \ldots, i_L)$ and $(j_0, \ldots, j_L)$ are L-paths labeled i and j, respectively, set $q_{ij} = p_{i_L j_L}$ if $(i_1, \ldots, i_L) = (j_0, \ldots, j_{L-1})$ and $p_{i_L j_L} > 0$; otherwise, set $q_{ij} = 0$. Also, let $p^{(0)} = p$ and define a row vector $p^{(L)} = (q_i) \in \mathbb{R}^{N_L}$ for each $L > 0$ as follows: whenever $(i_0, \ldots, i_L)$ is an L-path labeled i, set $q_i = p_{i_L}$ if $i_0 = \cdots = i_{L-1} = 0$ and $i_L \neq 0$; otherwise, set $q_i = 0$. Then the pair $(\mathbf{P}^{(L)}, p^{(L)})$ defines the *L-path Markov chain generated by* $(\mathbf{P}, p)$ with transition probability matrix $\mathbf{P}^{(L)}$ and initial distribution $p^{(L)}$. Consequently, if $\mathcal{T}_{\mathcal{K}}$ is as in (16) with $\mathbf{\Theta}$ replaced by $\mathbf{\Theta}(\mathbf{P}, p)$, then the closed-loop system, given by the triple $(\mathcal{T}_{\mathcal{K}}, \mathbf{P}^{(L)}, p^{(L)})$, is a Markovian jump linear system whose representation is of the form (18) for each realization $\boldsymbol{\theta}_L$ of $(\mathbf{P}^{(L)}, p^{(L)})$.

Definition 8: The controller $(\mathcal{K}, \mathbf{\Theta}(\mathbf{P}, p)_L)$ is said to be an *admissible (L-path dependent) synthesis (of order* n_K*)* for the system $(\mathcal{T}, \mathbf{P}, p)$ if the closed-loop system $(\mathcal{T}_{\mathcal{K}}, \mathbf{P}^{(L)}, p^{(L)})$ is almost surely uniformly exponentially stable and almost surely uniformly strictly contractive.

Theorem 5: Let $\mathcal{T}$ be as in (12); let $(\mathbf{P}, p)$ be a Markov chain. Suppose that $n_K \geq n$. There exists an admissible finite-path dependent synthesis of order n_K for the system $(\mathcal{T}, \mathbf{P}, p)$ if and only if there exist a nonnegative integer M and an indexed family $\bigcup_{\mathcal{I} \in \mathcal{M}_M^-(\mathbf{P}, p)} \{(\mathbf{R}_{\mathcal{I}}, \mathbf{S}_{\mathcal{I}})\}$ of pairs of symmetric positive definite matrices $\mathbf{R}_{\mathcal{I}}, \mathbf{S}_{\mathcal{I}} \in \mathbb{R}^{n \times n}$ such that (20) holds for all M-paths $(i_0, \ldots, i_M) \in \mathcal{M}_M(\mathbf{P}, p)$. Moreover, if (20) holds for all $(i_0, \ldots, i_M) \in \mathcal{M}_M(\mathbf{P}, p)$, then there exist a nonnegative integer $L \leq M$ and controller gain matrices $\mathbf{K}_{(i_{M-L}, \ldots, i_M)} \in \mathbb{R}^{(n_K + m_2) \times (n_K + l_2)}$ such that (19) holds for $(i_0, \ldots, i_M) \in \mathcal{M}_M(\mathbf{P}, p)$, where the matrices $\mathbf{X}_{\mathcal{I}}$ are reconstructed via (21) for $\mathcal{I} \in \mathcal{M}_M^-(\mathbf{P}, p)$.

V. Conclusion

The paper treated Markovian jump linear systems in the discrete-time domain, and developed an exact condition for almost sure uniform stabilization and disturbance attenuation. This condition naturally gives rise to finite-path dependent controllers, and admits an efficient algorithm for optimal controller synthesis in the form of a sequential semidefinite program. The computational complexity of the algorithm can grow exponentially in the number of past modes that the optimal controller recalls. This limitation is due to the problem nature, and considered unavoidable.

It is not difficult to see that our result allows one to minimize the disturbance attenuation level path-by-path over all admissible switching paths of a given length, instead of minimizing a single performance level. This gives a refined notion of optimality, and there are examples where finite-path dependent controllers outperform the usual mode dependent controllers under this notion. For a detailed discussion of this point, the reader is referred to [15].

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