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Performance analysis of saturated systems via two forms of differential inclusions

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***Abstract*— In this paper we develop a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems in a general configuration. The only assumptions we make about the system are local stability and well-posedness of the algebraic loop. Problems to be considered include the estimation of the domain of attraction, the reachable set under a class of norm-bounded disturbances and the nonlinear L_2 gain. The regional analysis is established upon an effective treatment of the algebraic loop and the deadzone function. This treatment yields two forms of differential inclusions, a polytopic differential inclusion (PDI) and a norm-bounded differential inclusion (NDI), for the description of the original system. The corresponding conditions for stability and performance are derived as Linear Matrix Inequalities (LMIs).**

keywords: saturation, deadzone, nonlinear $\mathcal{L}_2$ gain, reachable set, domain of attraction, Lyapunov functions.

I. Introduction

A. Background

Saturation is ubiquitous in engineering systems and is the most studied in the literature as compared to other types of nonlinearities. Intensified efforts have been devoted to systems with saturation since the earlier 1990s due to a few breakthroughs [32], [23], [30]. Saturation exists in different parts of a control system, such as the actuator, the sensor, the controller and within the plant. Most of the efforts have been devoted to actuator saturation which involves fundamental control problems such as time-optimal control, constrained controllability and global/semi-global stabilization. These problems have been addressed in great depth, e.g., in [14], [19], [18], [23], [24], [29], [30], [32], [33], among which [14], [19], [18] consider exponentially unstable systems.

Another major trend in the study of saturated systems can be categorized as a Lyapunov approach. In this approach, some quantitative measures of stability and performance, such as the size of the domain of attraction, the convergence rate, and the L_2 gain, are characterized by using Lyapunov functions or storage functions. Then the design parameters (e.g., of a controller or of an anti-windup compensator) are incorporated into an optimization problem to optimize these quantitative measures. This trend is mostly fueled by the numerical success in solving convex optimization problems with linear matrix inequalities (LMIs) (e.g., see [1]). This is a general approach which can be applied to deal with systems with saturations and deadzones occurring at different locations. The first papers that use LMI-based methods to deal with saturated systems include [13], [28], [5], [22], [25], where [13], [28], [5] consider state feedback design and [25], [22] analyze anti-windup systems. Since then, extensive LMI-based algorithms have been developed for analysis and design of saturated systems (see, e.g., [3], [2], [8], [6], [14], [16], [15], [9], [11], [10], [26], [27], [35].)

There are mainly two steps involved in the Lyapunov approach. The first step is to bound the saturation function or the deadzone function with a sector so that the original system can be cast into the general framework of absolute stability, or can be described with a linear differential inclusion (LDI). The second step applies available tools from absolute stability theory or from general Lyapunov approaches for LDIs, such as the circle criterion or the LMI characterizations of stability and performance in [1]. Because of the two-step framework, the effectiveness of a particular method depends on how the original system is transformed into LDIs and what kind of analysis tools for LDIs are used. In many works involving anti-windup compensation, global sectors are used to describe saturation/deadzone functions. It is well known that a global sector can be very conservative for regional analysis and can only be applied when the closed-loop system is globally stable or to detect global stability. In some other works, regional LDI descriptions (some based on local sectors) are derived to reduce the conservatism (see, e.g., [3], [2], [8], [13], [6], [16], [15], [22], [28]). Along this direction, the regional LDI description introduced in [16], [15] has proved very effective and easy to manipulate. It has been used successfully for different configurations or for different purposes in [3], [2], [8], [6], [17], [21].

B. Problem formulation

With all the recent developments and effective tools mentioned in the previous section, we are now able to address some stability and performance problems for systems with saturation/deadzone in the following general form:

$$\left\{\begin{array}{rcl} \dot{x} &=& Ax + B_q q + B_w w \\ y &=& C_y x + D_{yq} q + D_{yw} w \\ z &=& C_z x + D_{zq} q + D_{zw} w \\ q &=& \mathrm{dz}(y)\ . \end{array}\right. \qquad (1)$$

where $x \in \mathbb{R}^n, q, y \in \mathbb{R}^m, w \in \mathbb{R}^r, z \in \mathbb{R}^p$ and "dz" is the standard vector-valued deadzone function. This system can be graphically depicted as in Fig. 1, where w is the exogenous input or disturbance and z is the performance

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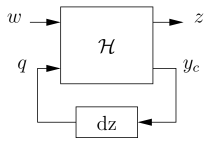

Fig. 1. Compact representation of a system with saturation/deadzone.

output. Many linear systems with saturation/deadzone components can be transformed into the above general form through loop transformation. This general form has been used to study anti-windup systems in [9], [22], [27], [35]. When $D_{yq} \neq 0$, the system contains an algebraic loop, which may present some difficulties in analysis. In many other works, it is assumed that $D_{yq} = 0$. However, it was shown in [27] that the algebraic loop can be purposely introduced into the anti-windup configuration to reduce the global L_2 gain. The importance of the parameter D_{yq} will also be illustrated in an example at the end of this paper.

We note that most of the previous works imposed different assumptions on the system, such as exponential stability of the original open-loop plant in an anti-windup configuration (e.g., [9], [27], [35]). In these works, the global sector $[0, I]$ is used to describe the deadzone function. In some other works such as [3], [2], [8], [6], [16], [15], [17], [4], regional LDI descriptions are used to reduce the conservatism. In these works, the algebraic loop is absent ($D_{yq} = 0$) and the disturbance does not enter the deadzone function, i.e., $D_{yw} = 0$.

A recent attempt was made in [34] to perform regional analysis on the general form without the assumption on stability of the open-loop plant. The main idea, which had also been suggested in some other works, was to use a smaller sector $[0, K]$ with $K < I$ to bound the deadzone function. However, this idea would not work on the general form if $D_{yw} \neq 0$. As it can be seen from the second equation in (1), if the L_2 norm of w is bounded, the L_∞ norm of y is not necessarily bounded. Hence there exists no $K < I$ to bound the deadzone function even at $x = 0$. After all, as commented in [16], [17], even in the absence of w, this kind of sector description is not only hard to manipulate, but also has a much restricted degrees of freedom as compared to the regional LDI description initiated in [16], which will be extended in this paper to deal with the general situation where $D_{yq} \neq 0$ and $D_{yw} \neq 0$.

The only assumptions that we will make about the system (1) is a necessary local stability assumption (A is Hurwitz) and the well-posedness of the algebraic loop, which will be made precise in Section II. These were also the only assumptions made in our recent paper [21].

By using quadratic Lyapunov functions, we address in this paper the following problems for system (1):

1. Estimation of the domain of attraction (in the absence of w) by using invariant ellipsoids.
2. With a given bound on the L_2 norm of w, i.e, $\|w\|_2 \leq s$ for a given s, we would like to determine a set S as small as possible so that under the condition $x(0) = 0$, we have $x(t) \in S$ for all t. This set S will be considered as an estimate of the reachable set.
3. With $\|w\|_2 \leq s$ for a given s, we would like to determine a number $\gamma > 0$ as small as possible, so that under the condition $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$. Performing this analysis for each $s \in (0, \infty)$, we obtain an estimate of the nonlinear L_2 gain.

To address these problems systematically, we will first provide an effective treatment of the algebraic loop and the deadzone function in Section II. In particular, the necessary and sufficient condition for the well-posedness of the algebraic loop will be made explicit. Moreover, we will derive two forms of differential inclusions to describe the original system (1). The first one is a polytopic differential inclusion (PDI) involving a certain adjustable parameter or nonlinear function. This parameter or nonlinear function offers extra degrees of freedom associated with a local region under consideration. It will be optimized in junction with the Lyapunov functions in the final analysis problems. The second differential inclusion is a norm-bounded differential inclusion (NDI) which is derived from the PDI. The NDI is more conservative than the PDI but may be more numerically tractable for some cases.

In Section III, we will apply quadratic Lyapunov functions via the PDI and the NDI to characterize stability and performance of the original system (1). In Section IV, we use a numerical example to demonstrate the effectiveness of this paper's results and the relationship between them. The proofs are omitted due to space constraints.

Notation

- $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] = \{k_1, k_1 + 1, \cdots, k_2\}$.
- $\mathrm{sat}(\cdot)$: The standard saturation function. For $u \in \mathbb{R}^m$, $[\mathrm{sat}(u)]_i = \mathrm{sign}(u_i)\min\{1, |u_i|\}$.
- $\mathrm{dz}(u)$: The deadzone function, $\mathrm{dz}(u) = u - \mathrm{sat}(u)$.
- $\mathrm{co}\, S$: The convex hull of a set S.
- $\mathcal{K}$: The set of diagonal matrices with 0 or 1 at each diagonal.
- $\mathrm{He}X$: For a square matrix X, $\mathrm{He}X := X + X^T$.
- $\mathcal{E}(P)$: For $P \in \mathbb{R}^{n \times n}, P = P^T > 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^T P x \leq 1\}$.
- $\mathcal{L}(H)$: For $H \in \mathbb{R}^{m \times n}$, $\mathcal{L}(H) := \{x \in \mathbb{R}^n : |Hx|_\infty \leq 1\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, for a given $s > 0$, we have (see, e.g., [16]),

$$s\mathcal{E}(P) \subset \mathcal{L}(H) \Longleftrightarrow \begin{bmatrix} 1/s^2 & H_\ell \\ H_\ell^T & P \end{bmatrix} \geq 0 \qquad (2)$$

for all $\ell \in I[1, m]$, where H_ℓ is the ℓth row of H.

II. Two forms of parameterized differential inclusions

Algebraic loops in linear systems can be easily solved (if they are well-posed). For system (1), the presence of the deadzone function makes the algebraic loop much harder to deal with. Theoretically, an explicit solution can be derived as a piecewise linear function by partitioning the vector space $\mathbb{R}^m$ into 3^m cells. However, the complexity of the partition even for $m = 2$ or 3 makes the solution almost impossible to manipulate. In this paper, we would like to use convex sets to bound all the possible solutions. By doing that, we obtain differential inclusion descriptions for the original system (1) and make it more approachable with Lyapunov methods.

Recall that the deadzone function belongs to the $[0, I]$ sector, i.e., $\mathrm{dz}(y) = \Delta y$ for some diagonal matrix $\Delta \in \mathbb{R}^{m\times m}$ such that $0 \le \Delta \le I$. Let $\mathcal{K}$ be the set of diagonal matrices whose diagonal elements are either 1 or 0. Then $\mathrm{co}\,\mathcal{K}$ is the set of diagonal Δ satisfying $0 \le \Delta \le I$. There are 2^m matrices in $\mathcal{K}$ and we denote them by $K_i, i = 1, 2, \cdots, 2^m$. Then we have

$$\mathrm{dz}(y) \in \mathrm{co}\{K_i y : \; i \in I[1, 2^m]\}.$$

This relation holds for all $y \in \mathbb{R}^m$ but could be conservative over a local region where the system operates. In [16], [15], a flexible description was introduced for dealing with the saturated state feedback $\mathrm{sat}(Fx)$ (see also [7] and references therein, where the same idea is exploited). This description can be easily adapted for the deadzone function. The main idea behind this description is the following simple fact:

Fact 1: Suppose $v \in [-1, 1]$. For any $u \in \mathbb{R}$, we have $\mathrm{sat}(u) \in \mathrm{co}\{u, v\}$. Equivalently, for the deadzone function, we have $\mathrm{dz}(u) \in \mathrm{co}\{0, u - v\}$, i.e., $\mathrm{dz}(u) = \delta(u - v)$ for some $\delta \in [0, 1]$.

This simple fact has also been used in [8] to analyze the nonlinear L_2 gain for a special case of (1), where D_{yq}, D_{yw}, D_{zq} and D_{zw} are all zero. For the general case where D_{yq} may be nonzero, we have the following algebraic loop,

$$y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw} w. \quad (3)$$

This algebraic loop is said to be well-posed if there exists a unique solution y for each $C_y x + D_{yw} w$. A sufficient condition for the algebraic loop to be well-posed is the existence of a diagonal matrix $W > 0$ such that $2W - D_{yq}W - WD_{yq}^T > 0$ (see, e.g., [9], [27], [31]). In what follows, we give a precise characterization of the well-posedness of the algebraic loop.

Claim 1: Assume that ϕ is the deadzone function or the saturation function. Then $y = D\phi(y) + v$ has a unique solution for every $v \in \mathbb{R}^m$ if and only if $\det(I - D\Delta) \neq 0$ for all $\Delta \in \mathrm{co}\,\mathcal{K}$.

Using similar arguments as those in page 57-58 in [1], it can be shown that

$$\{(I - D_{yq}\Delta)^{-1} : \; \Delta \in \mathrm{co}\,\mathcal{K}\} \subset \underset{i\in I[1,2^m]}{\mathrm{co}} \{(I - D_{yq}K_i)^{-1}\}, \quad (4)$$

$$\{(I - \Delta D_{yq})^{-1}\Delta : \; \Delta \in \mathrm{co}\,\mathcal{K}\} \subset \underset{i\in I[1,2^m]}{\mathrm{co}} \{(I - K_i D_{yq})^{-1}K_i\}, \quad (5)$$

$$\{\det(I - D_{yq}\Delta) : \Delta \in \mathrm{co}\,\mathcal{K}\} = \underset{i\in I[1,2^m]}{\mathrm{co}} \{\det(I - D_{yq}K_i)\}. \quad (6)$$

The relation (5) will be used to bound the solution of the algebraic loop with a polytope. The relation (6) implies that $\det(I - D_{yq}\Delta) \neq 0$ for all $\Delta \in \mathrm{co}\,\mathcal{K}$ if and only if $\det(I - D_{yq}K_i) \neq 0$ and have the same sign for all $i \in I[1, 2^m]$. Hence we have the following criterion for the well-posedness of the algebraic loop.

Claim 2: The algebraic loop (3) is well-posed if and only if $\det(I - D_{yq}K_i) \neq 0$ and have the same sign for all $i \in I[1, 2^m]$.

The condition in Claim 2 can be easily verified. In what follows, we assume that this well-posedness condition is satisfied. For $i \in I[1, 2^m]$, denote $T_i = (I - K_i D_{yq})^{-1}K_i$, $A_i = A + B_q T_i C_y$, $B_i = B_w + B_q T_i D_{yw}$, $C_i = C_z + D_{zq}T_i C_y$, $D_i = D_{zw} + D_{zq}T_i D_{yw}$.

Proposition 1: Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a given map and let h_ℓ be the ℓth component of h. For system (1), if $|h_\ell(x)| \le 1$ for all $\ell \in I[1, m]$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \underset{i\in I[1,2^m]}{\mathrm{co}} \left\{ \begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix} \right\}. \quad (7)$$

By taking $h(x) = 0$ in (7), we obtain a polytopic linear differential inclusion (PLDI) representation which holds globally for the original system (1). A nonzero term $h(x)$ is used to inject additional degrees of freedom in some subset of the state space to reduce conservatism in regional analysis. With quadratic Lyapunov functions, we choose $h(x) = Hx$, where H can be used as an optimizing parameter. When using non-quadratic Lyapunov functions, a nonlinear $h(x)$ may be more effective (see [20]).

The polytopic differential inclusion (PDI) (7) involves 2^m vertexes. It may present numerical difficulties when m is large (e.g., $m > 6$) and the order of the system is high. To reduce computational burden, we may use a more conservative description, namely, a norm bounded differential inclusion (NDI) to approximate it, which is based on the following result.

Claim 3: Let M be a positive diagonal matrix. Suppose that $2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1} = S^2$, where S is symmetric and nonsingular. Then

$$\begin{aligned} &\mathrm{co}\{(I - K_i D_{yq})^{-1}K_i : \; i \in I[1, 2^m]\} \\ &\quad \subset \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} : \|\Omega\| \le 1\}, \end{aligned} \quad (8)$$

where $\|\Omega\|$ is the spectral norm of Ω. Furthermore, a vertex of the left hand side is on the boundary of the right hand side.

Proposition 2: Assume that there exists a diagonal $M > 0$ and a symmetric nonsingular S such that

$$S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1}.$$

Let $H \in \mathbb{R}^{m\times n}$ be given. For $\Omega \in \mathbb{R}^{m\times m}$, denote

$$\begin{aligned} \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} &= \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \\ &+ \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} \begin{bmatrix} C_y - H \\ D_{yw} \end{bmatrix}^T \end{aligned} \quad (9)$$

For system (1), if $|Hx|_\infty \le 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \; \|\Omega\| \le 1 \right\}. \quad (10)$$

We call (10) the norm bounded differential inclusion (NDI) for (1). If $m = 1$, then the two sets in (8) are the same and the NDI is the same as the PDI. If $m > 1$, the NDI is generally strictly larger than the PDI. We also note that to obtain the NDI, there must exist a positive definite diagonal matrix M such that $2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1} > 0$, which is a stronger requirement than well-posedness.

III. Lyapunov stability and performance analysis

A. Some general results for linear differential inclusions

In [1], extensive results were established for stability and performance analysis of LDIs by using quadratic Lyapunov functions. Consider the LDI

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi \right\}, \quad (11)$$

where Φ is a given convex set of matrices. The following lemma can be established similarly to the corresponding results in [1] by extending a polytopic Φ to a general Φ.

Lemma 1: Given $P = P^T > 0, \gamma > 0$ and let $V(x) = x^T P x$. Along the trajectories of (11),

1. we have $\dot{V} < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$, if

$$A^T P + PA < 0 \quad \forall A \in \begin{bmatrix} I & 0 \end{bmatrix} \Phi \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (12)$$

2. we have $\dot{V} \le w^T w$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\mathrm{He} \begin{bmatrix} PA & PB \\ 0 & -I/2 \end{bmatrix} \le 0 \quad \forall \begin{bmatrix} A & B \end{bmatrix} \in \begin{bmatrix} I & 0 \end{bmatrix} \Phi.$$

3. we have $\dot{V} + \frac{1}{\gamma^2} z^T z \le w^T w$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\mathrm{He} \begin{bmatrix} PA & PB & 0 \\ 0 & -I/2 & 0 \\ C & D & -\gamma^2 I/2 \end{bmatrix} \le 0 \; \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi. \quad (13)$$

The condition in item 1 guarantees that the ellipsoid $\mathcal{E}(P)$ is contractively invariant in the absence of w. It will be used for the estimation of the domain of attraction. The condition in item 2 guarantees that if $\|w\|_2 \le s$, then under the initial condition $x(0) = 0$, we will have $x(t) \in s\mathcal{E}(P)$ for all $t > 0$. This will be used to determine the reachable set under a class of norm-bounded disturbances. Item 3 gives a condition for γ to be a bound for the L_2 gain, i.e., $\|z\|_2 \le \gamma \|w\|_2$ for all w and $x(0) = 0$. The result in item 3 can also be found in [12]. For the case where Φ is a polytope, we only need to verify the conditions at its vertexes.

Combining Lemma 1 with the two differential inclusion descriptions, we will obtain different methods for the analysis of the original system (1). The crucial point is to guarantee that the PDI (7) (or the LDI (10)) is valid for all times under the class of disturbances and the set of initial $x(0)$'s under consideration. We are mainly concerned about the existence of a matrix H, such that $|Hx(t)|_\infty \le 1$ for all t. To this end, we will construct a quadratic function $V(x) = x^T P x$, $P = P^T > 0$, and use Lemma 1 to guarantee that $x(t) \in s\mathcal{E}(P) \subset \mathcal{L}(H)$ for all $t \ge 0$.

B. Analysis based on the polytopic differential inclusion

When $h(x) = Hx$, the PDI (7) can be written as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \underset{i \in I[1,2^m]}{\mathrm{co}} \left\{ \begin{bmatrix} A_i - B_q T_i H & B_i \\ C_i - D_{zq} T_i H & D_i \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\}. \quad (14)$$

which corresponds to (11) with

$$\Phi = \mathrm{co} \left\{ \begin{bmatrix} A_i - B_q T_i H & B_i \\ C_i - D_{zq} T_i H & D_i \end{bmatrix} : i \in I[1, 2^m] \right\}. \quad (15)$$

We will restrict our attention to a certain ellipsoid $s\mathcal{E}(P)$. For the purpose of presenting the results in terms of linear matrix inequalities, we state the results using $Q = P^{-1}$ and $Y = HQ$. To apply the PDI description within the ellipsoid $s\mathcal{E}(P) = s\mathcal{E}(Q^{-1})$, we need to ensure that $s\mathcal{E}(P) \subset \mathcal{L}(H)$ so that $|Hx|_\infty \le 1$ for all $x \in s\mathcal{E}(P)$, which is equivalent to (recall from (2)),

$$\begin{bmatrix} 1/s^2 & H_\ell \\ H_\ell^T & P \end{bmatrix} \ge 0 \quad \ell \in I[1, m], \quad (16)$$

where H_ℓ is the ℓth row of H. Multiplying from left and right with $\mathrm{diag}\{I, Q\}$, we obtain

$$\begin{bmatrix} 1/s^2 & Y_\ell \\ Y_\ell^T & Q \end{bmatrix} \ge 0, \quad \ell \in I[1, m]. \quad (17)$$

Theorem 1: Given $Q \in \mathbb{R}^{n \times n}, Q = Q^T > 0$. Let $V(x) = x^T Q^{-1} x$. Consider system (1).

1. If there exists $Y \in \mathbb{R}^{m \times n}$ satisfying (17) with $s = 1$ and for all $i \in I[1, 2^m]$,

   $$QA_i^T + A_i Q - Y^T T_i^T B_q^T - B_q T_i Y < 0, \quad (18)$$

   then $\dot{V} < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$ and $w = 0$, i.e., $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.
2. Given $s > 0$. If there exists $Y \in \mathbb{R}^{m \times n}$ satisfying (17) and for all $i \in I[1, 2^m]$,

   $$\mathrm{He} \begin{bmatrix} A_i Q - B_q T_i Y & B_i \\ 0 & -I/2 \end{bmatrix} \le 0, \quad (19)$$

   then $\dot{V} \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t > 0$.
3. Given $\gamma, s > 0$. If there exists $Y \in \mathbb{R}^{m \times n}$ satisfying (17) and for all $i \in I[1, 2^m]$,

   $$\mathrm{He} \begin{bmatrix} A_i Q - B_q T_i Y & B_i & 0 \\ 0 & -I/2 & 0 \\ C_i Q - D_{zq} T_i Y & D_i & -\gamma^2 I/2 \end{bmatrix} \le 0, \quad (20)$$

   then $\dot{V} + \frac{1}{\gamma^2} z^T z \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma \|w\|_2$.

The three parts in Theorem 1 can be respectively used to estimate the domain of attraction, the reachable set and the L_2 gain for system (1). For these purposes, we may formulate corresponding optimization problems with linear matrix inequality (LMI) constraints.

Problem 1: Estimation of the domain of attraction. For the purpose of enlarging the estimation of the domain of attraction, we may choose a shape reference set X_R (see e.g., [14], [16], [15]) and maximize a scaling $\alpha > 0$ such that $\alpha X_R \subset \mathcal{E}(Q^{-1})$, with Q satisfying (17) and (18). The optimizing parameters are Q and Y. When X_R is a polygon or an ellipsoid, the resulting optimization problem has LMI constraints.

Problem 2: Estimation of the reachable set. Under the condition (17) and (19), an estimate of the reachable set is given by $s\mathcal{E}(Q^{-1})$. Since smaller estimates are desirable, we may formulate an optimization problem to minimize the size of $s\mathcal{E}(Q^{-1})$. There are different measures of size for ellipsoids, such as the trace of Q and the determinant of Q, among which the trace of Q is a convex measure and is much easier to handle. In practical application, we may be interested to know the range of a certain state or an output during the operation of the system. For instance, given a row vector $C \in \mathbb{R}^{1 \times n}$, we would like to estimate the maximal value of $|Cx(t)|$ for all $t \ge 0$. Since $x(t) \in s\mathcal{E}(Q^{-1})$, the maximal value of $|Cx(t)|$ is less than

$$\begin{aligned} \bar{\alpha} &:= (\max\{x^T C^T C x : x^T (s^2 Q)^{-1} x \le 1\})^{1/2} \\ &= \min\{\alpha : C^T C \le \alpha^2 (s^2 Q)^{-1}\} \\ &= \min\{\alpha : CQC^T \le \alpha^2 / s^2\}. \end{aligned}$$

To minimize $\bar{\alpha}$, we can minimize α such that $CQC^T \leq \alpha^2/s^2$ with Q satisfying (17) and (19). With α determined this way, we have $|Cx(t)| \leq \alpha$ for all $t > 0$. We may choose different C's, such as $C_i, i = 1, 2, \cdots, N$, and obtain a bound α_i on $|C_i x(t)|$ for each i. The polytope formed as $\{x \in \mathbb{R}^n : \ |C_i x| \leq \alpha_i, i = 1, \cdots, N\}$ will also be an estimate of the reachable set.

Problem 3: Estimation of the nonlinear L_2 gain. The problem of minimizing a bound on the L_2 gain directly follows from item 3 of Theorem 1 by minimizing γ along with parameters Q and Y satisfying (17) and (20). For each $s > 0$, denote $\gamma^*(s)$ as the minimal γ, then we have

$$\|z\|_2 \leq \gamma^*(\|w\|_2)\|w\|_2,$$

for all w. In other words, $\gamma^*(s)$ serves as an estimate for the nonlinear L_2 gain.

C. Analysis based on the norm-bounded differential inclusion

The following result establishes an LMI-based technique for stability and performance analysis of system (1), via the NDI description (9), (10).

Theorem 2: Given $Q \in \mathbb{R}^{n\times n}, Q = Q^T > 0$. Let $V(x) = x^T Q^{-1} x$. Consider system (1).

1. If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (17) with $s = 1$ and
$$\mathrm{He}\begin{bmatrix} AQ & B_q U \\ C_y Q - Y & -U + D_{yq} U \end{bmatrix} < 0, \qquad (21)$$
then $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.
2. Given $s > 0$. If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (17) and
$$\mathrm{He}\begin{bmatrix} AQ & B_w & B_q U \\ 0 & -I/2 & 0 \\ C_y Q - Y & D_{yw} & -U + D_{yq} U \end{bmatrix} \leq 0, \qquad (22)$$
then $\dot{V} \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t > 0$.
3. Given $\gamma, s > 0$. If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (17) and
$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_q U \\ 0 & -I/2 & 0 & 0 \\ C_z Q & D_{zw} & -\gamma^2 I/2 & D_{zq} U \\ C_y Q - Y & D_{yw} & 0 & -U + D_{yq} U \end{bmatrix} \leq 0, \qquad (23)$$
then $\dot{V} + \frac{1}{\gamma^2} z^T z \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $\|z\|_2 \leq \gamma \|w\|_2$.

As with Theorem 1, different optimization problems with LMI constraints can be formulated for stability and performance analysis of the original system (1) based on the three parts of Theorem 2. Since the NDI is a more conservative description than the PDI and since Theorems 1 and 2 are developed from the same framework, it is easy to see that the analysis results from using Theorem 2 are more conservative than those from using Theorem 1. The advantage of Theorem 2 is that the conditions involve fewer LMIs (but with higher dimensions, i.e., $+m$ more than those in Theorem 1).

We should note that the results in Theorem 1 were established in [21] through a quite different approach. The approach taken in this paper helps us to understand the relationship between the results based on two different types of differential inclusions, and allows for the subsequent developments proposed in [20].

IV. An Example Study

Consider system (1) with the following parameters:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

We use the two methods in Theorems 1 and 2 to estimate the nonlinear L_2 gain. The resulting estimates are plotted in Fig. 2, where the solid curve results from applying quadratic Lyapunov functions to the NDI description (Theorem 2) and the dashed one results from applying quadratic functions to the PDI description (Theorem 1). Additional results on this same example by using non-quadratic Lyapunov functions are reported in [20], where it is shown that non-quadratic functions can further reduce the conservatism.

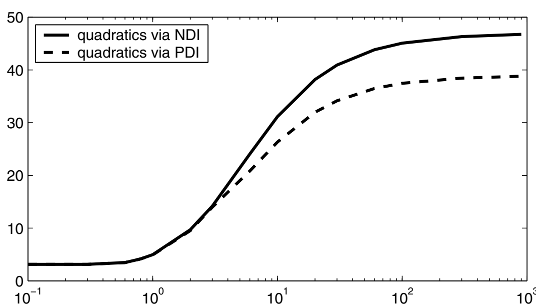

Fig. 2. Different estimates of the nonlinear L_2 gain: Case 1.

Both curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value will be an estimate of the global L_2 gain. As expected, the results from PDI are always better than those from NDI. In what follows, we present several scenarios through some adjustment of the plant parameters.

Case 2: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$, then the global L_2 gain by using NDI is unbounded (or, global stability is not confirmed), while that by using PDI is 170.1473. Improved results are obtained by using non-quadratic Lyapunov functions (see [20]).

Case 3: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$, then the global L_2 gain by using NDI or PDI is unbounded. Global L_2 gain can be established by using non-quadratic Lyapunov functions (see [20]).

The above three situations also show how the stability and performance results by the same method can be affected by the parameter D_{yq} which describes the algebraic loop. As discussed in [27], this parameter can be adjusted through anti-windup compensation.

Case 4: Next we replace the matrix A with its transpose and take $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$. The two different bounds on the L_2 gain are plotted in Fig. 3.

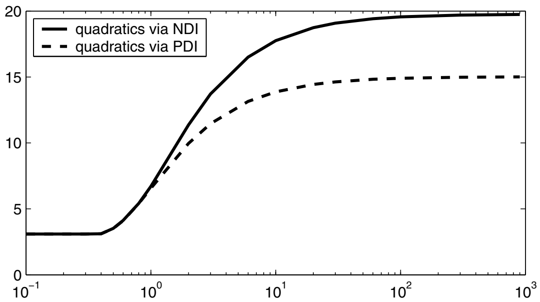

Fig. 3. Different estimates of the nonlinear L_2 gain: Case 4.

Due to space limitation, we will not present computational results about the estimation of the domain of attraction or the estimation of the reachable set. It is interesting however to point out that Case 2 suggests that the estimate of the domain of attraction by using NDI is bounded while that by using PDI is the whole state space. Similar interpretations can be given to the other cases.

V. Conclusions

For a general system with saturation or deadzone components, regional stability and performance analysis relies on an effective regional treatment of the algebraic loop and the deadzone function. This paper provides such a treatment which yields two forms of parameterized differential inclusions. Applying available tools based on quadratic Lyapunov functions to these differential inclusions, we obtained conditions for stability and performance in the form of LMIs.

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