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Causal and Stable Input/Output Structures on Multidimensional Behaviours

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Abstract

In this work we study multidimensional (nD) linear differential behaviours with a distinguished independent variable, called "time". We define in a natural way causality and stability of input/output structures with respect to this distinguished direction. We make an extension of some results in the theory of partial differential equations, demonstrating that causality is equivalent to a property of the transfer matrix. We also quote results which in effect characterize time-autonomy for the general systems case. Stability is likewise characterized by a property of the transfer matrix.

I. Introduction

In this paper we are concerned with questions of causality and stability for systems defined by partial differential equations. We consider these problems in the framework of multidimensional or nD behaviours. To date, the theory of multidimensional or nD behaviours has almost entirely considered the independent variables on an equal footing. However, in an apparent majority of applications, particularly in the case of systems given by partial differential equations, one of the independent variables, "time", is distinguished and plays a special role. Recent work [1], [2], [3] attempts to develop nD behavioural theory in this less symmetrical and more applicable situation.

This consideration is particularly significant when we discuss a concept such as stability, which is naturally associated with the passage of "time". Stability of course may be divided into two concepts: stability with respect to initial conditions (i.e. stability of an autonomous behaviour), and input/output stability. The current work was motivated by consideration of the first concept, but has led only to a (partial) characterization of the second!

Space limitations only allow us to list the main references. The notation $\{[n]\}$ refers to the corresponding reference number in [4].

II. Behaviours, classical spaces, and pole structure

We begin by briefly reviewing some concepts and results from the theory of nD behaviours, see e.g. [5] for general background on the continuous nD case. Here we consider solutions to behaviours in the classical spaces from the theory of distributions, so begin by recalling these and associated spaces. We denote the classical spaces by $\mathcal{C}^\infty$ (smooth functions), $\mathcal{D}'$ (distributions), $\mathcal{C}_0^\infty$ (compactly smooth functions), $\mathcal{E}'$ (compactly supported distributions), $\mathcal{S}$ (rapidly decreasing functions) and $\mathcal{S}'$ (tempered distributions). Here all functions and functionals are taken to be complex-valued. Recall that rapidly decreasing functions are those functions which decay faster than any polynomial grows. Following $\{[4]\}$ we define also, for any of the classical spaces $\mathcal{W}$, the spaces:

$$\mathcal{W}_+ := \{w \in \mathcal{W} \mid \text{supp}\, w \subseteq \mathbb{R}^n_+\} \quad (1)$$
$$\mathcal{W}_- := \{w \in \mathcal{W} \mid \text{supp}\, w \subseteq \mathbb{R}^n_-\} \quad (2)$$
$$\mathcal{W}_\oplus := \mathcal{W}/\mathcal{W}_-, \quad (3)$$
$$L\mathcal{W}_\ominus := \mathcal{W}/\mathcal{W}_+ \quad (4)$$

Here $\mathbb{R}_+ := \mathbb{R}^{n-1} \times [0, +\infty)$ and $\mathbb{R}_- := \mathbb{R}^{n-1} \times (-\infty, 0]$.

The spaces $\mathcal{S}_+$ and $\mathcal{S}'_\oplus$ will prove particularly important here, where $\mathcal{S}'_\oplus$ is in fact equal to the dual space of $\mathcal{S}_+$.

Denote by $\mathbb{C}[s]$ the polynomial ring in n indeterminates $s = s_1, \cdots, s_n$ with complex coefficients. We associate with any polynomial matrix $R = R(s) \in \mathbb{C}^{g\times q}$ the differential operator $R(\delta) := R(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $x_1, \cdots, x_n$ being independent variables in the space $\mathbb{R}^n$. This operator maps $\mathcal{W}^q$ to $\mathcal{W}^g$ for any of the spaces $\mathcal{W}$ listed above (the action on factors $\mathcal{W}_\oplus$, $\mathcal{W}_\ominus$ being induced in the obvious way).

For any of the spaces $\mathcal{W}$ discussed above, and for a polynomial matrix $R \in \mathbb{C}[s]^{g\times q}$, denote as usual

$$\ker_{\mathcal{W}} R = \{w \in \mathcal{W}^q \mid R(\delta)w = 0\} \quad (5)$$
$$\text{im}_{\mathcal{W}} R = \{w \in \mathcal{W}^g \mid \exists\, l \in \mathcal{W}^q \text{ s.t. } w = R(\partial)l\} \quad (6)$$

In this situation, we say that R is a kernel representation matrix of the behaviour $\mathcal{B} = \ker_{\mathcal{R}} R$. $\mathcal{W}$ is referred to as the signal space; the signal space of a behaviour is taken to be $\mathcal{D}'$ unless otherwise specified.

For the operator $R(\partial)$ or behaviour $\ker_{\mathcal{D}'} R$, the associated system module or module of formal quantities is defined as $\mathcal{M} := \mathbb{C}[s]^{1\times q}/\mathbb{C}[s]^{1\times g}R$. In particular, the behaviour $\mathcal{B}$ (for any signal space $\mathcal{W}$) may be identified with $\text{Hom}_{\mathbb{C}[s]}(\mathcal{M}, \mathcal{W})$, see e.g. [5].

Given a polynomial matrix $R \in \mathbb{C}[s]^{g\times q}$, recall the standard definition of a universal or minimal left annihilator in such a case. Then the "Fundamental Principle" of Ehrenpreis/Palamodov states that $\text{im}_{\mathcal{W}} R = \ker L$ for $\mathcal{W} = \mathcal{D}'$ or $\mathcal{W} = \mathcal{C}^\infty$. Equivalently, these two signal spaces (modules) are injective. This property is a major component of a very rich relationship between system modules $\mathcal{M}$ and behaviours

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$\mathcal{B}$, introduced into behavioural theory in [5]. We will also use standard facts and results concerning the associated primes of $\mathcal{M}$, see e.g. {[3]} for a detailed treatment.

Let $\mathcal{B} = \ker_{\mathcal{D}'} R$ with $R \in \mathbb{C}[s]^{g\times q}$; denote by $\mathcal{V}(\mathcal{B})$ the characteristic variety

$$\mathcal{V}(\mathcal{B}) := \{\zeta \in \mathbb{C}^n \mid \text{rank } R(\zeta) < q\} \tag{7}$$

which is well known to depend only on $\mathcal{B}$ and to be equal to the variety of the ideal

$$\text{ann } \mathcal{M} := \{r \in \mathbb{C}[s] \mid rx = 0 \text{ for all } x \in \mathcal{M}\} \tag{8}$$

The points of $\mathcal{V}(\mathcal{B})$ are precisely the frequencies ζ for which $\mathcal{B}$ admits polynomial exponential trajectories $p(x)exp(\langle\zeta, x\rangle)$, p a polynomial function, see e.g. {[18]} [6] for a discussion in the behavioural context.

The definition of time autonomy due to [1] is as follows.

Definition 1: A behaviour $\mathcal{B}$ is called time-autonomous if any trajectory is determined by its restriction to the half-space $\{x \in \mathbb{R}^n \mid x_n < 0.\}$ The behaviour is autonomous if the characteristic variety is not all of $\mathbb{C}^n$.

Thus for a behaviour $\mathcal{B}$ with signal space $\mathcal{D}'$, time-autonomy is equivalent to the absence of non-zero solutions in $\mathcal{D}'_+$, so means that if a trajectory is zero in the "past" ($\mathbb{R}^n_-$) it must remain zero in the "future" ($\mathbb{R}^n_+$).

Non-zero solutions over $\mathcal{D}'_+$ or $\mathcal{C}^\infty_+$ (or more generally in a specified half-space) are null solutions. As one case, results characterizing their existence for the distributional and smooth systems cases can be found in {[16]}.

We now recall some results from [6] concerning the pole structure of multidimensional behaviours. Recall first that a (free) input/output structure (x, y) on a behaviour $\mathcal{B}$ with a general signal space $\mathcal{W}$ is a partition of the system variables into m input variables u and p output variables y with the properties that (i) the projection of the behaviour onto the u variables equals $\mathcal{W}^m$ (we say, the variables u are free over $\mathcal{W}$), and (ii) the zero-input behaviour is autonomous, i.e. has no free variables.

For a given kernel representation, writing the system equations in the form $P(\partial)y = Q(\partial)u$ we equivalently have that P has full column rank and the rank of $(-Q, P)$ is equal to the rank of Q. When these conditions apply, there is a unique rational function matrix G with $PG = Q$, called the transfer matrix.

The controllable part $\mathcal{B}^c$ of $\mathcal{B}$, defined as the (unique) maximal controllable sub-behaviour of $\mathcal{B}$, possesses the same input/output structures as $\mathcal{B}$, and admits the same transfer matrix with respect to any such input/output structure. Note also (see [6] (Theorem 5.3)) that the zero input behaviour of the controllable part has a special structure. In particular, if $\mathcal{M}'$ denotes the system module associated to the zero-input behaviour then the associated primes of $\mathcal{M}'$ are all principal, and the ideal ann $\mathcal{M}'$ is generated by the least common denominator of the transfer matrix. We call a finitely generated module with the property that its associated primes are all principal a principal module.

The pole variety, controllable pole variety, and uncontrollable pole variety of $\mathcal{B}$ (with a specified input/output structure) are defined respectively as $\mathcal{V}(\mathcal{B}_{0,y}), \mathcal{V}((\mathcal{B}^c)_{0,y})$ and $\mathcal{V}(\mathcal{B}/\mathcal{B}^c)$. The points of the uncontrollable variety have an interpretation as input decoupling zeros, as discussed in {[39]}, which also establishes relationships between these sets.

III. Stability of autonomous behaviours

In this section we consider an autonomous behaviour $\mathcal{B}$ given by a kernel representation matrix R, which necessarily has full column rank q. We assume furthermore that one of the independent variables "time" (t) is distinguished; without loss of generality we will always take this to be the last variable listed in the coordinate system for $\mathbb{R}^n$. Under what conditions should $\mathcal{B}$ be referred to as a "stable" behaviour?

Stability in this context should mean that $\mathcal{B}$ contains no physically reasonable trajectories which grow in time at an unacceptably fast rate in some sense (e.g. which are unbounded). We might call this "stability with respect to the initial conditions".

Consider the heat or diffusion equation in one spatial variable:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \tag{9}$$

This system was used recently in [2], [3] to motivate an alternative signal space to $\mathcal{C}^\infty$, $\mathcal{D}'$; here we will consider it in a similar spirt. We find that the characteristic variety of the system (9) or of its behaviour $\mathcal{B}$, is:

$$\mathcal{V}(\mathcal{B}) = \{(\eta, \xi) \in \mathbb{C}^2 \mid \eta^2 = \xi\}$$

and the behaviour contains trajectories of the form

$$exp(\Re(\eta)x + \Re(\eta^2)t)exp(\iota\Im(\eta)x + \iota\Im(\eta^2)t)$$

for all $\eta \in \mathbb{C}$. Hence $\mathcal{B}$ contains trajectories which are unbounded on the positive t-axis, corresponding to the choices $\Re(\eta^2) > 0$, and so is unstable in the sense introduced in e.g. {[20]}. However, note that if $\Re(\eta^2) > 0$ then $\Re(\eta) \neq 0$, i.e. any solution which is unbounded on the $+t$ axis is also unbounded (indeed, exponentially growing) on the x-axis. In other words, the only way to get unbounded temporal behaviour in this system is to start with exponentially growing initial spatial data! Indeed, we would prefer to consider the heat equation as "stable"; with no external input of heat, heat should diffuse in time and never blow up. In this paper, we take the view that the initial data and trajectories which are exponentially growing spatially are physically unrealistic. With these considerations in mind we introduce the following concept.

Definition 2: An autonomous behaviour $\mathcal{B}$, or its characteristic variety $\mathcal{V} = \mathcal{V}(\mathcal{B})$, is said to satisfy (CV) (read "Characteristic Variety (condition)") if

$$\mathcal{V} \cap \mathcal{X}^+ = \emptyset, \mid \mathcal{X}^+ := \iota\mathbb{R}^{n-1} \times \overline{\mathbb{C}^+} \tag{CV}$$

where $\overline{\mathbb{C}^+}$ denotes the closed right-half plane.

We say that $\mathcal{B}$ or $\mathcal{V}$ satisfies (WCV) (read, $W =$ "$Weak$") if the same holds but for the open right-half plane $\mathbb{C}^+$ instead of $\overline{\mathbb{C}^+}$. We also say that a polynomial or ideal satisfies (CV) or satisfies (WCV) if the corresponding condition is satisfied by the variety of the polynomial/ideal.

For later use we also define $\mathcal{X}^- := \imath\mathbb{R}^{n-1} \times \overline{\mathbb{C}^-}$ where $\overline{\mathbb{C}^-}$ denotes the closed left-half plane.

We note that a behaviour satisfies (CV) if, and only if, it contains no polynomial exponential trajectories which are bounded at $t = 0$ (corresponding to the spatial frequency components being imaginary) but which do not decay along the positive $t-$axis (corresponding to the temporal frequency components being in $\overline{\mathbb{C}^+}$.) This observation applies equally well to both complex-and real-valued trajectories. We therefore think of points of $\mathcal{X}^+$ as unstable frequencies. Similarly, a behaviour satisfies (WCV) if and only if it contains no polynomial exponential trajectories which are bounded at $t = 0$ but grow faster than a polynomial in the $+t-$direction. Note that the behaviour defined by the heat equation certainly satisfies (WCV), as if η is imaginary then $\Re(\eta^2) < 0$ gives rise to a trajectory which is exponentially decaying in time.

As a working definition we consider an autonomous behaviour to be stable when it satisfies (CV). This attempts to capture the idea that a behaviour is unstable when it contains trajectories which are well-behaved at $t = 0$ but do not decay to 0 as $t \to +\infty$. In the case of hyperbolic systems, note that condition (WCV) implies the Gårding condition, which is necessary for hyperbolicity of an autonomous system given by a single polynomial. Here is the condition:

$$\{\Re(\xi) \,|\, \exists \eta \in \imath\mathbb{R}^{n-1},\, \xi, \eta \in \mathcal{V}\} \subseteq \mathbb{R} \text{ is bounded above} \quad (10)$$

We now discuss hyperbolicity, giving the definition for the systems case which is more complex than the better known definition for a single polynomial. The following definition is identical to one of the equivalent definitions given in {[16]}, adjusted only in respect of the fact that our systems are defined via $P(\partial)w = 0$, whereas {[16]} uses the more standard $P((1/\imath)\partial)w = 0$. Also we have specialized the definition to hyperbolicity in a fixed direction.

Definition 3: A system $P(\partial)w = 0$, operator $P(\partial)$, associated system module $\mathcal{M}$, or behaviour $\mathcal{B}$, is called hyperbolic (in the direction t) if for every associated prime I of $\mathcal{M}$, we can find a constant $0 < c < 1$ such that (where $\Re(\eta, \xi)$ denotes the real part vector of the complex $n + 1$ tuple (η, ξ))

$$\Re(\xi) \le c|\Re(\eta, \xi)| + c^{-1} \text{ for every } (\eta, \xi) \in \mathcal{V}(I) \quad (11)$$

The following result (adapted from {[16]}) links the to the more familiar one for a single polynomial.

Theorem 1: Let $P \in \mathbb{C}[s]^{g\times q}$, and let $\mathcal{M}$ be the system module, i.e. $\mathcal{M} = \mathbb{C}[s]^{1\times q}/\mathbb{C}[s]^{1\times g}P$. Suppose that $\mathcal{M}$ is principal. Then $P(\partial)$ is hyperbolic if, and only if, $(0, \cdots, 0, 1)$ is a non-characteristic direction for the system $P(\partial)y = 0$, and also the Gårding condition (11) holds for the characteristic variety of the system.

We remark that hyperbolicity is equivalent to solvability of the "non-characteristic" Cauchy problem in many different formulations {[9]}, {[10]}, which is of great importance and deserves investigation in the context of control systems theory. Essentially, hyperbolicity allows the unique continuation of initial data in a large class on $t = 0$ to trajectories on the half-space $t \ge 0$. We will note in the next section its connections to causality.

Note also that hyperbolic behaviours are in particular time-autonomous (in the general case this is a consequence of Theorem 4.2 in {[4]}). Next, we link the condition (WCV) to hyperbolicity.

Lemma 1: Let $P(\partial)$ be a partial differential operator with kernel $\mathcal{B}$ and system module $\mathcal{M}$. Suppose that $\mathcal{M}$ is principal, and that $\mathcal{B}$ satisfies (WCV) and is time-autonomous. Then the system is hyperbolic, and therefore admits a solution to the non-characteristic Cauchy problem.

In the case where $\mathcal{M}$ is principal, the property (CV) together with time-autonomy is of course a much stronger property than hyperbolicity; for example in two dimensions the kernel of the operator $(\partial/\partial t - 1)$ is hyperbolic but does not satisfy (WCV). The relationship between these two properties will become clearer when we examine stable input/output structures. This however will require us to consider causality in the continuous space-time input/output framework.

IV. CAUSAL INPUT/OUTPUT STRUCTURES

We are interested in this section with the question of when a given input/output structure is causal. Following {[38]} for the discrete case (in which the past and future are defined with respect to a cone), we introduce the following definition of causality.

Definition 4: Suppose that (u, y) is an input/output structure on $\mathcal{B}$ and $\mathcal{B}_{0,y}$ is time-autonomous. Then the input/output structure is said to be causal (with respect to $\mathcal{C}^\infty$) if for any smooth input u with support in $\mathbb{R}^n_+$, there exists a smooth output y (necessarily unique) with support in $\mathbb{R}^n_+$, such that $(u, y) \in \mathcal{B}$.

Hyperbolicity is intimately connected to causality, which can be demonstrated by results in {[16]}.

The next lemma shows that under a simple assumption, all the trajectories of $\mathcal{B}$ with support in $\mathcal{D}'_+$ are contained in the controllable part. We will then use this to demonstrate that $\mathcal{B}$ and $\mathcal{B}^c$ have the same causal input/output structures.

Lemma 2: Let $\mathcal{B}$ be a behaviour in $(\mathcal{D}')^q$. If $\mathcal{B}/\mathcal{B}^c$ is time-autonomous, then $\mathcal{B} \cap (\mathcal{D}')^q = \mathcal{B}^c \cap (\mathcal{D}'_+)^q$.

Also we have the following result (Corollary 4.5 in [4]).

Lemma 3: Let $\mathcal{B}$ be a behaviour with controllable part $\mathcal{B}^c$ and a given input/output structure (u, y) (which is necessarily an input/output structure on $\mathcal{B}^c$ also). Then:

1) $\mathcal{B}_{0,y}$ is time-autonomous if, and only if, both $\mathcal{B}/\mathcal{B}^c$ and $(\mathcal{B}^c)_{0,y}$ are.
2) Under the equivalent conditions of claim 1, (u, y) is a causal input/output structure on $\mathcal{B}$ if and only if it is a causal input/output structure on $\mathcal{B}^c$.

One consequence of this last result is that, when causality of an input/output structure is defined (i.e. when the zero-input behaviour is time-autonomous), whether or not it holds is determined purely by the controllable part of the behaviour and therefore by the transfer matrix. This motivates the following:

Definition 5: Call a transfer matrix G causal if its least common denominator is hyperbolic, stable if this polynomial obeys condition (CV), and weakly stable if this polynomial obeys (WCV).

Notice that for $n = 1$, stability of G agrees with the classical concept, and causality of G is automatic.

Suppose we are given a behaviour $\mathcal{B}$ with input/output structure (u, y) and transfer matrix G. Due to Lemma 2.3 in [4], G is causal if and only if $(\mathcal{B}^c)_{0,y}$ is hyperbolic. Similarly, G is stable if, and only if, $(\mathcal{B}^c)_{0,y}$ obeys (CV). The following result (Theorem 4.7 in [4]) shows that causality of G corresponds to causality of the corresponding input/output structures, when the latter are defined.

Theorem 2: Suppose that $\mathcal{B}$ is a behaviour with an input/output structure such that $\mathcal{B}_{0,y}$ is time-autonomous. Then the input/output structure is causal with respect to $\mathcal{C}^\infty$ if, and only if, the associated transfer matrix is causal. These conditions imply that the input/output structure is causal with respect to $\mathcal{D}'$.

In particular, Theorem 2 establishes that causality of a given input/output structure may be tested (when it is defined) merely by looking at the least common denominator d of the transfer matrix G. In fact, since the prior condition of time-autonomy enforces that $(\mathcal{B}^c)_{0,y}$ be time-autonomous and therefore that $(0, \cdots, 0, 1)$ be non-characteristic for d, we have that (u, y) is causal if, and only if, d satisfies the Gårding condition. Unfortunately, it is not immediately clear how the condition may be tested.

V. Convolution operators and ideal convexity

Before we tackle the subject of stable input/output structures, we need to explore two areas of background material. This first is convolution operators and Fourier transforms on the classical and other related spaces, as developed in {[4]}. This will lead us to a necessary and sufficient condition for input/output stability of the system $p(\partial)y = u$. The second area is "ideal-convexity" of a region in complex space, which is a necessary property for the extension of certain results from polynomials to ideals (and thereby general systems).

A. Convolution operators on $\mathcal{S}, \mathcal{S}'$

The following material is largely taken from {[4]} (Secs. 1.1–1.2]). For $s \in \mathbb{N}$ and $l \in \mathbb{R}$, let $\mathcal{C}_{(l)}^{(s)}$ denote the space of $s-$times continuously differentiable functions f on $\mathbb{R}^n$ with finite Hölder norm

$$|f|_{(l)}^{(s)} := \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, |\alpha| \le s} (1 + x^2)^{l/2} |\partial^\alpha f(x)| \tag{12}$$

Here $|\alpha|$ denotes the total of the components of α and δ^α is a shorthand for the operator $p(\partial)$, where $p = (s_1^{\alpha_1} s_2^{\alpha_2}, \cdots, s_n^{\alpha_n})$. The following elementary lemma is not in {[4]} (Lemma 5.1 in [4]) but will prove useful:

Lemma 4: If $f, g \in \mathcal{C}_{l/2}^{(s)}$ for some s, l, then $fg \in \mathcal{C}_l^{(s)}$.

Recall that $\mathcal{S}$ is defined as the intersection of all the spaces $\mathcal{C}_{(l)}^{(s)}$. Also of interest is the set

$$\mathcal{L} := \bigcap_s \bigcup_l \mathcal{C}_{(l)}^{(s)}, \tag{13}$$

which is the set of all smooth functions, each derivative of which grows no faster than some power of x (which power may depend on the derivative), and

$$\mathcal{O} := \bigcup_l \bigcap_s \mathcal{C}_{(l)}^{(s)}, \tag{14}$$

the set of all smooth functions, each derivative of which grows no faster than some power of x which is independent of the derivative. ("$\mathcal{M}$" is used rather than "$\mathcal{L}$" in {[4]}). $\mathcal{O}$ may be thought of as the set of (at most) slowly growing smooth functions; its dual space is denoted by $\mathcal{O}'$ and may be thought of as the space of rapidly decreasing distributions. Clearly $\mathcal{S} \subseteq \mathcal{O} \subseteq \mathcal{L} \subseteq \mathcal{C}^\infty$.

Still following {[4]}, we now introduce the spaces $\mathcal{C}_{(l)+}^{(s)}$ of all functions in $\mathcal{C}_{(l)}^{(s)}$ with support in $\mathbb{R}_+^n$, and define $\mathcal{S}_+,\ \mathcal{S}'_+,\ \mathcal{S}'_\oplus$ etc as before ((1)–(4)). $\mathcal{O}_+$ and $\mathcal{O}'_+$ are defined analogously. We have that $\mathcal{S}'_\oplus$ is the dual space of $\mathcal{S}_+$ and $\mathcal{S}'_+$ is dual to $\mathcal{S}_\oplus$.

Now for any Banach space B of functions $(\nu, \sigma) \in \mathbb{R}^{(n-1)+1}$ with norm $\phi \mapsto |\phi|_B$, denote by B^+ the space of functions f of $(\nu, \xi) \in \mathbb{R}^{n-1} \times \mathbb{C}$, $\xi = \sigma + \imath\rho$, with the following properties:

1) For each $\rho \le 0$, the functions $f_\rho = f(\cdot, \cdot + \imath\rho)$ are in B, and the map $(-\infty, 0] \mapsto B,\ \rho \mapsto f_\rho$ is continuous.
2) For each $\nu \in \mathbb{R}^{n-1}$, the functions $f_\nu = f(\nu, \cdot)$ are functions holomorphic in $\mathbb{C}_-$.
3) The norm $\sup_{\rho<0} |f_\rho|_B$ is finite.

A space B^- may be defined analogously, by changing the sign of ρ in conditions 1 and 3, and changing $\mathbb{C}_-$ for $\mathbb{C}_+$ in condition 2. Note that if $f(s) \in B^+$ then $F(-s) \in B^-$ and vice versa, provided that B is preserved by the same operation.

Now we define

$$\mathcal{S}^+ \;:=\; \bigcap_{s,l} \mathcal{C}_{(l)}^{(s)+}, \tag{15}$$

$$\mathcal{L}^+ \;:=\; \bigcap_s \bigcup_l \mathcal{C}_{(l)}^{(s)+} \tag{16}$$

Spaces $\mathcal{S}^-$ and $\mathcal{L}^-$ may be defined analogously, and again if $f(s) \in \mathcal{L}^+$ then $f(-s) \in \mathcal{L}^-$ and vice versa.

The interest in these spaces comes from the following collection of points, from {[4]} (except where proof is given in [4]):

Lemma 5: 1) $\mathcal{L}^+$ is closed under multiplication.
2) $\mathcal{S}^+$ is closed under multiplication by elements of $\mathcal{L}^+$, and this multiplication rule is associative.
3) $\mathcal{F}(\mathcal{S}_+) = \mathcal{S}^+$ and $\mathcal{F}(\mathcal{O}'_+) = \mathcal{L}^+$.
4) Let p be a polynomial. The equation

$$p(\partial)y = u$$

is uniquely solvable for $y \in \mathcal{S}'_+$ for any $u \in \mathcal{S}'_+$, if, and only if, p has no roots in $\mathcal{X}^+ = \mathbb{R}^{n-1} \times \overline{\mathbb{C}_+}$.

5) The equation
$$p(\partial)y = u$$
is uniquely solvable for $y \in \mathcal{S}'_+$ for any $u \in \mathcal{S}'_+$ if, and only if, p has no roots in $\mathbb{R}^{n-1} \times \overline{\mathbb{C}_+}$.

6) Any polynomial p is in $\mathcal{L}^+$, and the function $1/p(\imath\zeta)$ of ζ is in $\mathcal{L}^+$ if, and only if, p has no roots in $\mathcal{X}^+$.

Clearly all the claims of Lemma 5 can be "time-reversed" to give corresponding results for $\mathcal{L}^-, \mathcal{S}_-, \mathcal{X}^-$ etc.

Claims 4 and 5 of Lemma 5 are our first input/output stability results. Claim 4, for example, states that if the input is both spatially and temporally rapidly decreasing (the first condition being a reasonable prior assumption on physical signals and the second meaning that it is "stable"), and has zero past, then there exists a causal response with the same properties, if, and only if, a certain condition (in fact (CV)) holds on p. Moreover, by time-autonomy (which can be assumed *a-priori*), there cannot be any different causal system response, i.e. all causal responses are stable. Our main goal in what follows will be to generalize this result to the general system case.

B. Ideal Convexity

In order to generalize the results in the previous section to systems, we will need certain properties of polynomials with respect to the set $\mathcal{X}^+$ to extend to ideals.

Definition 6: We call a set $S \subseteq \mathbb{C}^n$ codimension $k-$convex, $k = 1, \cdots, n$, if for any codimension k prime ideal J we have
$$\mathcal{V}(J) \cap S = \emptyset \Rightarrow \exists\, f \in J : \mathcal{V}(f) \cap S = \emptyset \tag{17}$$
We say that S is ideal-convex if property (17) holds for any (not necessarily prime) ideal J.

The first of these properties was introduced in {[32]}, in which it is shown that the closed unit polydisc is codimension $k-$convex for all k. Codimension 1-convexity is trivial, since any prime ideal of codimension 1 in $\mathbb{C}[s]$ is principal (it must contain an irreducible polynomial and so must be equal to the codimension 1 prime ideal generated by that polynomial). It was also observed in {[32]} that if $\mathcal{S}$ is codimension $k-$convex for $k = 1, \cdots, n$, then S is ideal convex; this essentially due to claim 2 of the following simple but important result.

Theorem 3: Let $S \subseteq \mathbb{C}^n$ 'be one of the sets: $\imath\mathbb{R}^n, \mathcal{X}^+, \mathcal{X}^-$. We'have:

1) S is codimension $n-$convex.
2) If the minimal prime divisors of an ideal satisfy (17), then the ideal itself also does.
3) For $n = 2$, S is ideal-convex.

An important open question is whether the sets S in the preceding theorem are actually ideal-convex for all n. We will see below that this question has major implications for input/output stability.

VI. Stable input/output structures

Having finally done all the necessary groundwork, we can now consider input/output stability. We consider this only for input/output structures which are a priori causal (with respect to $\mathcal{C}^\infty$ or $\mathcal{D}'$ according to the type of stability required); thus, in particular, we assume that $\mathcal{B}_{0,y}$ is time-autonomous.

Definition 7: Let $\mathcal{B}$ be a behaviour with associated input/output structure (u, y), where $\mathcal{B}_{0,y}$ is time-autonomous and the input/output structure is causal. Call this i/o structure stable (with respect to $\mathcal{S}$ (resp. $\mathcal{S}'$)) if for any $u \in (\mathcal{S}_+)^m$ (resp. $u \in (\mathcal{S}_+)^m$) and $y \in (\mathcal{C}^\infty_+)^p$ (resp. $y \in (\mathcal{D}'_+)^p$) for which $(u, y) \in \mathcal{B}$, we must have $y \in (\mathcal{S}_+)^p$ (resp. $y \in (\mathcal{S}'_+)^p$).

Thus, roughly speaking, an input/output structure is stable if any causal output response to a stable input is itself stable. Since it is reasonable to assume *a priori* that our input/output structure is causal, the existence of a $y \in (\mathcal{C}^\infty_+)^p$ corresponding to a $u \in (\mathcal{S}_+)^m$ is guaranteed. Moreover, if $\mathcal{B}_{0,y}$ is *a priori* time-autonomous this y is unique, and so in this case the input/output structure is stable with respect to $\mathcal{S}$ (resp. $\mathcal{S}'$) if, and only if, the variables u are free over the signal space $\mathcal{S}$ (resp. $\mathcal{S}'$). Fortunately, using methods analogous to those in the proof of Theorem 6.4 in [4], we can now characterize freeness of variables over $\mathcal{S}_+$ using the structure theory developed in the last section. The results for $\mathcal{S}_+$ are, however, restricted to the special cases when $n \leq 2$ or P is a single polynomial (i.e. there is a single system equation), due to the difficulty of proving ideal-convexity for $n > 2$, whereas those for $\mathcal{S}'_+$ give sufficient conditions for freeness only.

Now we have the following result (Theorem 7.2 in [4]).

Theorem 4: Let
$$\mathcal{B} := \{(u, y) \in (\mathcal{D}')^{m+p} \mid P(\partial)y = Q(\partial)u\}$$
be a behaviour with given input/output structure, and transfer matrix G. Suppose that either P is a single polynomial or $n \leq 2$. Then the following are equivalent.

1) The variables u are free over $\mathcal{S}_+$ in $\mathcal{B}$.
2) $\ker_{\mathcal{S}'_\oplus} P^* \subseteq \ker_{\mathcal{S}'_\oplus} Q^*$.
3) There exists a polynomial r with no roots in $\mathcal{X}^+$, and a polynomial matrix L, such that $G = \frac{1}{r}L$.
4) $\mathcal{B}$ has no controllable poles in $\mathcal{X}^+$.
5) $G(\imath\zeta) \in (\mathcal{L}^+)^{p+m}$.

Remark 1: A topic for future research is to seek further characterizations for a principal module, other than computing its associated primes. This could well lead to generalizations of Theorems 6.4 in [4] and 4 here with less restrictive assumptions.

Let us consider the case of single polynomials in the equivalent conditions $2 \equiv 3 \equiv 5$ in Theorem 4; take q and $p \neq 0$ rather than their adjoints for ease of notation. We have that $\ker_{\mathcal{S}'_\oplus} p \subseteq \ker_{\mathcal{S}'_\oplus} q$ if, and only if, $p/\gcd(p, q)$ has no roots in $\mathcal{X}^-$, if, and only if, $(q/p)(\imath\zeta) \in \mathcal{L}^-$. From this we may define an action of $\mathcal{L}^-$ on $\mathcal{S}'_\oplus$: given any $u \in \mathcal{S}'_\oplus$ we can choose an arbitrary $v \in \mathcal{S}'_\oplus$ with $u = p(\partial)v$; this is possible as $\mathcal{S}'_\oplus$ is a divisible $\mathbb{C}[s]-$module due to divisibility of $\mathcal{S}'$. Now $y := q(\partial)v \in \mathcal{S}_{\oplus'}$ is uniquely determined by u, due to the condition $\ker_{\mathcal{S}'_\oplus} p \subseteq \ker_{\mathcal{S}'_\oplus} q$, and we have $p(\partial)y = q(\partial)u$. We now give a generalization of claim 5 of Lemma 5, (stated as Corollary 7.3 in [4]) which gives sufficient conditions for freeness of variables over $\mathcal{S}'_+$. We suspect that these conditions are also necessary.

Lemma 6: Let

$$\mathcal{B} := \{(u, y) \in (\mathcal{D}')^{m+p} \mid P(\partial)y = Q(\partial)u\}$$

be a behaviour with given input/output structure, and transfer matrix G, and suppose'that the denominators of G have no roots in $\mathbb{R}^{n-1} \times \mathbb{C}_+$. Then the variables u are free over $\mathcal{S}'_+$ in $\mathcal{B}$.

Note that the conditions of Corollary 6 are particularly met when the equivalent conditions of Theorem 4 are satisfied. One consequence of this corollary is that, when G is as specified, given any input u which is a Dirac delta in one component and zero in the others (and so in $(\mathcal{S}'_+)^m$), there is a corresponding causal output in $(\mathcal{S}'_+)^p$. If we assume time-autonomy of $\mathcal{B}_{0,y}$, then these causal outputs are unique, and we may collect them into a matrix called the impulse response matrix H_{imp}. When G is further stable, the input-to-output map over $\mathcal{S}_+$, which exists due to Theorem 4, is then given by applying H_{imp} as a convolution operator. However, as shown in the proof $5 \Rightarrow 1$ of Theorem 4, it can also be given by Fourier transformation, multiplication by $G(\imath\zeta)$, and inverse Fourier transformation. Thus H_{imp} is indeed the inverse Fourier transform of the transfer matrix G. Moreover, by Lemma 5 we have $\mathcal{L}^+ = \mathcal{F}(\mathcal{O}'_+)$, so we have $H_{imp} \in (\mathcal{O}'_+)^{p\times m}$.

Our next result shows that, as for causality, when stability with respect to $\mathcal{S}$ of an input/output structure is defined, it is characterized purely in terms of the transfer matrix. This result is however restricted to the cases $n \leq 2$ or P is a single polynomial. For the case of stability with respect to $\mathcal{S}'$, no such restriction is needed, but only a sufficient condition is obtained.

Theorem 5: Let $\mathcal{B}$ be a behaviour with a given input/output structure (u, y) such that $\mathcal{B}_{0,y}$ is time-autonomous, and associated transfer matrix G. If G is weakly stable, then (u, y) is causal with respect to $\mathcal{D}'$ and stable with respect to $\mathcal{S}'$. Moreover, suppose that either $n \leq 2$ or $\mathcal{B}_{0,y}$ is defined by a single polynomial. Then (u, y) is both causal with respect to $\mathcal{C}^\infty$, and stable with respect to $\mathcal{S}$, if, and only if, G is stable, or equivalently, if, and only if, $G(\imath\zeta) \in (\mathcal{L}^+)^{p\times m}$.

Note that Theorem 5 effectively states that an input/output structure is both causal and stable (with respect to $\mathcal{C}^\infty$ and $\mathcal{S}$ respectively) if, and only if, the zero-input behaviour $(\mathcal{B}^c)_{0,y}$ of the controllable part satisfies (CV), i.e. if, and only if, $\mathcal{B}$ has no controllable unstable poles. It is pleasing that input/output stability is determined by the poles of the system, as in the 1D case, and that the condition for input/output stability is precisely that which has been proposed for stability of the autonomous behaviour $(\mathcal{B}^c)_{0,y}$. Also, observe that stability with respect to $\mathcal{S}$ is stronger than stability with respect to $\mathcal{S}'$.

Whilst we have taken time-autonomy as a prior condition for the definition of causal and therefore stable (with respect to $\mathcal{S}$, input/output structures, it is in fact a consequence of these two properties. For if $y \in \mathcal{B}_{0,y}$ has support in $\mathbb{R}^n_+$, then by stability (0 being a stable input!), $y \in (\mathcal{S}_+)^p \subseteq \mathcal{S}^p$. If P is a kernel matrix representation matrix of $\mathcal{B}_{0,y}$, then it has a non-zero highest order minor r, and now we find that $r(\partial)y = 0$, which as $y \in \mathcal{S}$ necessitates $y = 0$ (e.g. by taking Fourier transforms). Thus $\mathcal{B}_{0,y}$ is time-autonomous.

As is the case for causality, Theorem 5 in particular implies that stability is determined by the properties of a single polynomial d, the least common denominator of the transfer matrix. To ascertain stability of the input/output structure with respect to $\mathcal{S}$, we need only test whether d obeys the condition (CV), i.e. whether the roots of d intersect the set $\mathcal{X}^+$. This test amounts to checking whether a set of real algebraic equations and inequalities has a solution, and so may be solved by quantifier elimination theory (e.g. {[1]}). An important open question is whether a simpler algorithm may be developed, making special use of the structure of $\mathcal{X}^+$.

VII. Conclusions

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