Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Equivalence to a (Strict) Feedforward Form of Nonlinear Discrete-Time Single-Input Control Systems

Claude H. Moog and Ülle Kotta

Abstract— **It is proved that a discrete-time nonlinear system in a strict feedforward form is state equivalent to a string of forward shifts. An algorithm-based checkable necessary and sufficient conditions for a discrete-time nonlinear system to be transformable into a strict feedforward form are given. Moreover, sufficient conditions for an accessible system to be transformable via static state feedback to the feedforward form are given.**

I. INTRODUCTION

The paper addresses the problem of transforming the discrete-time single-input nonlinear system by state transformation (respectively, by state feedback and state transformation) into an upper-triangular form, referred to as strict feedforward (respectively, feedforward) form. A property, crucial in applications, of systems in (strict) feedforward form is that one can construct for them a stabilizing feedback [1], [2]. It is therefore natural to ask which systems are (state or feedback) equivalent to (strict) feedforward form.

The only paper on discrete-time system transformation into feedforward form, known to authors, is [3], where the state equivalence to feedforward systems without inputs has been studied using the one-dimensional invariant subspaces. The above paper can be understood as an extension of a geometric description of systems in feedforward form for continuous-time systems [4]. The conditions of [4], unlike those in [3], although being intrinsic, are not checkable.

A literature on continuous-time system transformation into feedforward form is more numerous. In [5], the problem of transforming an affine control system into (strict) feedforward form via a coordinate transformation was studied. A step-by-step constructive method to bring system into feedforward form or strict feedforward form has been proposed in papers [6], [7] and [8], respectively. In [9] the feedback equivalence (respectively, state equivalence) to the strict feedforward form has been characterized by the existence of a sequence of infinitesimal symmetries (respectively, strong infinitesimal symmetries) of the system. The paper [10] is an extension of paper [4].

In this paper we are going to study for discrete-time nonlinear systems the state equivalence to strict feedforward form and feedback equivalence to feedforward form. In the first problem we show that existence of a strict feedforward form is nothing but state equivalence to a linear system. This is a completely unexpected result when referring to the case of continuous time systems [12]. The required state transformations are explicitly provided thanks to an algorithm. For the second problem we give sufficient conditions.

II. PROBLEM STATEMENT AND MATHEMATICAL TOOLS

In this paper we consider discrete-time nonlinear systems described by the state equations

$$x^+ = f(x, u) \tag{1}$$

where the state $x \in I\!R^n$, the control $u \in I\!R$, and $f : I\!R^{n+1} \to I\!R^n$ is a real analytic mapping. Note that the superscripts "+" and "−" denote, respectively, forward and backward shift. Precise definitions are given below. We assume that system (1) is generically submersive, that is, generically

$$\text{rank}\frac{\partial f(x, u)}{\partial(x, u)} = n$$

holds.

Definition 1: *System (1) is said to be in the* strict feedforward form *if $f(\zeta, u)$ takes the following form*

$$f(\zeta, u) = \begin{bmatrix} f_1(\zeta_2, \ldots, \zeta_n, u) \\ f_2(\zeta_3, \ldots, \zeta_n, u) \\ \vdots \\ f_{n-1}(\zeta_n, u) \\ f_n(u) \end{bmatrix} \tag{2}$$

The submersivity assumption yields $\partial f_i / \partial \xi_{i+1} \not\equiv 0$ for $i = 1, \ldots, n-1$.

Problem 1: (State equivalence to strict feedforward form) Given a single-input nonlinear system described by equations of the form (1), find, if possible, a coordinate transformation $z = T(x)$, defined on a neighborhood X^0 of x^0, such that in the new coordinates z, the system is in the strict feedforward form.

Definition 2: *System (1) is said to be in the* feedforward form *if $f(\zeta, u)$ takes the following form*

$$f(\zeta, u) = \begin{bmatrix} f_1(\zeta_1, \zeta_2, \ldots, \zeta_n, u) \\ f_2(\zeta_2, \zeta_3, \ldots, \zeta_n, u) \\ \vdots \\ f_n(\zeta_n, u) \end{bmatrix} \tag{3}$$

Problem 2: (Feedback equivalence to feedforward form) Given a single-input nonlinear system described by equations of the form (1), find, if possible, a static state feedback control law $u = \alpha(x, v)$ and a coordinate transformation $z = T(x)$, defined on a neighborhood X^0 of x^0 such that,

This work has been done when Ü. Kotta was visiting IRCCyN

C.H. Moog is with IRCCYN, CNRS/Ecole Centrale de Nantes, Université de Nantes, 1 rue de la Noë, BP 92101, 44321 Nantes Cedex 3, France Claude.Moog@irccyn.ec-nantes.fr

Ü. Kotta is with the Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia kotta@cc.ioc.ee

in the new coordinates z, the closed-loop system $x^+ = f(x, \alpha(x, v))$ is in feedforward form.

Below we recall the mathematical tools introduced in [11]. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\{x, \delta^k u, k \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by

$$\delta\zeta(x, u) = \zeta(f(x, u), u^+). \tag{4}$$

Under submersivity assumption of system (1) the pair $(\mathcal{K}, \delta)$ is a difference field, and up to an isomorphism, there exists an unique pair $(\mathcal{K}^*, \delta^*)$, called the inversive closure of $(\mathcal{K}, \delta)$, such that $\mathcal{K} \subset \mathcal{K}^*$, $\delta^* : \mathcal{K} \to \mathcal{K}^*$ is an automorphism and the restriction of $\delta^*$to $\mathcal{K}$ equals to δ. By abuse of notation, we will assume that $(\mathcal{K}^*, \delta^*)$ is given and we will use the same symbol to denote $(\mathcal{K}, \delta)$ and its inversive closure. Thus the backward shift operator $\delta^{-1} : \mathcal{K} \to \mathcal{K}$ is well defined. Over the field $\mathcal{K}$ one can define a difference vector space $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator δ induces a forward-shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by $\sum_i a_i \mathrm{d}\phi_i \to \sum_i \delta a_i \mathrm{d}(\delta\phi_i)$, $a_i; \phi_i \in \mathcal{K}$. The relative degree r of a one form $\omega \in \mathcal{E}$ is defined to be the smallest integer such that $\Delta^r \omega \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. If such an integer does not exist, we set $r = \infty$. A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$\begin{array}{rcl} \mathcal{H}_1 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \\ \mathcal{H}_{k+1} & = & \{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\} \quad k \geq 1. \end{array} \tag{5}$$

Obviously, $\mathcal{H}_k$ is a subspace of one-forms with relative degrees equal to k or higher than k. It is easy to see that sequence (5) is decreasing. Denote by k^* the smallest integer such that

$$\begin{array}{l} \mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} \\ = \mathcal{H}_{k^*+2} = \ldots =: \mathcal{H}_\infty \end{array} \tag{6}$$

Recall that system (1) is accessible if and only if $\mathcal{H}_\infty = 0$ [11].

III. STATE EQUIVALENCE TO STRICT FEEDFORWARD FORM

Surprisingly, a system in strict feedforward form is state equivalent to a linear system in a canonical string of forward shifts. This is completely different from the continuous time case. Continuous time strict feedforward systems are "generically" not state equivalent to linear systems. A subclass of strict feedforward systems has been identified in [12] that are state equivalent to a chain of integrators. More precisely, our statement is as follows.

Theorem 1: *System (2) in the strict feedforward form can be transformed via state coordinate change into the form*

$$\begin{array}{rcl} z_1^+ & = & z_2 \\ & \vdots & \\ z_{n-1}^+ & = & z_n \\ z_n^+ & = & u \end{array} \tag{7}$$

Proof: The transformations $\zeta = \Phi^{-1}(z)$, given below will do the job. First, define

$$\zeta_n = f_n(u^-),$$

where we take $u^- := z_n$ in order to get $z_n^+ = u$. Next, define

$$\zeta_{n-1} = f_{n-1}(\cdot, u^-) \circ \zeta_n^-,$$

where in $\zeta_n^- = f_n(z_n^-)$, we take $z_n^- = z_{n-1}$ in order to get $z_{n-1}^+ = z_n$.

Furthermore, for $1 \leq i \leq n-1$, we define

$$\zeta_i = f_i(\cdot, \zeta_{i+2}^-, \ldots, \zeta_n^-, u^-) \circ \zeta_{i+1}^-$$

where in

$$\zeta_{i+1}^- = f_{i+1}(f_{i+2}(\ldots(f_n(z_{i+1}^-), z_{i+1}), \ldots, z_{i+2}), \\ \ldots, f_n(z_{n-1}), z_n)$$

we take $z_{i+1}^- = z_i$ to get $z_i^+ = z_{i+1}$. Finally, we define

$$\zeta_1 = f_1(\cdot, \zeta_3^-, \ldots, \zeta_n^-, u^-) \circ \zeta_2^-,$$

where in

$$\zeta_2^- = f_2(f_3(\ldots(f_n(z_2^-), z_2), \ldots, z_3), \ldots, f_n(z_{n-1}), z_n)$$

we take $z_2^- = z_1$to get $z_1^+ = z_2$. ■

As announced before, an analogous result is not true for continuous time systems. The system

$$\begin{array}{rcl} \dot{x}_1 & = & x_3 \\ \dot{x}_2 & = & x_3^2 \\ \dot{x}_3 & = & u \end{array}$$

is in strict feedforward form and it is not fully linearizable.

Thanks to Theorem 1, it becomes an easy task to check the existence of a strict feedforward form, as stated below.

Theorem 2: *System (1) is state equivalent to* strict feedforward form *iff all the steps of Algorithm 1 can be completed for* $z_1, \ldots, z_n$.

Proof: Note first that by Theorem 1, without loss of generality, system (1) with $f(\zeta, u)$ given in the form (2), can be transformed by state transformation $z = T(\zeta)$ into the special case of strict feedforward form (7).

Algorithm 1: Step 1

Check if $\dim \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x^+\} \cap \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u\} = 0$. If this is the case, then the first step cannot be accomplished and the algorithm terminates. Otherwise, pick an integrable one-form

$$\omega_n \in \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x^+\} \cap \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u\}.$$

Note that this is always possible since the subspace $\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x^+\} \cap \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u\}$ is one-dimensional and $\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u\}$ is obviously integrable. Note also, that if we choose another one-form $\tilde{\omega}_n \in \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x^+\} \cap \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u\}$, this will not be independent of ω_n. The one-form ω_n can be expressed as

$$\omega_n = \mathrm{d}u = \sum_{i=1}^{n} (a_n^i)^+ \mathrm{d}x_i^+$$

By integrating ω_n we get $T_n(x^+) = u$. Now, if we define

$$z_n = T_n(x)$$

we get $z_n^+ = u$.

Step k ($k = 2, \dots, n$). Check if $\dim_{\mathcal{K}}\{dx^+\} \cap sp_{\mathcal{K}}\{dz_{n+2-k}\} = 0$. If this is true, then the kth step cannot be accomplished and the algorithm terminates. Otherwise, pick an integrable one-form $\omega_{n+1-k} \in sp_{\mathcal{K}}\{dx^+\} \cap sp_{\mathcal{K}}\{dz_{n+2-k}\}$

$$\omega_{n+1-k} = dz_{n+2-k} = \sum_{i=1}^{n}(a^i_{n+1-k})^+ dx_i^+.$$

By integrating ω_{n+1-k} we get $T_{n+1-k}(x^+) = z_{n+2-k}$. If we define

$$z_{n+1-k} = T_{n+1-k}(x)$$

we get $z^+_{n+1-k} = z_{n+2-k}$.

If all the steps of Algorithm 1 can be accomplished, then we get

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 \\ &\vdots \\ z_{n-1}^+ &= z_n \\ z_n^+ &= u. \end{aligned}$$

■

Note that transformability property does not depend on the choices made at each step since we deal in each step with one-dimensional subspaces.

Example 1

$$\begin{aligned} x_1^+ &= (x_2 - x_3)u^2 \\ x_2^+ &= x_1/x_3 + (x_2 - x_3)u \\ x_3^+ &= (x_2 - x_3)u \end{aligned}$$

Applying the 1st step of the algorithm, we calculate

$$sp_{\mathcal{K}}\{dx^+\} \cap sp_{\mathcal{K}}\{du\}$$
$$= sp_{\mathcal{K}}\{-u dx_3^+ + dx_1^+\} = sp_{\mathcal{K}}\{u(x_2 - x_3)du\}.$$

Choosing

$$\omega_3 = du = \frac{1}{x_3^+}dx_1^+ - \frac{x_1^+}{(x_3^+)^2}dx_3^+$$

we get

$$z_3 = \frac{x_1}{x_3}$$

Next, as

$$sp_{\mathcal{K}}\{dx^+\} \cap sp_{\mathcal{K}}\left\{\frac{1}{x_3}dx_1 - \frac{x_1}{x_3^2}dx_3\right\} = sp_{\mathcal{K}}\{dx_2^+ - dx_3^+\}$$
$$= sp_{\mathcal{K}}\left\{\frac{1}{x_3}dx_1 - \frac{x_1}{x_3^2}dx_2\right\}$$

we can define

$$z_2 = x_2 - x_3.$$

Finally, taking $z_1 = x_3^2/x_1$ we get

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 \\ z_3^+ &= u \end{aligned}$$

Example 2

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= (1 + x_1)u \end{aligned}$$

For this example $\dim[sp_{\mathcal{K}}\{dx^+\} \cap sp_{\mathcal{K}}\{u\}] = 0$, therefore this system cannot be transformed into the strict feedforward form.

IV. FEEDBACK EQUIVALENCE TO FEEDFORWARD FORM

Theorem 3: *Accessible system of the form (1) is feedback equivalent to the feedforward form, if the largest integrable subspace of $\mathcal{H}_{n-1}$ has dimension greater than or equal to 1.*

Proof: Assume that the dimension of the largest integrable subspace is equal at least to 1. Then $\mathcal{H}_{n-1} = sp_{\mathcal{K}}\{\omega_1, d\zeta_2\}$ with ω_1 not necessarily exact and $\mathcal{H}_1$ can be given as $\mathcal{H}_1 = sp_{\mathcal{K}}\{d\zeta_1, d\zeta_2, d\delta\zeta_2, \dots, d\delta^{n-2}\zeta_2\}$. Then, clearly, we can define ζ_1 and ζ_2 by integrating the exact one-forms $d\zeta_1$ and $d\zeta_2$ and choose $\zeta_i := \zeta^+_{i-1}$ for $3 \le i \le n$. This choice will yield the structure

$$\begin{aligned} \zeta_1^+ &= F(\zeta_1, \dots, \zeta_n, u) \\ \zeta_2^+ &= \zeta_3 \\ &\vdots \\ \zeta_{n-1}^+ &= \zeta_n \\ \zeta_n^+ &= \alpha(\zeta, u) \end{aligned} \tag{8}$$

Using a feedback $\alpha(\zeta, u) = v$ we can bring the system into the required form, since

$$\begin{aligned} \zeta_1^+ &= F(\zeta_1, \dots, \zeta_n, \alpha^{-1}(\zeta, v)) \\ \zeta_2^+ &= \zeta_3 \\ &\vdots \\ \zeta_{n-1}^+ &= \zeta_n \\ \zeta_n^+ &= v \end{aligned}$$

is a special case of (3). ■

Obviously, the condition in Theorem 3 is fulfilled for $n \le 3$. The sufficient condition in Theorem 3 is fully computable since the largest integrable of $\mathcal{H}_{n-1}$ can be computed as the limit of the so-called derived flag of $\mathcal{H}_{n-1}$.

Note that condition of Theorem 3 is not necessary as demonstrated by the following counterexample.

Example 3

$$\begin{aligned} x_1^+ &= u + x_1x_2 \\ x_2^+ &= x_2x_3 \\ x_3^+ &= ux_3x_4 \\ x_4^+ &= u + x_4 \end{aligned} \tag{9}$$

Compute

$$\begin{aligned} \mathcal{H}_{n-1} \;=\; & \mathcal{H}_3 = sp_{\mathcal{K}}\{x_3^2[u^{--}(2u^- - x_4) \\ +\; & u^-(-u^- + x_4)]dx_1 \\ -\; & u^{--}u^-(u^- - x_4)[(u^-)^2x_2 + x_3 \\ -\; & u^-x_2x_4]dx_3 + u^-x_3[x_3(u^- - x_4) \\ +\; & u^{--}((u^-)^2x_2 - x_3 - 2u^-x_2x_4 \\ +\; & x_2x_4^2)]dx_4, \\ & x_3[u^{--}(2u^- - x_4) + u^-(-u^- + x_4)]dx_2 \\ +\; & u^-x_2(u^- - x_4)dx_3 + x_2x_3(-u^- + x_4)dx_4\} \end{aligned}$$

After lengthy computations, one checks that the limit of the derived flag of $\mathcal{H}_{n-1} = \mathcal{H}_3$ is zero: in plain words, there is no nonzero exact form in $\mathcal{H}_3$. The largest integrable subspace of $\mathcal{H}_3$ has dimension 0, but obviously (9) is in feedforward form.

V. CONCLUSIONS

In this paper we addressed the problem of the intrinsic characterization of discrete-time nonlinear strict feedforward systems. Since discrete-time feedforward systems can be transformed via state transformation to a string of forward shifts, this class of models cannot be regarded as a subclass of truly nonlinear systems. Moreover, we presented an algorithm-based necessary and sufficient conditions for a system to be transformable into a strict feedforward form. In this paper we have obtained only sufficient conditions for discrete-time nonlinear accessible system to be feedback equivalent to a feedforward form. Remaining open problems are the characterization of state equivalence to a feedforward form as well as the complete characterization of feedback equivalence to a feedforward form.

VI. ACKNOWLEDGMENTS

This work was partially supported by ESF Grant N 5405.

REFERENCES

[1] T. Ahmed-Ali, F. Mazenc and F. Lamnabhi-Lagarrigue, "Disturbance attenuation for discrete-time feedforward nonlinear systems", *in Stability and Stabilization of Nonlinear Systems* (Eds. D. Aeyels, F. Lamnabhi-Lagarrigue and A. J. van der Schaft), Lecture Notes in Control and Inf. Sci, vol. 246, 1999, pp 1-17.

[2] F. Mazenc and H. Nijmeijer, Forwarding in discrete-time nonlinear systems, *Int J. Control*, vol. 71, 1998, pp 823-835.

[3] E. Aranda-Bricaire and C.H. Moog, Invariant distributions and the feedforward form for discrete-time nonlinear systems, *Systems and Control Letters*, vol. 52, 2004, pp 113-122.

[4] A. Astolfi and F. Marenc, "A geometric characterization of feedforward forms", *in Proc. of the Int. Symp. MTNS-2000*, Perpignan, France, 2000.

[5] A. Marigo, "Constructive necessary and sufficient conditions for strict triangularizability of driftless nonholonomic systems", *in Proc. of the 34th IEEE CDC*, 1999, Arizona, pp 2138-2143.

[6] I.A. Tall and W. Respondek, "Transforming a single-input nonlinear system to a feedforward form via feedback", *in Nonlinear Control in the Year 2000* (Eds. A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek), Springer, vol. 2, 2000, pp 527-542.

[7] I.A. Tall and W. Respondek, Feedback equivalence to a strict feedforward form of nonlinear single-input control systems, *Int. J. Control*, accepted for publication.

[8] I.A. Tall and W. Respondek, "Feedback equivalence to feedforward forms for nonlinear single-input control systems", *in Dynamics, Bifurcations and Control* (Eds. F. Colonius and I. Grüne), Lecture Notes in Control and Inf. Sci, Springer, vol. 273, Berlin-Heidelberg, 2002, pp 269-286.

[9] W. Respondek and I.A. Tall, "Strict feedforward form and symmetries of nonlinear control systems, *in Proc. of the 43rd IEEE CDC*, 2004, Bahamas, pp 1611-1616.

[10] G. Kaliora and A. Astolfi, "On the equivalence to feedforward forms", *in Proc. of European Control Conference*, 2003.

[11] E. Aranda-Bricaire, Ü. Kotta and C.H. Moog, Linearization of discrete-time systems, *SIAM J. Control and Optimization*, vol. 34, 1996, No 6, pp 1999-2023.

[12] M. Kristic, Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems, *IEEE Trans. on Automatic Control*, vol. 49, No 10, pp 1668-1682.