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Dynamical Adaptive Synchronization

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Abstract— **The notions of dynamical synchronization and adaptive dynamical synchronization problems are introduced. The algorithm solving adaptive synchronization problem for a subclass of Lurie systems with exciting input is proposed. The performance and potentialities of proposed solutions are demonstrated by two examples related to formation control and self-organization of swarm systems.**

I. Introduction

THE synchronization phenomenon was firstly described by Huygens [20], who found that a pair of pendulum clocks attached to a common beam synchronize their oscillations. During the last century synchronization was in the center of attention in vibrational mechanics (the theory of vibroactuators [2], [3], [4]). Beginning with [32], synchronization of chaotic systems becomes a popular research topic [6], [7], [8], [14], [28], [36]. Another line of research deals with common problem of synchronization of network of nonlinear oscillators [4], [33]. The synchronization theory finds its applications in different areas of science and technology, like mobile robots [37], [41] and robot manipulators [24], [35], general mechanical systems [11], [30], vibrational technologies [2], [4], information transmission and encoding [6], [13], etc.

There are two basic synchronization schemes: *master-slave* and *cooperative (mutual)* schemes. In master-slave systems there exists a leader (master) to which the other systems are supposed to coordinate to. In cooperative systems all systems interact at the same level of hierarchy, system behavior being the result of interactions between all individual systems (see papers [4], [16], [19], [27], [33] and references therein). Master-slave synchronization design problem is sometimes closely related to the general observer design problem in control theory [29]. Two other important notions are *full (or complete) synchronization* (i.e. when all state trajectories of the synchronized systems asymptotically converge to each other [4], [14], [28]) and *partial synchronization* (when it is necessary to ensure convergence only for part of systems variables [33], [36]). If the models of systems to be synchronized contain parametric uncertainties and adaptation techniques are to be designed then the problem is called *adaptive synchronization problem* [4], [11].

In this paper we will consider only master-slave synchronization scheme. Based on the theory of observer or adaptive observer design this scheme typically provides convergence of synchronization error to zero, that results in legibly defined position of followers with respect to the leader. For example in formation control problem [31], [40] each vehicle has a fixed position with respect to formation reference point (which can be viewed as a virtual leader for formation of vehicles). However, it frequently happens in nature or in human operated systems in similar situations that synchronization error obeys some dynamics giving an additional maneuverability, dexterity and performance to the system. E.g. schools of fishes, flocks of birds or human operated group of aircrafts have not hard defined formation during manoeuvres and each element of group can freely oscillate around its position with respect to the leader. The same situation appears in pursuer-evader problem, where the pursuer, for example helicopter as in paper [18], should exactly follow the evader car at the given height and, to increase its safety, human operated helicopter draws circles around the evader. It is worth to stress, that oscillatory movement is a movement on the border of stability that gives additional maneuverability to pursuer. The design philosophy of some systems can be renewed from this point of view.

Another example is our galaxy which exhibits synchronization between its subsystems. E.g. the sun can be considered as the leader, while planets are coordinated to the sun with oscillating synchronization error with some amplitude, satellites, in turn, are synchronized with their own planets and synchronization error again obeys oscillatory differential equations. Such a type of synchronization can be called *dynamical synchronization*.

The precise definition of adaptive dynamical synchronization problem is given in Section 3. Sections 4 and 5 contain the main results and their applications.

II. Preliminaries

Let us consider nonlinear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}),\ \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is state vector; $\mathbf{u} \in R^m$ is input vector; $\mathbf{y} \in R^p$ is output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0,0) = 0$. Euclidean norm will be denoted as $|\mathbf{x}|$, and $\|\mathbf{u}\|_{[t_0,t]}$ denotes the L_∞^m norm of the input ($\mathbf{u}(t)$ is measurable and locally essentially bounded function $\mathbf{u}: R_+ \to R^m$,

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$R_+ = \{\tau \in R : \tau \geq 0\}$):

$$\|\mathbf{u}\|_{[t_0,T]} = ess\sup\{|\mathbf{u}(t)|, t \in [t_0, T]\},$$

if $T = +\infty$ then we will simply write $\|\mathbf{u}\|$. We will denote as $\mathcal{M}_{R^m}$ the set of all such Lebesgue measurable inputs $\mathbf{u}$ with property $\|\mathbf{u}\| < +\infty$ and $\mathcal{M}_\Omega$ will be the set of inputs $\mathbf{u}(t) \in \Omega \subset R^m$ for almost all $t \geq 0$, where Ω is a compact set. For initial state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ let $\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$ if all other arguments of solution are clear from the context; $\mathbf{y}(t, \mathbf{x}_0, \mathbf{u}) = \mathbf{h}(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u}))$), which is defined on some finite interval $[0, T)$; if $T = +\infty$ for every initial state $\mathbf{x}_0$ and $\mathbf{u} \in \mathcal{M}_{R^m}$, then system is called forward complete. We will also use a weaker property of system (1), which is closely connected to forward completeness: system (1) has unboundedness observability (UO) property, if for each state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ such that $T < +\infty$ necessarily

$$\lim_{t \to T} \sup |\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| = +\infty.$$

In other words it is possible to observe any unboundedness of the state. The contrapositive statement of this property says that, if $\sup_{t \in [0,T)} |\mathbf{y}(t)| < +\infty$, then $\mathbf{x}(T)$ is defined, so boundedness of UO output means forward completeness. The necessary and sufficient conditions for forward completeness and UO properties were investigated in [1]. Distance in R^n from given point $\mathbf{x}$ to set $\mathcal{A}$ is denoted as $|\mathbf{x}|_{\mathcal{A}} = \text{dist}(\mathbf{x}, \mathcal{A}) = \inf_{\boldsymbol{\eta} \in \mathcal{A}} |\mathbf{x} - \boldsymbol{\eta}|$ and $|\mathbf{x}|_0 = |\mathbf{x}|$ is standard Euclidean norm.

As usually, continuous function $\sigma : R_+ \to R_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is additionally radially unbounded; and continuous function $\beta : R_+ \times R_+ \to R_+$ is from class $\mathcal{KL}$, if it is from class $\mathcal{K}$ for the first argument for any fixed second, and it is strictly decreasing to zero by the second argument for any fixed first one.

Definition 1 [17], [39]. *A UO system (1) is input-to-output stable (IOS), if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such, that*

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| \leq \beta(|\mathbf{x}_0|, t) + \gamma(\|\mathbf{u}\|),\ t \geq 0$$

holds for all $\mathbf{x}_0 \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_{R^m}$. □

In works [17], [39] necessary and sufficient conditions for verification of IOS property are presented.

III. Statement of Dynamical Synchronization Problem

Let us consider two systems

$$\dot{\mathbf{x}}_l = \mathbf{f}_l(\mathbf{x}_l, \boldsymbol{\theta}_l, \mathbf{d}_l);\ \mathbf{y}_l = \mathbf{h}_l(\mathbf{x}_l, \mathbf{d}_l);\ \boldsymbol{\psi}_l = \boldsymbol{\eta}_l(\mathbf{x}_l);\quad (2)$$

$$\dot{\mathbf{x}}_f = \mathbf{f}_f(\mathbf{x}_f, \boldsymbol{\theta}_f, \mathbf{d}_f, \mathbf{u});\ \mathbf{y}_f = \mathbf{h}_f(\mathbf{x}_f, \mathbf{d}_f);\ \boldsymbol{\psi}_f = \boldsymbol{\eta}_f(\mathbf{x}_f),\quad (3)$$

where system (2) describes dynamics of the leader and system (3) corresponds to the follower, $\mathbf{x}_l \in R^{n_l}$, $\mathbf{x}_f \in R^{n_f}$ are state vectors of the systems; $\mathbf{y}_l \in R^{p_l}$, $\mathbf{y}_f \in R^{p_f}$ define output vector variables available for measurements; $\boldsymbol{\psi}_l \in R^r$, $\boldsymbol{\psi}_f \in R^r$ are synchronizable output vectors; $\mathbf{d}_l \in R^{k_l}$, $\mathbf{d}_f \in R^{k_f}$ are vectors of external disturbances, $\mathbf{d} = (\mathbf{d}_l, \mathbf{d}_f) \in R^k$, $k = k_l + k_f$, $\mathbf{d} \in \mathcal{M}_{R^k}$; $\boldsymbol{\theta}_l \in R^{q_l}$, $\boldsymbol{\theta}_f \in R^{q_f}$ are vectors of may be unknown constant parameters of models (2) and (3); $\mathbf{u} \in R^m$ is a control input of the follower, $\mathbf{u} \in \mathcal{M}_{R^m}$. Vector functions $\mathbf{f}_l, \mathbf{f}_f, \mathbf{h}_l, \mathbf{h}_f, \boldsymbol{\eta}_l, \boldsymbol{\eta}_f$ are assumed to be smooth enough to ensure existence and uniqueness of system (2), (3) solutions at the least locally, which for given initial conditions $\mathbf{x}_l^0 \in R^{n_l}$, $\mathbf{x}_f^0 \in R^{n_f}$ we denote as $\mathbf{x}_l(t, \mathbf{x}_l^0, \boldsymbol{\theta}_l, \mathbf{d}_l)$ and $\mathbf{x}_f(t, \mathbf{x}_f^0, \boldsymbol{\theta}_f, \mathbf{d}_f, \mathbf{u})$. Introduce the reference system

$$\dot{\mathbf{e}} = \mathbf{f}_e(\mathbf{e}),\quad (4)$$

where $\mathbf{e} \in R^r$ is state vector and vector function $\mathbf{f}_e$ is locally Lipschitz continuous. It is supposed, that system (4) is forward complete and an invariant non empty closed set $\mathcal{A}$ is globally asymptotically stable [23] for (4).

The *dynamical synchronization problem* is to design a controller

$$\dot{\mathbf{x}}_c = \mathbf{f}_c(\mathbf{y}_l, \mathbf{y}_f),\ \mathbf{u} = \mathbf{h}_c(\mathbf{x}_c, \mathbf{y}_l, \mathbf{y}_f)\quad (5)$$

such, that UO system (2), (3), (5) is IOS with respect to output $|\boldsymbol{\varepsilon}|_{\mathcal{A}}$, $\boldsymbol{\varepsilon} = \boldsymbol{\psi}_l - \boldsymbol{\psi}_f$ for given $\boldsymbol{\theta}_l \in R^{q_l}$, $\boldsymbol{\theta}_f \in R^{q_f}$ and input $\mathbf{d} \in \mathcal{M}_{R^k}$ (here $\mathbf{x}_c \in R^{n_c}$ is the controller state vector and functions $\mathbf{f}_c$ and $\mathbf{h}_c$ admit the same requirements as in (2), (3)). Variable $\boldsymbol{\varepsilon}$ denotes the synchronization error. If IOS property of system (2), (3), (5) with respect to output $|\boldsymbol{\varepsilon}|_{\mathcal{A}}$ and input $\mathbf{d} \in \mathcal{M}_{R^k}$ should hold uniformly in $\boldsymbol{\theta}_l \in R^{q_l}$, $\boldsymbol{\theta}_f \in R^{q_f}$, then such problem is called *adaptive dynamical synchronization problem*.

If $\mathbf{d}(t) \equiv 0$, $t \geq 0$, then dynamical synchronization problem is equivalent to orbital stabilization [15] of the output $\boldsymbol{\varepsilon}$ and solutions $\mathbf{e}(t)$ of system (4). From another point of view this problem can be formulated in terms of explicit or implicit reference model approach or output regulation problem.

IV. Main Result

Let us consider a subclass of Lurie models for description of the leader and follower systems:

$\dot{\mathbf{x}}_l = \mathbf{A}\mathbf{x}_l + \varphi_l(\mathbf{y}_l) - \mathbf{B}_l(\mathbf{y}_l)\boldsymbol{\theta}_l + \mathbf{d}_l$; $\mathbf{y}_l = \mathbf{C}\mathbf{x}_l$; $\boldsymbol{\psi}_l = \mathbf{x}_l$; (6)

$\dot{\mathbf{x}}_f = \mathbf{A}\mathbf{x}_f + \varphi_f(\mathbf{y}_f) + \mathbf{B}_f(\mathbf{y}_f)\boldsymbol{\theta}_f + \mathbf{d}_f + \mathbf{u}$; $\mathbf{y}_f = \mathbf{C}\mathbf{x}_f$; $\boldsymbol{\psi}_f = \mathbf{x}_f$, (7)

where $\mathbf{x}_l, \mathbf{x}_f \in R^n$, $\mathbf{y}_l, \mathbf{y}_f \in R^p$, $\mathbf{d}_l, \mathbf{d}_f \in R^n$, $\boldsymbol{\theta}_l \in R^{q_l}$, $\boldsymbol{\theta}_f \in R^{q_f}$, $\mathbf{u} \in R^n$ have the same meaning as in the previous section. Matrices $\mathbf{A}$ and $\mathbf{C}$ have dimensions $(n \times n)$ and $(p \times n)$ respectively; vector functions $\varphi_l : R^p \to R^n$, $\varphi_f : R^p \to R^n$ and matrix functions $\mathbf{B}_l : R^p \to R^{n \times q_l}$, $\mathbf{B}_f : R^p \to R^{n \times q_f}$ are continuous and locally Lipschitz; $\mathbf{d} \in \mathcal{M}_{R^n}$. The leader system (6) is forward complete. Here for simplicity we will consider the full state synchronization problem. Then synchronization error $\boldsymbol{\varepsilon} = \mathbf{x}_l - \mathbf{x}_f$ admits the differential equation:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{A}\boldsymbol{\varepsilon} + \varphi_l(\mathbf{y}_l) - \varphi_f(\mathbf{y}_f) - \mathbf{B}(\mathbf{y})\boldsymbol{\theta} + \mathbf{D} - \mathbf{u}, \quad (8)$$

where $\mathbf{y} = (\mathbf{y}_l^T\ \mathbf{y}_f^T)^T$, $\boldsymbol{\theta} = (\boldsymbol{\theta}_l^T\ \boldsymbol{\theta}_f^T)^T \in R^q$, $q = q_l + q_f$, $\mathbf{D} = \mathbf{d}_l - \mathbf{d}_f \in R^n$ and $\mathbf{B}(\mathbf{y}) = (\mathbf{B}_l(\mathbf{y}_l)\,\mathbf{B}_f(\mathbf{y}_f))$; $\mathbf{D} \in \mathcal{M}_{R^n}$. Let the reference system (4) have form

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \phi(\mathbf{C}\mathbf{e}) + \mathbf{D}_e, \quad (9)$$

where $\mathbf{e} \in R^n$ is state vector; $\phi : R^p \to R^n$ is locally Lipschitz continuous function; $\mathbf{D}_e \in R^n$ is auxiliary disturbing input in (9), $\mathbf{D}_e \in \mathcal{M}_{R^n}$. Let non empty closed time invariant set $\mathcal{A} \in R^n$ be given, and system (9) is UO and IOS with respect to output $|\mathbf{e}|_{\mathcal{A}}$ and input $\mathbf{D}_e$. Here opposite to Section 2 for purposes of control $\mathbf{u}$ design we introduce auxiliary input $\mathbf{D}_e$ and require a stronger stability property for system (9) (instead of global asymptotic stability of set $\mathcal{A}$).

Let us take control law in the form

$$\mathbf{u} = \varphi_l(\mathbf{y}_l) - \varphi_f(\mathbf{y}_f) - \mathbf{B}(\mathbf{y})\hat{\boldsymbol{\theta}} - \phi(\mathbf{y}_l - \mathbf{y}_f), \quad (10)$$

where $\hat{\boldsymbol{\theta}} \in R^q$ is adjustable vector of estimates of $\boldsymbol{\theta}$. Only variable $\mathbf{C}\boldsymbol{\varepsilon} = \mathbf{y}_l - \mathbf{y}_f$ is available for measurements. Then system (8), (10) can be rewritten as follows

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{A}\boldsymbol{\varepsilon} + \phi(\mathbf{C}\boldsymbol{\varepsilon}) + \mathbf{B}(\mathbf{y})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + \mathbf{D}. \quad (11)$$

It is obvious, that for case $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, if system (6), (7), (10) is UO for output $|\varepsilon|_{\mathcal{A}}$, then synchronization goal is reached and the system is IOS with respect to output $|\varepsilon|_{\mathcal{A}}$ and input $\mathbf{D} \in \mathcal{M}_{R^n}$ (due to system (11) for $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ is identical to (9), which possesses this property). That is more, from Definition 1 if item $\mathbf{B}(\mathbf{y}(t))(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta})$ is essentially bounded for all $t \ge 0$ and converges to zero with $t \to +\infty$, then we also obtain the solution of the posed dynamical synchronization problem.

Therefore, it is necessary to design an adaptation algorithm, which would adjust vector $\hat{\boldsymbol{\theta}}$ providing property

$$\lim_{t \to +\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta},$$

for any unknown value of vector $\boldsymbol{\theta} \in R^q$. Such problem was considered in papers [9] and [10] and was called *tuning to bifurcation*. In those papers it was supposed that $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ is a bifurcation point for the system and system behavior in this point lies on the border of stability. Unlike conventional adaptive control [12], [22], [26], where only boundedness of parameters estimation error $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is guaranteed and asymptotic stability of the system in point $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ is needed, in the papers [9] and [10] the adaptation algorithms are designed ensuring asymptotic stability with respect to the variable $\tilde{\boldsymbol{\theta}}$ without requirement of plant stability in bifurcation point.

To apply result from [10] let us fix the following requirements to the system.

Assumption 1. *For any $\boldsymbol{\theta}_l \in R^{q_l}$ system (6) is forward complete.* □

Assumption 2. *The UO system (9) is IOS with respect to output $|\mathbf{e}|_{\mathcal{A}}$, $\mathcal{A} \in R^n$ is non empty closed time invariant set of (9) for $\mathbf{D}_e(t) \equiv 0$, $t \ge 0$.* □

Assumption 3. *There exists matrix $\mathbf{K}$ of dimension $(n \times p)$ such, that system*

$$\dot{\mathbf{s}} = \mathbf{G}\mathbf{s} + \mathbf{u},\ \mathbf{s} \in R^n,\ \mathbf{u} \in R^n,\ \mathbf{G} = \mathbf{A} - \mathbf{K}\mathbf{C} \quad (12)$$

admits the following properties:

I. *For any initial conditions $\mathbf{s}_0 \in R^n$ and $\mathbf{u} \in \mathcal{M}_{R^n}$ solutions of system (12) are bounded*:

$$\|\mathbf{s}(t, \mathbf{s}_0, \mathbf{u})\| \le r_1 |\mathbf{s}_0| + r_2 \|\mathbf{u}\|,\ r_1, r_2 \in R_+.$$

II. *System (12) is IOS with respect to output $\mathbf{C}\mathbf{s}$ and input $\mathbf{u} \in \mathcal{M}_{R^n}$.* □

The first two assumptions were introduced before in informal manner, they come from dynamical synchronization problem statement. The third assumption introduces subsidiary system (12) with useful stability properties. For example, this assumption is satisfied for detectable pare of matrices $(\mathbf{A}, \mathbf{C})$. These suppositions allow to design the following adaptive observer for system (11):

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \phi(\mathbf{C}\boldsymbol{\varepsilon}) + \mathbf{K}(\mathbf{C}\boldsymbol{\varepsilon} - \mathbf{C}\mathbf{z}) + \mathbf{B}(\mathbf{y})\hat{\boldsymbol{\theta}} + \Delta; \quad (13)$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{G}\boldsymbol{\Omega} - \mathbf{B}(\mathbf{y}); \quad (14)$$

where $\mathbf{z} \in R^n$ is estimate of $\boldsymbol{\varepsilon}$; $\boldsymbol{\Omega} \in R^{n \times q}$ is auxiliary matrix variable, which help us to overcome high relative degree obstruction in system (11); $\Delta \in R^n$ corresponds to estimate of exciting input $\mathbf{D}$, $\Delta \in \mathcal{M}_{R^n}$. To solve the posed problem it is suggested to adjust the estimates $\hat{\boldsymbol{\theta}}$ of parameters $\boldsymbol{\theta}$ by the speed gradient algorithm [12]:

$$\dot{\hat{\boldsymbol\theta}} = \gamma \boldsymbol\Omega^T \mathbf{C}^T \left(\mathbf{C}\boldsymbol\varepsilon - \mathbf{C}\mathbf{z} - \mathbf{C}\boldsymbol\Omega\hat{\boldsymbol\theta}\right),\ \gamma > 0. \quad (15)$$

Therefore, controller (5) is described by (10), (13)–(15).

A s s u m p t i o n 4. *The smallest singular value* $a(t)$ *of matrix function* $\boldsymbol\Omega(t)^T \mathbf{C}^T$ *is* (μ,Δ)*-positive in average* (PA), $\mu > 0$, $\Delta > 0$. □

Property PA is introduced in the Appendix. It is a variant of persistent excitation property (as it is proven in Lemma A2). The above assumptions (as in [10]) will be utilized to justify convergence to zero of parametric error $\tilde{\boldsymbol\theta}$. Note that due to presence of exciting signal $\mathbf{d}$ Assumption 4 (as any other supposition about persistent excitation in the system) is a mild one. Let us formulate the main result of the paper.

T h e o r e m 1. *Let Assumptions 1–4 hold*; $\mathbf{d} \in \mathcal{M}_{R^n}$ *and* $\Delta \in \mathcal{M}_{R^n}$; $|\mathbf{B}(\mathbf{y}(t))| \le B$ *for almost all* $t \ge 0$, $B \in R_+$. *Then the system (6), (7), (10), (13)–(15) is IOS with respect to output* $|\varepsilon|_{\mathcal{A}}$ *and input* $(\mathbf{d},\Delta)$. ■

The proof is excluded due to space limitation. It is based on auxiliary error variable

$$\boldsymbol\delta = \boldsymbol\varepsilon - \mathbf{z} - \boldsymbol\Omega\boldsymbol\theta, \quad (16)$$

which dynamics obeys the following differential equation:

$$\begin{aligned}\dot{\boldsymbol\delta} &= \mathbf{A}\boldsymbol\varepsilon + \phi(\mathbf{C}\boldsymbol\varepsilon) + \mathbf{B}(\mathbf{y})(\hat{\boldsymbol\theta} - \boldsymbol\theta) + \mathbf{D} - \mathbf{A}\mathbf{z} - \phi(\mathbf{C}\boldsymbol\varepsilon) - \\ &\quad \mathbf{K}(\mathbf{C}\boldsymbol\varepsilon - \mathbf{C}\mathbf{z}) - \mathbf{B}(\mathbf{y})\hat{\boldsymbol\theta} - \Delta - (\mathbf{G}\boldsymbol\Omega - \mathbf{B}(\mathbf{y}))\boldsymbol\theta = \quad (17)\\ &= \mathbf{G}\boldsymbol\delta + (\mathbf{D} - \Delta).\end{aligned}$$

R e m a r k 1. Signal Δ is included in the right hand side of (13) to compensate influence of external disturbing signal $\mathbf{D}$ on dynamics of auxiliary error δ and to increase accuracy of value $\boldsymbol\theta$ estimation by adjustable parameters $\hat{\boldsymbol\theta}$. However, in this case desired IOS property depends on Δ. Thus, if it is possible, then it is better to add Δ in (10):

$$\mathbf{u} = \varphi_l(\mathbf{y}_l) - \varphi_f(\mathbf{y}_f) - \mathbf{B}(\mathbf{y})\hat{\boldsymbol\theta} - \phi(\mathbf{y}_l - \mathbf{y}_f) + \Delta,$$

to increase quality of synchronization and $\boldsymbol\theta$ estimation. □

V. Applications

In this section the problem of adaptive dynamical synchronization is illustrated by two examples, frequently used in fields of formation control and self-organization of swarm systems [5], [21].

A. Oscillatory synchronization

Let systems (6) and (7) be given as follows:

$$\dot{x}_1^l = x_2^l; \quad \dot{x}_2^l = -\omega^2 \sin(x_1^1) + d_l; \quad (18)$$

$$\dot{x}_1^f = x_2^f; \quad \dot{x}_2^f = u + d_f, \quad (19)$$

where $\mathbf{x}_l = (x_1^l, x_2^l)$, $\mathbf{x}_f = (x_1^f, x_2^f)$ are available for measurements; ω is unknown frequency of system (18). Right hand side of system (18) is bounded by norm of the system state and input, therefore, the system is forward complete (Assumption 1 holds). System (8) has the form:

$$\dot\varepsilon_1 = \varepsilon_2; \quad \dot\varepsilon_2 = -\omega^2 \sin(x_1^l) - u + D, \quad (20)$$

where $D = d_l - d_f$. Reference system is described by the pendulum equations:

$$\dot{e}_1 = e_2; \quad \dot{e}_2 = -\omega_e^2 \sin(e_1^l) + \phi(e_1, e_2) + D_e, \quad (21)$$

where control $\phi(e_1,e_2) = -k e_2 (H(e_1,e_2) - H_o)$ with $H(e_1,e_2) = 0.5 e_2^2 + \omega_e^2[1 - \cos(e_1)]$ ensures for pendulum (21) asymptotic stabilization of level H_o of energy H, robustly with respect to sufficiently small disturbance D_e [34]. Thus Assumption 2 is satisfied for such inputs D_e. Substituting control

$$u = -\hat\theta \sin(x_1^l) + \omega_e^2 \sin(x_1^l - x_1^f) - \phi(x_1^l - x_1^f, x_2^l - x_2^f)$$

in system (20) we obtain

$$\dot\varepsilon_1 = \varepsilon_2;\ \dot\varepsilon_2 = -\omega_e^2 \sin(\varepsilon_1) + (\hat\theta - \theta)\sin(x_1^l) + \phi(\varepsilon_1, \varepsilon_2) + D, \quad (22)$$

where $\theta = \omega^2$. To design adaptive observer (13), (14) for system (22) assume that only variables x_1^l and x_1^f are available for measurements. Thus,

$$\begin{aligned}&\dot{z}_1 = z_2 + K(\varepsilon_1 - z_1); K > 0;\\ &\dot{z}_2 = -\omega_e^2 \sin(\varepsilon_1) + \hat\theta \sin(x_1^l) + \phi(\varepsilon_1, \varepsilon_2) + K(\varepsilon_1 - z_1);\end{aligned} \quad (23)$$

$$\dot\Omega_1 = \Omega_2 - K\Omega_1; \quad \dot\Omega_2 = -K\Omega_1 - \sin(x_1^l); \quad \mathbf{G} = \begin{bmatrix} -K & 1 \\ -K & 0 \end{bmatrix}. \quad (24)$$

Matrix $\mathbf{G}$ is strict minimum phase and the system (12) is input-to-state stable [38]. Hence, Assumption 3 is satisfied too. Adaptation algorithm has the form

$$\dot{\hat\theta} = \gamma \Omega_1 (x_1^l - x_1^f - z_1 - \Omega_1 \hat\theta). \quad (25)$$

Signal $\Omega_1^2(t)$ is PA if signal $\sin(x_1^l(t))$ is persistently exciting, that is the case for any non zero initial conditions in (18) and non vanishing disturbance d_l. Therefore Assumption 4 holds, and all other conditions of Theorem 1 are satisfied.

B. Pendulums synchronization with desired angle offset

Let the leader and the follower be identical pendulums:

$$\dot{x}_1^l = x_2^l; \quad \dot{x}_2^l = -\omega^2 \sin(x_1^1) - r x_2^l + d_l; \quad (26)$$

$$\dot{x}_1^f = x_2^f; \quad \dot{x}_2^f = -\omega^2 \sin(x_1^f) - r x_2^f + u + d_l, \quad (27)$$

where $\mathbf{x}_l = (x_1^l, x_2^l)$, $\mathbf{x}_f = (x_1^f, x_2^f)$; variables x_1^l and x_1^f are available for measurements; ω and r are unknown frequency and known friction coefficient of systems (26), (27) respectively. Right hand side of system (26) is bounded by norm of the system state and input, therefore, the system is forward complete and Assumption 1 holds. Dynamics of synchronization error obeys equation:

$$\dot\varepsilon_1 = \varepsilon_2; \quad \dot\varepsilon_2 = -\omega^2[\sin(x_1^l) - \sin(x_1^f)] - r\varepsilon_2 - u + D. \quad (28)$$

Reference system is described by stable linear system:

$$\dot{e}_1 = e_2; \quad \dot{e}_2 = -r e_2 - e_1 + e_{1o} + D_e,$$

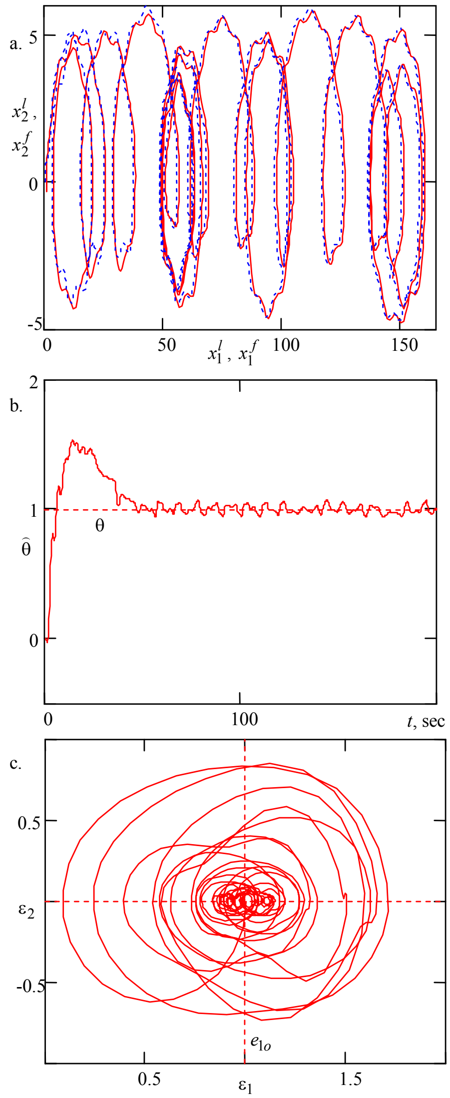

Fig. 1 . Trajectories of system (26), (27), (30), (31), (32).

which provides oscillations of pendulums (26), (27) with the same speed and with angle offset, that equals to e_{1o}. This system is again is input-to-state stable [38] and Assumption 2 is satisfied. Substituting control

$$u = -\hat{\theta}\,[\sin(x_1^l) - \sin(x_1^f)] + (x_1^l - x_1^f) - e_{1o}$$

in system (28) we obtain for $\theta = \omega^2$:

$$\dot{\varepsilon}_1 = \varepsilon_2\,; \quad \dot{\varepsilon}_2 = (\hat{\theta} - \theta)\,[\sin(x_1^l) - \sin(x_1^f)] - r\,\varepsilon_2 - \varepsilon_1 + e_{1o} + D, \tag{29}$$

Adaptive observer can be written as follows:

$$\dot{z}_1 = z_2\,;\ \dot{z}_2 = \hat{\theta}\,[\sin(x_1^l) - \sin(x_1^f)] - r\,z_2 - z_1 + e_{1o}\,; \tag{30}$$

$$\dot{\Omega}_1 = \Omega_2\,;\ \dot{\Omega}_2 = -\Omega_1 - r\,\Omega_2 - [\sin(x_1^l) - \sin(x_1^f)]\,. \tag{31}$$

Matrix $\mathbf{G}$ in this case coincides with linear part of reference model and it is strictly minimum phase. Thus, system (12) is input-to-state stable [38] and Assumption 3 holds. Adaptation algorithm has form

$$\dot{\hat{\theta}} = \gamma\,\Omega_1\,(x_1^l - x_1^f - z_1 - \Omega_1\,\hat{\theta})\,. \tag{32}$$

Signal $\Omega_1^2(t)$ is PA if signal $\sin(x_1^l) - \sin(x_1^f)$ is persistently exciting, but according to control goal variables x_1^l and x_1^f have non zero offset, which also should not be a multiple of 2π. Therefore Assumption 4 holds. Result of computer simulation of the system is presented in Fig. 1 for

$$\omega = 1,\ r = 0.1,\ e_{1o} = 1,\ d_l(t) = 2\sin(0.5t)\,,$$
$$d_f(t) = d_l(t) + 0.1\sin(0.3t)\,,\ \gamma = 0.5\,.$$

VI. Conclusion

The problem of adaptive dynamical synchronization is formulated and solved for a subclass of Lurie systems. The proposed solution is oriented to applications in fields of formation control, mechanical systems synchronization and self-organization of swarm systems. The potentialities of the solution is demonstrated via two examples.

Appendix

The next property is frequently used in adaptive control theory to establish identification ability of adaptation algorithms.

Definition A1 [12], [25]. *It is said, that essentially bounded matrix function* $\mathbf{R}(t)$, $t \ge 0$ *with dimension* $l_1 \times l_2$ *admits* (L, ϑ)*–persistent excitation (*PE*) condition, if there exist strictly positive constants* L *and* ϑ *such, that for any* $t \ge 0$

$$\int_t^{t+L} \mathbf{R}(s)\mathbf{R}(s)^T\,ds \ge \vartheta\,\mathbf{I}_{l_1}\,,$$

where $\mathbf{I}_{l_1}$ *denotes identity matrix of dimension* $l_1 \times l_1$. □

The following property was proposed in [8].

Definition A2. *Function* $a : R_+ \to R$ *is called* (μ, Δ)*–positive in average (PA), if for any* $t \ge 0$ *and any* $\delta \ge \Delta > 0$, $\mu > 0$,

$$\int_t^{t+\delta} a(\tau)\,d\tau \ge \mu\,\delta\,. \qquad \square$$

Lemma A1. *Let us consider time-varying linear dynamical system*

$$\dot{p} = -a(t)\,p + b(t)\,,\ t_0 \ge 0\,, \tag{A1}$$

where $p \in R$ *and functions* $a : R_+ \to R$, $b : R_+ \to R$ *are Lebesgue measurable,* b *is essentially bounded, function* a *is* (μ, Δ)*–PA for some* $\mu > 0$, $\Delta > 0$ *and essentially bounded from below, i.e. there exists* $A \in R_+$*, such, that:*

$$\operatorname{ess\,inf}\{a(t),\ t \ge t_0\} \ge -A\,.$$

Then solutions of system (A1) are defined for all $t \ge t_0$ *and they admit estimate*

$$
| p(t) | \le \begin{cases} | p(t_0) | e^{-\mu(t-t_0)+(A+\mu)\Delta} + \\ + \| b \| \max\{ A^{-1} e^{At_0}, \mu^{-1} e^{-\mu t_0} \}, \ A \ne 0; \\ | p(t_0) | e^{-\mu(t-t_0)+\mu\Delta} + \\ + \| b \| \max\{ \Delta, \mu^{-1} e^{-\mu t_0} \}, \qquad A = 0. \end{cases} \quad \blacksquare
$$

The following lemma establishes a connection between properties defined in definitions A1 and A2.

Lemma A2. *Suppose, that function* $a(t)$, $t \ge 0$ *admits* (L, ϑ)*–PE condition, then function* $a^2(t)$ *is* $(0.5\vartheta/L, L)$*– PA. Conversely, if function* $a^2(t)$, $t \ge 0$ *is* (ϑ, L)*–PA, then function* $a(t)$ *also possesses* $(L, L\vartheta)$*–PE condition too.* ■

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