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The Linear Periodic Output Regulation Problem

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***Abstract*— The problem of asymptotic output regulation for linear systems driven by time-varying, T-periodic exosystems is considered in this paper. Necessary and sufficient condition for its solvability based on the existence of periodic solutions of differential Sylvester equations are derived. These conditions constitute a generalization to the periodic case of the celebrated algebraic regulator equations of Francis. A general algorithm for the synthesis of an error-feedback regulator is given. For the special case of minimum-phase systems, it is shown that the regulator design can be carried out without the knowledge of the Floquet decomposition of the exosystem. The issue of robust regulation by error feedback is also briefly addressed.**

I. Introduction

The output regulation problem is one of the central themes in control theory. A complete solution for LTI systems has been known since the seminal works [5], [7], and since the trend-setting work of Isidori and Byrnes [9] for the case of nonlinear systems. More recently, considerable research efforts have been spent to extend the class of exosystems that can be dealt with using internal-model based design, including parameter-dependent linear systems [12], and special classes of nonlinear models [4]. As a fundamental step towards a more comprehensive theory, we consider in this paper the output regulation problem for time-invariant linear plant models driven by time-varying periodic exosystems. We give explicit conditions for the solvability of the problem which constitute a generalization of the celebrated regulator equations of Francis [7]. We shown that the solvability of the full-information problem is necessary and sufficient for the existence of an error-feedback regulator, and a general algorithm for a controller synthesis is readily obtained. In case the plant model is minimum-phase with respect to the regulated error, the regulator design can be carried out without the explicit knowledge of the Floquet factors of the exosystem, whose computation may be intractable for some exosystem models of interest. We also provide preliminary results on the design of robust regulators for minimum-phase systems, based on the concept of system immersion.

The paper is organized as follows: In Section II, we give some background and the formulation of the problem. Necessary and sufficient conditions for the existence of the solution are given in Section III. A general algorithm for the regulator synthesis is outlined in Section IV. Sections V and VI deal respectively with the design of a regulator for general minimum-phase plant models, and the construction of its robust version when appropriate conditions are met.

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II. Problem formulation

We consider LTI plant models of the form

$$\begin{array}{rcl} \dot{x} &=& Ax + Bu + d \\ e &=& Cx - r \end{array} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and regulated error $e \in \mathbb{R}^m$, satisfying the obvious assumption

Assumption 2.1: The pair (A, B) is stabilizable, and the pair (A, C) is detectable. ⋄

The disturbance $d \in \mathbb{R}^n$ to be rejected and the reference output $r \in \mathbb{R}^m$ to be tracked are generated by the time-varying exosystem

$$\begin{array}{rcl} \dot{w} &=& S(t)w \\ d &=& P(t)w \\ r &=& -Q(t)w \end{array} \tag{2}$$

with state $w \in \mathbb{R}^q$. The following assumption broadly characterizes the class of exosystems under consideration.

Assumption 2.2: The entries of the matrix-valued functions $S : \mathbb{R} \to \mathbb{R}^{q\times q}$, $P : \mathbb{R} \to \mathbb{R}^{n\times q}$, and $Q : \mathbb{R} \to \mathbb{R}^{m\times q}$ are smooth T-periodic functions, for some $T > 0$. ⋄

In what follows, the transition matrix of the system (2) is denoted by $\Phi_S(t, \tau)$, and the corresponding *monodromy matrix* by $\bar{\Phi}_S = \Phi_S(T, 0)$. Let $U(t) : t \to \mathbb{R}^{q\times q}$ and $R \in \mathbb{C}^{q\times q}$ denote the Floquet factors of $S(t)$, i.e.,

$$\bar{\Phi}_S = \mathrm{e}^{RT}\,, \quad U(t) = \Phi_S(t, 0)\mathrm{e}^{-Rt}\,,$$

with $U(t + T) = U(t)$ for all $t \in \mathbb{R}$. In particular,

$$\Phi_S(t, \tau) = U(t)\mathrm{e}^{R(t-\tau)}U^{-1}(\tau)\,. \tag{3}$$

While in general the matrix R has complex entries, it is always possible to obtain a Floquet factorization with a real matrix R simply redefining the period as $\bar{T} = 2T$ (see [6]). Since this involves no loss of generality for our purposes, we assume that $R \in \mathbb{R}^{q\times q}$. For the interconnection

$$\begin{array}{rcl} \dot{w} &=& S(t)w \\ \dot{x} &=& Ax + Bu + P(t)w \\ e &=& Cx + Q(t)w\,, \end{array} \tag{4}$$

we consider in this paper the design of a smooth T-periodic error-feedback controller of the form

$$\begin{array}{rcl} \dot{\xi} &=& F(t)\xi + G(t)e \\ u &=& H(t)\xi + K(t)e \end{array} \tag{5}$$

with state $\xi \in \mathbb{R}^{\nu}$, such that (i) the origin is an asymptotically stable equilibrium of the unforced closed-loop system

$$\begin{array}{rcl} \dot{x} & = & (A + BK(t)C)x + BH(t)\xi \\ \dot{\xi} & = & F(t)\xi + G(t)Cx \,, \end{array} \tag{6}$$

and (ii) the trajectories of the closed-loop system (4)-(5) originating from any initial condition $(w_0, x_0, \xi_0) \in \mathbb{R}^{q+n+\nu}$ satisfy $\lim_{t\to\infty} e(t) = 0$. It is customary to exclude from the analysis the presence of converging trajectories of (2), for which the solution of the problem (ii) is a trivial consequence of the closed-loop stability requirement (i). Therefore, we make the following additional assumption.

Assumption 2.3: The eigenvalues of the monodromy matrix $\bar{\Phi}_S$ have magnitude greater or equal to one. ◇

By virtue of (3), the above assumption implies that

$$\|\Phi_S(t,\tau)\| \geq \mu\, \mathrm{e}^{\sigma(t-\tau)}$$

for some $\mu > 0$ and $\sigma \geq 0$.

III. Solvability of the problem

The following lemma characterizes the class of stabilizing controllers of the form (5) capable of achieving regulation.

Lemma 3.1: Assume that the controller (5) asymptotically stabilizes the origin of (6). Then, the same controller yields asymptotic regulation of the error $e(t)$ if and only if the *unique T-periodic solution* $X(t) = (\Pi'(t), \Sigma'(t))'$ of the differential Sylvester equation (DSE)

$$\begin{array}{rcl} \dot{\Pi} + \Pi S & = & (A + BKC)\Pi + BH\Sigma + P + BKQ \\ \dot{\Sigma} + \Sigma S & = & F\Sigma + GC\Pi + GQ \end{array}$$

satisfies $C\Pi(t) + Q(t) = 0$ for all $t \in [0, T)$.

Proof: Denote with $A_{\mathrm{cl}}(t)$ and $P_{\mathrm{cl}}(t)$ the matrices (omitting the argument t for brevity)

$$A_{\mathrm{cl}} = \begin{pmatrix} A + BKC & BH \\ GC & F \end{pmatrix}, \quad P_{\mathrm{cl}} = \begin{pmatrix} P + BKQ \\ GQ \end{pmatrix},$$

and let $\mathbf{x} = \mathrm{col}(x, \xi)$. Since by assumption the transition matrix $\Phi_{A_{\mathrm{cl}}}(t,\tau)$ of (6) satisfies

$$\|\Phi_{A_{\mathrm{cl}}}(t,\tau)\| \leq \kappa \mathrm{e}^{-\lambda(t-\tau)}$$

for some $\kappa, \lambda > 0$, Lemma 1.1 in the appendix guarantees that there exists a unique solution $X(t)$ of the DSE

$$\dot{X}(t) + X(t)S(t) = A_{\mathrm{cl}}(t)X(t) + P_{\mathrm{cl}}(t)\,, \quad X(t_0) = X_0$$

satisfying $X(t+T) = X(t)$ for all $t \in \mathbb{R}$. Since $X(t)$ is bounded, the transformation $\tilde{\mathbf{x}} := \mathbf{x} - X(t)w = \mathrm{col}(\tilde{x}, \tilde{\xi})$ is a Lyapunov transformation, yielding a closed-loop system in the form

$$\begin{array}{rcl} \dot{w} & = & S(t)w \\ \dot{\tilde{\mathbf{x}}} & = & A_{\mathrm{cl}}(t)\tilde{\mathbf{x}}\,. \end{array} \tag{7}$$

The solution of (7) from arbitrary initial conditions $(w_0, \tilde{\mathbf{x}}_0)$ at t_0 generates the error trajectory

$$e(t) = C\tilde{x}(t, t_0, \tilde{x}_0) + [C\Pi(t) + Q(t)]w(t, t_0, w_0)\,.$$

Letting $t = t_0 + kT$, $k \in \mathbb{N}$, and keeping in mind that $\lim_{t\to\infty} \tilde{x}(t, t_0, \tilde{\mathbf{x}}_0) = 0$, one obtains

$$\lim_{k\to\infty} e(t_0 + kT) = [C\Pi(t_0) + Q(t_0)]w_0$$

and therefore $\lim_{k\to\infty} e(t_0 + kT) = 0$ if and only if $w_0 \in \ker(C\Pi(t_0) + Q(t_0))$. Then, the result follows from the arbitrariness of $w_0 \in \mathbb{R}^q$ and $t_0 \in [0, T)$ ■

An equivalent characterization of Lemma 3.1, which constitutes the counterpart of the celebrated result by Francis [7] to the case of T-periodic systems is stated as follows.

Proposition 3.2: Assume that the controller (5) is such that the system (6) is asymptotically stable. Then, asymptotic regulation is achieved if and only if there exist T-periodic matrix-valued functions $\Pi(t)$, $\Gamma(t)$, and $\Sigma(t)$ satisfying

$$\begin{array}{rcl} \dot{\Pi}(t) + \Pi(t)S(t) & = & A\Pi(t) + B\Gamma(t) + P(t) \\ 0 & = & C\Pi(t) + Q(t) \end{array} \tag{8}$$

and

$$\begin{array}{rcl} \dot{\Sigma}(t) + \Sigma(t)S(t) & = & F(t)\Sigma(t) \\ \Gamma(t) & = & H(t)\Sigma(t) \end{array} \tag{9}$$

for all $t \in [0, T)$. ◇

The proof of Proposition 3.2 follows directly from that of Lemma 3.1, and need not be repeated. The differential equation (8) is the extension to the T-periodic case of the regulator equations of [7]. It is easy to show that the existence of a periodic solution of (8) is a necessary and sufficient condition for the solvability of the *full information* output regulation problem by means of a memoryless control law of the form

$$u = Kx + L(t)w\,,$$

where $L(t)$ satisfies $L(t+T) = L(t)$ for all $t \in \mathbb{R}$. Necessity of (8) can be proven following arguments similar to those employed in the proof of Lemma 3.1, while sufficiency is easily shown selecting K in such a way that $A + BK$ is Hurwitz, and taking $L(t) = \Gamma(t) - K\Pi(t)$. What remains to be seen is if, analogously to the time-invariant case, the existence of a solution of the regulator equations (8) alone is *sufficient* for the existence of a regulator.

IV. Regulator synthesis

In this section, we will show that the solvability of the regulator equations (8) is a sufficient condition for the synthesis of a regulator. Following [10], we look for a solution under a slightly more restrictive hypotheses than the mere detectability of (A, C), that however does not result in any loss of generality. Specifically, we replace Assumption 2.2 with the following:

Assumption 4.1: The pair (A, B) is stabilizable, and the pair $(A^{\mathrm{a}}(t), C^{\mathrm{a}}(t))$, where

$$A^{\mathrm{a}}(t) = \begin{pmatrix} S(t) & 0 \\ P(t) & A \end{pmatrix}, \quad C^{\mathrm{a}}(t) = \begin{pmatrix} Q(t) & C \end{pmatrix},$$

is detectable. ◇

The fact that there is no loss of generality in considering Assumption 4.1 in place of Assumption 2.2 is made clear by the following proposition, which is an extension of [10, Prop. 1.4.1] to our case.

Proposition 4.2: Assume that (A, C) is detectable, but $(A^{\rm a}(t), C^{\rm a}(t))$ is not. Denote with $x^{\rm a}$ the state of the augmented plant, $x^{\rm a} = {\rm col}(w, x)$. Then, there exists a periodic Lyapunov transformation $\tilde{x}^{\rm a} = T^{\rm a}(t)x^{\rm a}$ such that in the new coordinates the augmented system matrices have the form

$$\tilde{A}^{\rm a} = \begin{pmatrix} \tilde{S} & 0 \\ \tilde{P} & A \end{pmatrix} = \left(\begin{array}{c|cc} S_{11} & S_{12} & 0 \\ \hline 0 & S_{22} & 0 \\ 0 & P_2 & A \end{array}\right) = \begin{pmatrix} \tilde{A}^{\rm a}_{11} & \tilde{A}^{\rm a}_{12} \\ 0 & \tilde{A}^{\rm a}_{22} \end{pmatrix},$$

$$\tilde{B}^{\rm a} = B^{\rm a} = \left(\begin{array}{c} 0 \\ \hline 0 \\ B \end{array}\right) = \begin{pmatrix} 0 \\ \tilde{B}^{\rm a}_2 \end{pmatrix},$$

$$\tilde{C}^{\rm a}(t) = \begin{pmatrix} \tilde{Q}(t) & C \end{pmatrix} = \left(\begin{array}{c|cc} 0 & Q_2(t) & C \end{array}\right) = \begin{pmatrix} 0 & \tilde{C}^{\rm a}_2(t) \end{pmatrix},$$

and the pair $(\tilde{A}^{\rm a}_{22}, \tilde{C}^{\rm a}_2(t))$ is detectable. ◇

The result shows that there is no loss of generality in considering Assumption 4.1 in place of Assumption 2.2. As a matter of fact, according to Proposition 4.2, there exists a realization of the exosystem such that a certain number of component of the exosystem state (those corresponding to the matrix S_{11}) do not affect neither the plant nor the regulated error. The unobservable exosystem dynamics can be factored out, yielding a reduced exosystem for which Assumption 4.1 holds. We are now in the position to state that the solvability of the regulator equations is a sufficient condition for the synthesis of a regulator. Due to space limitations, the proof is omitted.

Theorem 4.3: Let Assumptions 2.3 and 4.1 hold. Then, there exists a controller of the form (5) that solves the output regulation problem if there exist T-periodic matrix-valued functions $\Pi(t)$ and $\Gamma(t)$ that solve (8). ◇

V. Regulator design for minimum-phase systems

The previous section shows that, under mild detectability assumptions, the solvability of the full-information problem is a necessary as well as a sufficient condition for the existence of a solution to the error-feedback problem, in a fashion that is completely analogous to the LTI case. While this result is of methodological importance, the construction of the regulator outlined in the previous section may not be entirely satisfactory, as it suffers from the potentially serious drawback of requiring the explicit knowledge of the transition matrix of the exosystem, as well as the reconstructibility Gramian of the augmented system. For this reason, we have referred to the content of Section IV as a regulator *synthesis* rather than a regulator *design*. In general, stabilization methods for linear periodic systems require *de facto* the explicit use of the real Floquet-Lyapunov decomposition (see [11] for an exposition of recent results, and references therein). For the problem at issue here, however, the particular structure of (4) suggests that the stabilization of the plant model may be achieved in some cases with a time-invariant controller, and the use of Floquet factors can be avoided. The approach we follow is to look for a controller that is decomposed into the parallel interconnection of a time-invariant error-feedback stabilizer and a T-periodic internal model of the exosystem, whose role is solely that of reconstructing asymptotically the feedforward control $u = \Gamma(t)w(t)$, where $\Gamma(t)$ is given by the second equation in (9). To this end, we restrict our attention to the case of SISO plant models (1) and exosystem models (2) satisfying the following assumptions.

Assumption 5.1: The system (1), where $m = 1$, is minimum phase and has relative degree one. ◇

Assumption 5.2: The exosystem (2) is neutrally stable. ◇

Neither the assumption of relative degree one or the single-input single-output case for the plant model are restrictive. Systems with higher relative degree can be dealt with by using dynamic extensions or high-gain observers. The assumption on the exosystem is indeed quite natural, as it corresponds to the case in which every trajectory $w(t)$ generated by the exosystem is T-periodic. Consequently, the monodromy matrix of $S(t)$ is the identity, and its Floquet factors are $R = 0$ and $U(t) = \Phi_S(t, 0)$.

It is well known that, if Assumption 5.1 holds, the interconnected system (4) can be put in the following form

$$\begin{aligned} \dot{w} &= S(t)w \\ \dot{z} &= A_{11}z + A_{12}y + P_1(t)w \\ \dot{y} &= A_{21}z + a_{22}y + P_2(t)w + bu \\ e &= y + Q(t)w, \end{aligned}$$

where the matrix A_{11} is Hurwitz and $b \neq 0$ by assumption. Consequently, Lemma 1.1 guarantees the existence of a T-periodic solution $\Xi(t)$ of the Sylvester differential equation

$$\dot{\Xi}(t) + \Xi(t)S(t) = A_{11}\Xi(t) + P_1(t) - A_{12}Q(t).$$

Changing coordinates as $\tilde{z} = z - \Xi(t)w$, $e = y + Q(t)w$, one obtains

$$\begin{aligned} \dot{w} &= S(t)w \\ \dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}e \\ \dot{e} &= A_{21}\tilde{z} + a_{22}e + b[u - \Gamma(t)w], \end{aligned} \tag{10}$$

where

$$\Gamma(t) = \tfrac{1}{b}[A_{12}Q(t) - A_{21}\Xi(t) - P_2(t) - \dot{Q}(t) - Q(t)S(t)].$$

Note that it possible to assume without loss of generality that the pair $(S(t), \Gamma(t))$ is completely observable, as the trajectories $w(t)$ which lie in the unobservable subspace of $(S(t), \Gamma(t))$ do no affect the trajectories $(\tilde{z}(t), e(t))$ of (10), and can be factored out by means of a canonical decomposition [1].

When the exosystem is disconnected, the equilibrium $(\tilde{z}, e) = (0, 0)$ of (10) can be rendered asymptotically stable by the application of the static output feedback control $u_{\rm st} = -{\rm sign}(b)k\,e$, if the gain $k > 0$ is chosen sufficiently large. The system (10) is then augmented with a T-periodic internal model of the form

$$\begin{aligned} \dot{\xi} &= F(t)\xi + G(t)u \\ u_{\rm im} &= H(t)\xi \end{aligned} \tag{11}$$

with $\xi \in \mathbb{R}^q$, and the control input is selected as $u = u_{\rm st} + u_{\rm im}$. The design of the internal model proceeds as follows. Fix arbitrarily $\alpha > 0$, and choose

$$F(t) = -\alpha I - S'(t)\,.$$

It can be easily verified that the transition matrix of $F(t)$ is given by

$$\Phi_F(t,\tau) = {\rm e}^{-\alpha(t-\tau)}\Phi_S'(\tau,t)\,,$$

and thus the T-periodic system (11) is asymptotically stable, since its monodromy matrix is $\bar\Phi_F = {\rm e}^{-\alpha T}I$.

Proposition 5.3: Assume that $(S(t),\Gamma(t))$ is completely observable. Then, a T-periodic solution $L_\infty(t)$ of the Sylvester differential equation

$$\dot L(t)+L(t)S(t) = F(t)L(t)+\Gamma'(t)\Gamma(t)\,,\; L(t_0) = L_0 \quad (12)$$

exists, is unique, and is nonsingular for all $t \in [0,T)$.

Proof: Existence and uniqueness of $L_\infty(t)$ follow directly from Lemma 1.1. To prove that $L_\infty(t)$ is nonsingular for any t, note that since $L_\infty(t+T) = L_\infty(t)$ for all $t \in \mathbb{R}$, necessarily

$$L_\infty(t) = {\rm e}^{-\alpha T}L_\infty(t) + M_\alpha(t,t+T)\,,$$

where the matrix $M_\alpha(t,t{+}T)$ is analogous to the one defined in (**??**), and reads in this case as

$$M_\alpha(t,t+T) = \int_t^{t+T} {\rm e}^{-\alpha(t+T-\tau)}\Phi_S'(\tau,t+T)\Gamma'(\tau)\Gamma(\tau)\Phi_S(\tau,t+T){\rm d}\tau\,.$$

It is clear that $L_\infty(t)$ is invertible for all $t \in [0,T)$ if and only if so is $M_\alpha(t,t+T)$. In turn, $M_\alpha(t,t+T)$ is nonsingular for all t if and only if so is the reconstructibility Gramian $M(t,t+T)$ of $(S(t),\Gamma(t))$. Due to the fact that by assumption the transition matrix of $S(t)$ satisfies $\Phi(t+T,t) = I$ for all t, it is not difficult to show that

$$M(t,t+T) = \Phi_S'(0,t)M(0,T)\Phi_S(0,t)\,,\quad \forall\, t \in [0,T)$$

and that

$$M(0,kT) = kM(0,T)\,,\quad \text{for all } k = 1,2,\ldots$$

Since the pair $(S(t),\Gamma(t))$ is assumed to be completely observable, the reconstructibility matrix $M(0,qT)$ is nonsingular, being q the order of the exosystem (see [2], [3]), and this completes the proof. ∎

The result of Proposition 5.3 allows to define the periodic Lyapunov transformation $\bar w = L_\infty(t)w$ yielding a system which is topologically equivalent to (2), given by

$$\dot{\bar w} = (F(t) + \Gamma'(t)\Gamma(t)N_\infty(t))\,\bar w\,, \qquad (13)$$

where, for ease of notation, we have denoted the inverse of $L_\infty(t)$ by $N_\infty(t)$. Next, the matrices $G(t)$ and $H(t)$ of the internal model (11) are selected as $G(t) = \Gamma'(t)$ and $H(t) = \Gamma(t)N_\infty(t)$, to obtain the controller

$$\begin{aligned}\dot\xi &= (F(t) + \Gamma'(t)\Gamma(t)N_\infty(t))\xi - {\rm sign}(b)k\Gamma'(t)e\\ u &= \Gamma(t)N_\infty(t)\xi - {\rm sign}(b)k\,e\,. \end{aligned}\qquad (14)$$

Following [12], the change of coordinates

$$\chi = \xi - \bar w - \tfrac{1}{b}\Gamma(t)e$$

after easy algebraic manipulations yields the interconnection of (10) and (14) in the form

$$\begin{aligned}\dot\chi &= F(t)\chi + J_1(t)\tilde z + J_2(t)e\\ \dot{\tilde z} &= A_{11}\tilde z + A_{12}e \\ \dot e &= b\Psi_\infty(t)\chi + A_{21}\tilde z + (a_{22} + \Psi_\infty(t)\Gamma'(t) - k|b|)e\,,\end{aligned}\qquad (15)$$

where (omitting the argument t for brevity)

$$J_1 = -\tfrac{1}{b}\Gamma'A_{21}\,,\; J_2 = \tfrac{1}{b}(F-a_{22}I)\Gamma' - \tfrac{1}{b}\dot\Gamma'\,,\; \Psi_\infty = \Gamma N_\infty\,.$$

The next proposition shows that the controller (14) solves the output regulation problem.

Proposition 5.4: There exists a number $k^\star > 0$ such that for all $k \geq k^\star$ the system (15) is asymptotically stable. ⋄

Proof: First of all, note that (15) is still a T-periodic system, and that $J_1(t)$, $J_2(t)$, $\Gamma(t)$, and $\Psi_\infty(t)$ are all continuous and bounded functions. Moreover, it is clear that the zero dynamics of (15) with respect to e, that is, the system

$$\begin{aligned}\dot\chi &= F(t)\chi + J_1(t)\tilde z\\ \dot{\tilde z} &= A_{11}\tilde z\end{aligned}$$

is asymptotically stable. Therefore, standard arguments can be invoked to show that (15) is rendered asymptotically stable choosing $k > 0$ sufficiently large. ∎

Note that the construction of the regulator (14) still requires the knowledge of $N_\infty(t)$, which once again requires $M_\alpha(t,t+T)$ to be computed explicitly and inverted.

A. Certainty-equivalence implementation

In what follows, we look for an implementation of the controller (14) that does not require the a priori computation of $N_\infty(t)$. In regard to this, we apply the principle of *certainty equivalence*, and look for a suitable estimate to replace $N_\infty(t)$ in (14). It can be verified by direct substitution that $N_\infty(t)$ is indeed the solution of the Riccati-type differential equation

$$\dot N(t) = S(t)N(t) - N(t)F(t) - N(t)\Gamma(t)'\Gamma(t)N(t) \quad (16)$$

corresponding to the initial condition

$$N(t_0) = L_\infty^{-1}(t_0) = (1-{\rm e}^{-\alpha T})M_\alpha^{-1}(t_0,t_0+T)\,,$$

and thus equation (16) may be used in principle to obtain $N_\infty(t)$. However, it should be kept in mind that $N_\infty(t)$ is an attracting solution for (16) in forward time only when the inverse of $L(t)$ makes sense. As a matter of fact, the trajectory $N(t)$ of (16) starting from an arbitrary nonsingular initial condition $N(t_0) \in \mathbb{R}^{q\times q}$ exhibits a finite escape time at $t_1 > t_0$, whenever the solution $L(t)$ of (12) originating from $L(t_0) = N^{-1}(t_0)$ is such that $\det L(t_1) = 0$. Therefore, a correct initialization of (16) is crucial to obtain a trajectory $N(t)$ which is well defined and converges to $N_\infty(t)$. Note that equation (12) does generate a trajectory which converges exponentially to $L_\infty(t)$ from any initial condition, and therefore an initial condition $L(t_0)$ which is "close" to $L_\infty(t_0)$ can be computed from numerical simulations, looking at the solution $L(t_0+kT)$ for $k \in \mathbb{N}$ large enough. Using open loop simulations, it is also possible to compute bounds on the norm of $L_\infty(t)$ and $N_\infty(t)$ for all $t \in [0,T)$. In particular, it is reasonable to assume that an initial condition $L(t_0) = L_0$ for (12) and a positive number ε_0 can be determined such that the solution $L(t)$ originating from L_0 at $t = t_0$ satisfies

$$\sigma_{\min}(L(t)) \geq \varepsilon_0\,, \quad \text{for all } t \geq t_0\,,$$

where $\sigma_{\min}(L(t))$ stands for the smallest singular value of $L(t)$. The given bound ensures that the solution of (16) originating from $N(t_0) = L_0^{-1}$ does not have a finite escape time, converges to $N_\infty(t)$, and satisfies

$$\|N(t)\|_2 \leq \varepsilon_0^{-1}\,, \quad \text{for all } t \geq t_0\,.$$

The dynamic controller (14) is then replaced by

$$\begin{aligned}\dot N &= S(t)N - NF(t) - N\Gamma(t)'\Gamma(t)N\,,\ N(t_0) = L_0^{-1}\\ \dot\xi &= (F(t) + \Gamma'(t)\Gamma(t)N(t))\xi - \mathrm{sign}(b)k\Gamma'(t)e\\ u &= \Gamma(t)N(t)\xi - \mathrm{sign}(b)k\,e\,,\end{aligned} \qquad (17)$$

yielding the closed-loop system

$$\begin{aligned}\dot\chi &= F(t)\chi + J_1(t)\tilde z + J_2(t)e\\ \dot{\tilde z} &= A_{11}\tilde z + A_{12}e\\ \dot e &= b\Psi(t)\chi + A_{21}\tilde z + (a_{22} + \Psi(t)\Gamma'(t) - k|b|)e + d(t)\end{aligned} \qquad (18)$$

driven by the time-varying disturbance

$$d(t) = \Gamma(t)\tilde N(t)\bar w(t)\,,$$

where $\bar w(t)$ is the trajectory generated by (13). Note that, since $\bar w(t)$ is T-periodic and $\tilde N(t)$ converges to zero, the disturbance $d(t)$ is asymptotically vanishing for any initial condition of (13). Since $\Psi(t)$ is a bounded matrix-valued function of t (though not necessarily periodic), the result of Proposition 5.4 still holds for the system (18) when $d(t) \equiv 0$. Therefore, for $k > 0$ large enough, the time-varying linear system (18) is uniformly globally asymptotically stable. As the perturbation $d(t)$ vanishes asymptotically, the trajectories $(\chi(t), \tilde z(t), e(t))$ of (18) originating from any initial conditions $(\chi(t_0), \tilde z(t_0), e(t_0)) \in \mathbb{R}^{q+n}$ converge to the origin, and asymptotic regulation is achieved.

VI. Towards Robust Regulation

The last section of this paper is devoted to the problem of designing controllers to achieve asymptotic regulation in spite of possibly large parametric uncertainties in the plant model. In particular, we consider again the error system (10), and assume that the plant matrices depend on a vector μ of uncertain parameters, ranging over a compact subset $\mathcal{P}$ of $\mathbb{R}^p$. This yields a system of the form

$$\begin{aligned}\dot w &= S(t)w\\ \dot{\tilde z} &= A_{11}(\mu)\tilde z + A_{12}(\mu)e\\ \dot e &= A_{21}(\mu)\tilde z + a_{22}(\mu)e + b(\mu)[u - \Gamma(t,\mu)w]\,,\end{aligned} \qquad (19)$$

where the matrix $A_{11}(\mu)$ is Hurwitz, and $b(\mu) \geq b_0 > 0$ for all $\mu \in \mathcal{P}$. The pair $(S(t), \Gamma(t,\mu))$ is assumed to be completely observable for all $\mu \in \mathcal{P}$. It is also assumed that the vector fields of the system (19) depends continuously on μ. While asymptotic stabilization of the origin $(\tilde z, e) = (0,0)$ of the unforced system (19) can still be achieved by static high-gain feedback, it is clear that the applicability of the method developed in the previous section is precluded by the dependence of $\Gamma(t,\mu)$ on the unknown parameter vector μ. The solvability of the problem reposes precisely upon the possibility of reconstructing $\Gamma(t,\mu)w(t)$ independently of the actual value of μ. For linear time-invariant systems (and in some cases, for nonlinear systems), this is always possible, and follows from the fact that the Cayley-Hamilton theorem guarantees that the system with μ-dependent output

$$\dot w = Sw\,, \quad v = \Gamma(\mu)w$$

can always be *immersed* into an observable system independent of μ. For T-periodic systems, the situation is much more complicated, and such an immersion may not be found. Whenever this is possible, however, the construction of a robust regulator follows easily from the availability of a canonical realization of the internal model. We begin with the following definition.

Definition 6.1: The system

$$\begin{aligned}\dot w &= S(t)w\\ \dot\mu &= 0\\ v &= \Gamma(t,\mu)w\end{aligned} \qquad (20)$$

is *strongly immersible* into a finite dimensional T-periodic uniformly observable system if there exist an integer ℓ and T-periodic functions $a_0(t), a_1(t), \ldots, a_{\ell-1}(t)$ such that

$$\Delta_S^\ell\Gamma'(t,\mu) + \sum_{i=0}^{\ell-1} a_i(t)\Delta_S^i\Gamma'(t,\mu) = 0$$

for all $t \in [0,T)$, where $\Delta_S = S'(t) + \frac{\partial}{\partial t}I$. ◇

We refer to the above property as a *strong immersibility*, to stress the fact that the functions $a_i(t)$ are independent of μ. If (20) is strongly immersible, then its output trajectories can be generated as output trajectories of the system

$$\begin{aligned}\dot\eta &= \Phi(t)\eta\\ v &= \Gamma\eta\,,\end{aligned} \qquad (21)$$

where

$$\Phi(t) = \begin{pmatrix} 0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -a_0(t) & -a_1(t) & \cdots & -a_{\ell-1}(t)\end{pmatrix}, \ \Gamma' = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}.$$

Note that the original pair $(S(t), \Gamma(t,\mu))$ need not be uniformly observable to admit an immersion into a uniformly observable system. Clearly, (21) is uniformly observable, as its observability matrix is the identity, and thus there exists a periodic Lyapunov transformation $P_o(t)$ yielding a system in observer companion form [13], that is

$$\left[P_o(t)\Phi(t) + \dot P_o(t)\right]P_o^{-1}(t) = \Phi_o(t)\,, \quad \Gamma P_o^{-1}(t) = \Gamma\,,$$

where

$$\Phi_o(t) = \begin{pmatrix} -\alpha_{\ell-1}(t) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_1(t) & 0 & 0 & \cdots & 1 \\ -\alpha_0(t) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It can be easily verified that $P_o^{-1}(t) = \Theta_o(t)$, being $\Theta_o(t)$ the observability matrix of $(\Phi_o(t), \Gamma)$, and that the equation

$$\Theta_o(t)\left[\Phi_o(t)\Theta_o^{-1}(t) - \dot{\Theta}_o^{-1}(t)\right] = \Phi(t)$$

can always be solved for $\alpha_i(t)$, $i = 0, \ldots, \ell-1$. This result allows a systematic design of an internal model-based controller for the prototype system (19). Assume that (20) is strongly immersed, in the sense of Definition 6.1, into $(\Phi(t), \Gamma)$. Then, (20) is also immersed into $(\Phi_o(t), \Gamma)$, and system (19) can be replaced with

$$\begin{aligned} \dot{\eta} &= \Phi_o(t)\eta \\ \dot{\tilde{z}} &= A_{11}(\mu)\tilde{z} + A_{12}(\mu)e \\ \dot{e} &= A_{21}(\mu)\tilde{z} + a_{22}(\mu)e + b(\mu)[u - \Gamma\eta]. \end{aligned}$$

Let $\Phi_b \in \mathbb{R}^{\ell\times\ell}$ be in Brunowsky form, and let L_0 be such that $F := \Phi_b - L_0\Gamma$ is Hurwitz. Denote with $\alpha(t)$ the first column of $\Phi_o(t)$, that is,

$$\alpha(t) = \begin{pmatrix} -\alpha_{\ell-1}(t) & -\alpha_{\ell-2}(t) & \ldots & -\alpha_0(t) \end{pmatrix}'$$

and define $L(t) = \alpha(t) + L_0$. Clearly, the output injection matrix $L(t)$ exponentially stabilizes $(\Phi_o(t), \Gamma)$, as $\Phi_o(t) - L(t)\Gamma = F$. Consider the internal model-based controller

$$\begin{aligned} \dot{\xi} &= \Phi_o(t)\xi - kL(t)e \\ u &= \Gamma\xi - k\,e, \end{aligned} \qquad (22)$$

where $k > 0$ is a gain parameter, and change coordinate as $\chi = \xi - \eta - L(t)e$. This yields the closed-loop system

$$\begin{aligned} \dot{\chi} &= F\chi + J_1(t,\mu)\tilde{z} + J_2(t,\mu)e \\ \dot{\tilde{z}} &= A_{11}(\mu)\tilde{z} + A_{12}(\mu)e \\ \dot{e} &= b(\mu)\Gamma\chi + A_{21}(\mu)\tilde{z} + (a_{22}(\mu) + \Gamma L(t) - k)e, \end{aligned} \qquad (23)$$

where (omitting the arguments t and μ for brevity)

$$J_1 = -\tfrac{1}{b}LA_{21}, \quad J_2 = \tfrac{1}{b}(F - a_{22}I)L - \tfrac{1}{b}\dot{L}.$$

Since $J_1(t,\mu)$, $J_2(t,\mu)$, and $L(t)$ are T-periodic and continuous functions of t, and continuous functions of μ, the result of Proposition 5.4 applies for (23). In particular, one can easily prove the following.

Proposition 6.1: Let the compact set $\mathcal{P} \subset \mathbb{R}^p$ be given. Then, there exists a number $k^\star > 0$ such that for all $k \geq k^\star$ the system (23) is asymptotically stable for all $\mu \in \mathcal{P}$. ◇

APPENDIX

Lemma 1.1: Let $S(t) \in \mathbb{R}^{q\times q}$, $A(t) \in \mathbb{R}^{n\times n}$, and $P(t) \in \mathbb{R}^{n\times q}$ be continuous and T-periodic matrix-valued functions, for some $T > 0$. Assume that there exist constants $\kappa, \lambda, \mu > 0$, and $\sigma \geq 0$ such that

$$\|\Phi_A(t,\tau)\| \leq \kappa\,\mathrm{e}^{-\lambda(t-\tau)}, \quad \|\Phi_S(t,\tau)\| \geq \mu\,\mathrm{e}^{\sigma(t-\tau)}$$

for all $t, \tau \in \mathbb{R}$. Then, the matrix-valued function

$$X_\infty(t) = \int_{-\infty}^{t} \Phi_A(t,\tau)P(\tau)\Phi_S(\tau,t)\,\mathrm{d}\tau$$

is the unique solution of the Sylvester differential equation

$$\dot{X}(t) + X(t)S(t) = A(t)X(t) + P(t) \qquad (24)$$

satisfying $X(t+T) = X(t)$ for all $t \in \mathbb{R}$. ◇

Proof: First of all, the assumptions on the transition matrices of $A(t)$ and $S(t)$ guarantee that $X_\infty(t)$ is well defined. By direct differentiation and by application of Liebnitz's rule, it can be verified that $X(t)$ indeed satisfies the differential equation (24). Moreover, by virtue of (3)

$$\begin{aligned} X_\infty(t+T) &= \textstyle\int_{-\infty}^{t+T} \Phi_A(t+T,\tau)P(\tau)\Phi_S(\tau,t+T)\,\mathrm{d}\tau \\ &= \textstyle\int_{-\infty}^{t} \Phi_A(t+T,s+T)P(s+T)\Phi_S(s+T,t+T)\,\mathrm{d}s \\ &= \textstyle\int_{-\infty}^{t} \Phi_A(t,s)P(s)\Phi_S(s,t)\,\mathrm{d}s = X_\infty(t), \quad \forall\, t \in \mathbb{R}. \end{aligned}$$

Finally, fix t_0 arbitrarily, and let $X_0 = X_\infty(t_0)$ be the initial condition corresponding to the periodic solution $X_\infty(t)$. Denote by $\bar{X}(t) = X(t,t_0,\bar{X}_0)$ the solution of (24) originating from an arbitrary initial condition $\bar{X}_0 \neq X_0$. It can be easily verified that the explicit solution of (24) is given by

$$\begin{aligned} X(t) = {} & \Phi_A(t,t_0)X(t_0)\Phi_S(t_0,t) \\ & + \int_{t_0}^{t} \Phi_A(t,\tau)P(\tau)\Phi_S(\tau,t)\,\mathrm{d}\tau. \end{aligned}$$

Therefore, the difference $\tilde{X}(t) \triangleq X_\infty(t) - \bar{X}(t)$ satisfies

$$\tilde{X}(t) = \Phi_A(t,t_0)\tilde{X}(t_0)\Phi_S(t_0,t), \quad t \geq t_0$$

and thus $\lim_{t\to\infty}\tilde{X}(t) = 0$ no matter what $\bar{X}_0$ is. Since all trajectories of (24) are attracted to $X_\infty(t)$ in forward time, no periodic solution other than $X_\infty(t)$ may exist. ■

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