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Computation of the Excitability Index for Linear Oscillators

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***Abstract*— The problem of numerical evaluation of the excitability index for oscillatory systems is considered. It is shown that the speed-gradient excitation provides exact solution to the maximum energy problem for a second order linear oscillator over the infinite time interval. Upper and lower bounds for the total energy of the system in the steady-state oscillation mode and the excitability index are estimated. Exact value of the accessible system energy for the harmonic excitation case is found.**

I. INTRODUCTION

The problem of oscillation excitation via a bounded external force has various scientific and engineering applications. This problem can be treated as achieving maximum system energy, which is reachable in the case of bounded input signal. In order to measure excitablility of the system oscillations, the so-called excitability index (EI) has been recently introduced by A. Fradkov [1], [2], and the techniques of the excitability analysis, based on the EI computation, have been also developed [3], [4].

Computation of the EI is based on solving the optimal control problem, which is quite intricate in general. Simplification the EI computation can be achieved in the view of the fact, that the speed-gradient (SG) control laws provide a locally-optimal control, and the SG solution approaches the optimal one for the control action of small magnitude [5].

In spite of the fact that the excitability analysis is aimed at the study of nonlinear systems, examination of linear systems can also provide some useful information concerning the excitation problem. In the present paper the excitation of a linear oscillator via bounded control action over the infinite time interval is considered. It is proved that the SG method gives the optimal solution to the problem of maximization the total energy of the system on the infinite time interval. Therefore, in the considered case the SG method makes possible exact evaluation of the EI.

The paper is organized as follows. The brief introduction to the excitability analysis is presented in Sec. II. The SG method for energy-based control design and its implementation for excitation the linear oscillator are presented in Sec. 3. The optimal control strategy and its comparison with the SG-control are given in Sec. 4. In Sec. 5 the upper and lower limits of the total energy and the EI are estimated. Section 5 is devoted to the case of the harmonic excitation signal.

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II. EXCITABILITY ANALYSIS FOR NONLINEAR SYSTEMS

Excitability index (EI) was introduced in [1] as a measure of resonant properties of nonlinear systems as follows. Consider a system described by state-space equations

$$\dot{x} = F(x,u), \quad y = h(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, u, y are the scalar input and output, respectively. In order to create resonance mode in the nonlinear system (1) (to find small input force leading to significant changes in system output behavior), it was suggested in [1], [2], [4], to consider the following optimal control problems solution

$$Q^+(\gamma) = \limsup_{\substack{t\to\infty \\ t\geq 0}} \sup_{\substack{|u(s)|\leq\gamma, \\ 0\leq s\leq t, \\ x(0)=0}} |y(t)| \tag{2}$$

$$Q^-(\gamma) = \liminf_{\substack{t\to\infty \\ t\geq 0}} \sup_{\substack{|u(s)|\leq\gamma, \\ 0\leq s\leq t, \\ x(0)=0}} |y(t)| \tag{3}$$

If the system (1) is BIBO stable (bounded input implies bounded output) and $x = 0$ is the equilibrium of the unforced system then $Q^{\pm}(\gamma)$ are well defined. Apparently, the signal providing maximum excitation should depend not only on time but also on the system state, i.e. input signal should have a feedback form.

Following [4], let us consider the upper and lower excitability indices (EI) $E^+(\gamma)$, $E^-(\gamma)$ as

$$E^+(\gamma) = \frac{1}{\gamma}Q^+(\gamma), \tag{4}$$

$$E^-(\gamma) = \frac{1}{\gamma}Q^-(\gamma), \tag{5}$$

For some cases it is valid that $Q^+(\gamma) = Q^-(\gamma) = Q(\gamma)$ and, therefore, $E^+(\gamma) = E^-(\gamma) = E(\gamma)$. Obviously, $E^+(\gamma)$ coincides with input-output L_∞ gain [6] of the system (1).

For nonlinear systems $E^{\pm}(\gamma)$ are functions of γ that characterize excitability (resonance) properties of the nonlinear system. EI resembles a family of system gains for different levels of input. However, unlike the case of standard gain, its lower bounds are equally or even more important for measuring resonance ability of the system than the upper bounds.

The solutions to the problems (2), (3) are quite complicated in most cases. It was shown in [2] that approximate locally optimal speed-gradient (SG) solution can be used.

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The SG solution for the case when output y is the system energy is described below.

III. Speed-Gradient Method and Energy Control Problem

Let the control goal for the system (1) be expressed as the limit relation

$$y(t) \to \infty \quad \text{when} \quad t \to \infty, \tag{6}$$

In order to achieve the goal (6), the following SG-algorithm in the finite form may be applied [2], [7]:

$$u = -\Psi\big(\nabla_u \dot{h}(x,u)\big), \tag{7}$$

where $\dot{h} = \big(\partial h/\partial t\big) F(x,u)$ is the speed of changing $\big(h(x(t)\big)$ along the trajectories of (1), vector $\Psi(z)$ forms an acute angle with the vector z, i.e. $\Psi(z)^T z > 0$ when $z \neq 0$. The first step of the speed-gradient procedure is to calculate the speed $\dot{h}$. The second step is to evaluate the gradient $\nabla_u \dot{h}(x,u)$ with respect to controlling input u. Finally the vector-function $\Psi(z)$ should be chosen to meet the acute angle condition. The choice $\Psi(z) = \gamma z$, $\gamma > 0$ yields the *proportional* (with respect to speed-gradient) feedback

$$u = -\gamma\big(\nabla_u \dot{h}(x,u)\big), \tag{8}$$

while the choice $\Psi(z) = \gamma \operatorname{sign} z$ yields the *relay* algorithm

$$u = -\gamma \operatorname{sign}\big(\nabla_u \dot{h}(x,u)\big), \tag{9}$$

The underlying idea of the choice Eqs. (8), (9) is that moving along the antigradient of the speed $\dot{h}$ provides decrease of $\dot{h}$. It may eventually lead to negativity of $\dot{h}$ which, in turn, yields decrease of h and, eventually, achievement of the primary goal (6). However, to prove (6) some additional assumptions are needed, see [4], [7].

Consider now the problem of excitation for the second order linear oscillator. The plant (1) equation in this case can be written in the state-space form as

$$\dot{x} = Ax + Bu, \tag{10}$$

where $x = [\varphi, \dot{\varphi}]^T$ is the plant state vector, composed from the oscillator output φ and its time derivative, u denotes the control action. The matrices A and B are written in the canonical form of phase variable as

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\rho \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

where ω_0 stands for the natural frequency of the oscillator, $\rho \geq 0$ is the damping coefficient. The total energy $H(x)$ of the system (10), (11) can be written as the following quadratic form

$$H(x) = \frac{1}{2} x^T G x, \tag{12}$$

where the matrix G is a diagonal one, $G = \operatorname{diag}\{\omega_0^2, 1\}$. Even for this seemingly simple case the exact solution to the optimal control problem (2), at the best author's knowledge was not presented in the literature before. The reason is in that the L_∞ maximization problem (2) differs from conventional minimum norm problems and it is significantly more difficult [9]. Below we provide a solution to the problem for the second order linear oscillator.

Taking the plant output $y(t)$ in the form $y = h(x) = 0.5\big(H^* - H(x)\big)^2$, where H^* is its prescribed value, one obtains from (8), (9) the following *SG energy-control laws*:

$$u = -\gamma x_2 \big(H^* - H(x)\big), \tag{13}$$

$$u = -\gamma \operatorname{sign}(x_2) \operatorname{sign}\big(H^* - H(x)\big). \tag{14}$$

Choosing sufficiently large prescribed energy H^* in (14), [1] one gets that $\operatorname{sign}\big(H^* - H(t)\big) = \text{const}_t = 1$ and obtains the following *SG-excitation* algorithm:

$$u = \gamma \operatorname{sign}(x_2) \tag{15}$$

Let us find now the explicit solution of Eqs. (10), (15). It can be easily shown that if the system (10) is dissipative (i.e. $\rho > 0$), then the stable limit cycle appears. To find its parameters (the frequency and the amplitude), the *pointwise mapping* (Poincaré map) method can be applied [10]. Taking into account that $u = +\gamma$ as far as $x_2 > 0$ and $u = -\gamma$ for $x_2 < 0$, one gets the phase trajectories shown in Fig. 1. The point x_∞ corresponds to the steady-state solution of (10). That solution may be obtained by assumption that $u(t) \equiv -\gamma$. The curve $\mathscr{G}$ in Fig. 1 represents the limit cycle. Examination of

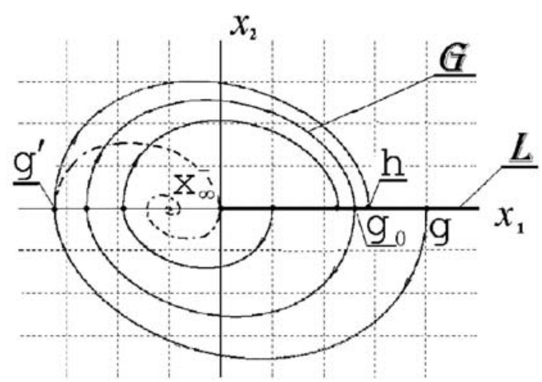

Fig. 1. Phase plot

Eq. (10) solutions shows that the time intervals T between two successive intersections of the phase trajectory and the positive semi-axis $\mathscr{L}$ can be found as $T = \dfrac{2\pi}{\beta}$, where $\beta = \dfrac{1}{2}\sqrt{4\omega_0^2 - \rho^2}$ is the natural oscillations frequency for the plant (10). Therefore the SG control action (15), as a function on t, is a *square waveform* of the period T and some phase shift ψ:

$$u(t) = \gamma \operatorname{sign}\big(\sin(\beta t + \psi)\big) \tag{16}$$

The oscillations amplitude may be found from the Poincaré map, plotted in Fig. 2.

[1] If the plant (10) is BIBO stable (i.e. $\rho > 0$) and the control action $u(t)$ is bounded, then $\|x(t)\|$ is bounded as well [8]. The total system energy (12) depends continuously on the system state $x(t)$, therefore there exists appropriate $H^* < \infty$.

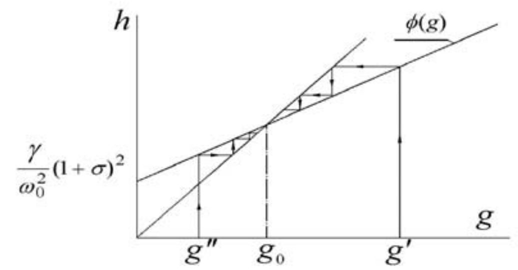

Fig. 2. Poincaré map

The *fixed point* g_0 of the Poincaré map yields the limit cycle amplitude (see Fig. 1). To find g_0 let us solve the equation $\phi(g_0) = g_0$, where $\phi(g)$ is the following Poincaré function:

$$\phi(g) = \gamma\omega_0^{-2}(1+\sigma)^2 + \sigma^2 g. \tag{17}$$

In Eq. (17) parameter σ denotes the *overshoot* of the oscillator (10) step response. It can be found as $\sigma = \exp\left(\frac{\alpha}{\beta}\pi\right)$, where $\alpha = -\frac{\rho}{2}$ is the real part of plant model (10) eigenvalues. Note, that $0 < \sigma < 1$ as $\rho > 0$. Finally, one obtains the limit cycle magnitude g_0 as

$$g_0 = \frac{\gamma}{\omega_0^2}\frac{1+\sigma}{1-\sigma}. \tag{18}$$

The magnitude g_0 can be easily found through the plant model (10) and the control action (15) parameters. It is known [5], that the control law (15) is locally-optimal in the sense of maximum of the system (1) output magnitude. As it is shown in the next Section, the law (15) is also globally optimal for the system (10), (11).

IV. Variational Solution for Excitation Problem

Let us apply the *maximum principle* to solve the posed problem of an energy-optimal oscillator excitation. Introduce the *performance index* as

$$J(u) = H\big(x(t_f)\big) = \frac{1}{2}x(t_f)^T G x(t_f),$$

where t_f stands for the end point, $t \in [0, t_f]$, $x(0) = 0$. Consider the *free end* optimization problem. The phase variables $x(t)$ and the control action $u(t)$ are constrained via the differential relation (10). Let the control action $u(t)$ be bounded, $|u(t)| \le \gamma$, where $\gamma > 0$ is a given constant. The quality index $J(u)$ does not depend explicitly on the system trajectory inside the optimization interval, therefore the problem under consideration is a *terminal* one. Let us now introduce the *Hamiltonian* $\mathscr{H}(x,u,\lambda)$ as

$$\mathscr{H}(x,u,\lambda) = \lambda(t)^T\big(Ax(t) + Bu(t)\big),$$

where $\lambda(t) \in \mathbb{R}^2$ is the vector of *dual variables*. According to the maximum principle, the optimal control action $u^*(t)$ must satisfy the following conditions:

– extremality on x

$$\dot\lambda(t)^T = -\frac{\partial\mathscr{H}(x,u,\lambda)}{\partial x}\bigg|_{u=u^*}, \tag{19}$$

– the maximum principle

$$u^*(t) = \max_{|u(t)|\le\gamma} \mathscr{H}(x,u,\lambda), \tag{20}$$

– the transversality condition $\lambda(t_f) = -Gx(t_f)$.

As it follows from (19), the dual variables satisfy the equation $\dot\lambda(t) = -A^T\lambda(t)$, or, componentwise,

$$\begin{cases} \dot\lambda_1(t) = \omega_0^2\lambda_2(t), \\ \dot\lambda_2(t) = -\lambda_1(t) + \rho\lambda_2(t). \end{cases} \tag{21}$$

Condition (20) gives the optimal control signal $u^*(t)$ as

$$u^*(t) = \gamma\,\mathrm{sign}\,\lambda_2(t). \tag{22}$$

The pair-wise boundary-value problem (10), (20)–(22) can not be solved explicitly, hence to find the optimal control function $u^*(t)$, let us apply some additional arguments.

From Eq. (21) it follows that the dual variable $\lambda_2(t)$ has a form

$$\lambda_2(t) = C\exp(-\alpha t)\sin(\beta t + \psi), \tag{23}$$

where constants C, ψ depend on boundary conditions, α and β are the real and the imaginary parts of plant model (10) eigenvalues. Substitution (23) into (22) leads to the following expression for the optimal control action:

$$u^*(t) = \gamma\,\mathrm{sign}\big(\sin(\beta t + \psi)\big). \tag{24}$$

It is worth to notice that the value of ψ is not important for the posed problem, because the oscillation parameters in the steady-state mode (as $t \to \infty$) do not depend on ψ. Thus, the optimal solution of the excitation problem is not unique.

Comparison of the formulas (16) and (24) shows that the SG control law (15) an optimal one for any $\gamma > 0$.

V. Evaluation of the Total Energy and the Excitability Index

The total energy (12) of the system can be represented as a sum of two terms: the *potential energy* $\Pi(x_1) = \omega_0^2 x_1^2/2$ and the *kinetic energy* $K(x_2) = x_2^2/2$. The supremum $\overline{\Pi}$ of the potential energy can be directly found from Eq. (18). By substitution, one obtains that

$$\overline{\Pi} = \frac{\gamma^2\big(1+e^{\pi/\mu}\big)^2}{2\omega_0^2\big(1-e^{\pi/\mu}\big)^2} \tag{25}$$

where $\mu = \beta/\alpha = -\sqrt{4\omega_0^2/\rho^2 - 1}$.

Examination of the first and the second time derivatives of $H_t = H\big(x(t)\big)$ along the system (10), (16) trajectories, shows that, asymptotically, H_t reaches its maximal value $\overline{H}$ as $x_2(t) = \bar x_2$, where $\bar x_2 = -\gamma\rho^{-1}\,\mathrm{sign}\,x_1$. Therefore, $\overline{\Pi}$ and $\overline{H}$ are different, but $\overline{\Pi}$ exceeds the lower bound of H_t in the limit cycle and, therefore, can be used as a lower estimate of $\bar H$. To get the upper estimate of $\bar H$, consider the arc of the limit cycle between the point $\big(\bar x_1, 0\big)$, $\bar x_1 = g_0$, and the point of extremum $\big(x_1(\bar{t}),\bar{x}_2\big)$, where $H\big(x(\bar{t})\big)=\overline{H}$. Since the solution for $\bar{t}$ can not be found explicitly, then let us assume that $\rho=0$ in (10). Some computations lead to the following estimate:

$$\overline{\Pi}<\overline{H}<=\frac{2\gamma^2}{2\omega_0^2\big(1-e^{\pi/\mu}\big)^2}\approx\frac{8\gamma^2\beta^2}{\pi^2\rho^2\omega_0^2}. \qquad (26)$$

Substitution $\overline{H}$ for $Q(\gamma)$ in Eq. (1) gives the following estimate of the upper excitability index:

$$E^+(\gamma)=E^+<\frac{\sqrt{2}}{\omega_0^2\big(1-e^{\pi/\mu}\big)}\approx\frac{2\sqrt{2}\beta}{\pi\rho\omega_0}. \qquad (27)$$

The obtained estimate is more accurate than that obtained by A.L. Fradkov [11] via the *energy balance method*.

Let us consider now the case of the oscillator (10) excitation by means of the harmonic input signal.

VI. Harmonic Excitation

Let us find the excitability index assuming that the admissible input signals are the harmonic ones. Let $u(t)=\gamma\sin\omega t$, where γ, ω stand for the excitation magnitude and frequency, respectively. In such a case the steady-state solutions of (10) can be written as $x_1(t)=A(\omega)\sin(\omega t+\psi)$, $x_2(t)=-\omega A(\omega)\cos(\omega t+\psi)$, where $A(\omega)$ is the gain-frequency characteristic of the system (10), $\psi=\psi(\omega)$ is a phase-response characteristic of the system. For the sake of the present study the value of ψ is unimportant and hereafter it is assumed that $\psi=0$. Then in the steady-state mode, the total energy H_t of the system is as follows:

$$\begin{aligned}H_t&=\gamma^2A(\omega)^2\big(\omega^2\cos^2(\omega t)+\omega_0^2\sin^2(\omega t)\big)/2\\&=\gamma^2A(\omega)^2\big(\omega^2+(\omega_0^2-\omega^2)\sin^2(\omega t)\big)/2. \qquad (28)\end{aligned}$$

It is seen that H_t oscillates about some average value with the frequency 2ω as far as $\omega\neq\omega_0$. If $\omega=\omega_0$, then H_t is constant on t. Let us find the maximum $\bar{H}=\sup_t H_t$. The time derivative of H_t is $\dot{H}_t=\omega A(\omega)^2\cos(\omega t)\sin(\omega t)\big(\omega^2-\omega_0^2\big)$. Hence has following extreme points: $t_1=k\pi/\omega$, $t_2=(k\pi+\pi/2)/\omega$, where $k=0,1,2,\ldots$. The corresponding values of H_t in these points are following:

$$H_1(\omega)=\omega^2\gamma^2A(\omega)^2/2,\quad H_2(\omega)=\omega_0^2\gamma^2A(\omega)^2/2 \qquad (29)$$

These values depend on the forcing frequency ω. Let us find the value of ω that ensures maximization of the total energy of the system for all possible harmonic input waveforms. Recall that for the plant (10) it is valid that $A(\omega)^2=\big((\omega_0^2-\omega^2)^2+\rho^2\omega^2\big)^{-1}$. Differentiation $H_1(\omega)$, $H_2(\omega)$ with respect to ω yields

$$\begin{aligned}\frac{\partial H_1(\omega)}{\partial\omega}&=\gamma^2\omega\big(\omega_0^4-\omega^4\big)/R(\omega),\\\frac{\partial H_2(\omega)}{\partial\omega}&=\gamma^2\omega_0^2\omega\big(2(\omega_0^2-\omega^2)-\rho^2\big)/R(\omega),\end{aligned}$$

where $R(\omega)=\big((\omega_0^2-\omega^2)^2+\rho^2\omega^2\big)^2$. For the oscillator (10) it is valid that $4\omega_0^2>\rho^2$ and $R(\omega)\neq0$ for all ω. Then there are following extreme points of $H_1(\omega)$, $H_2(\omega)$, as the functions on ω:

$$\omega_1=\omega_0,\quad \omega_2=\sqrt{\omega_0^2-\rho^2/2}. \qquad (30)$$

By substitution ω in (29) for ω_1, ω_2 from (30) one gets the extreme values for $H_1(\omega)$, $H_2(\omega)$ as

$$\overline{H}_1=\frac{\gamma^2}{2\rho^2},\quad \overline{H}_2=\frac{2\omega_0^2\gamma^2}{\rho^2(4\omega_0^2-\rho^2)}. \qquad (31)$$

It can be easily verified that $\overline{H}_2>\overline{H}_1$ and therefore,

$$\overline{H}_1=\frac{2\omega_0^2\gamma^2}{\rho^2\big(4\omega_0^2-\rho^2\big)},\quad E=\frac{\sqrt{2}\omega_0\gamma^2}{\rho\sqrt{4\omega_0^2-\rho^2}}. \qquad (32)$$

Note, that ω_2 in (30) can not be found if $\rho>\omega_0\sqrt{2}$. In that case the total energy is a monotone non-increasing function of ω.

VII. Conclusions

It is shown that the SG excitation provides exact solution to the maximum energy problem (2) for a second order linear oscillator. The L_∞ maximization problem (2) differs from conventional minimum norm problems and it is significantly more difficult [9]. Upper and lower bounds of the total energy of the system and the excitability index are estimated. Exact value of the accessible system energy for the harmonic excitation case is found.

VIII. ACKNOWLEDGMENTS

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