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Hybrid Static Output Feedback Stabilization of Two-Dimensional LTI Systems: A Geometric Method

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***Abstract*— For two-dimensional linear time-invariant (LTI) systems which are not stabilizable via a single static output feedback, we propose a hybrid stabilization strategy based on a geometric method. More precisely, we design two static output feedback gains and a switching law between the feedback gains so that the entire closed-loop system is asymptotically stable. The proposed switching law is composed of output-dependent switching and time-controlled switching. We demonstrate the design method with various examples, and show that in some cases the stabilizability depends on the region of the initial state, while in other cases the system is globally stabilizable.**

***Index Terms*—Two-dimensional LTI system, static output feedback, hybrid stabilization, asymptotic line, switching line, output-dependent switching, time-controlled switching.**

I. INTRODUCTION

Consider the linear time-invariant (LTI) control system described by equations of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu(t) \\ y(t) &=& Cx(t) \,, \end{array} \right. \tag{1.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control input, $y \in \Re^p$ is the measurement output, and A, B, C are constant matrices of suitable dimensions. We assume that the triple (A, B, C) is controllable and observable. The stabilization problem of the system (1.1) via a single static output feedback has been studied exhaustively; see the survey paper [1] and the references cited therein. However, when the system (1.1) is not stabilizable via a single static output feedback, it is necessary to consider hybrid stabilization method, where a family of static output feedbacks should be included. In the following, we present a motivation example, which was also discussed in [2], [3], [4].

Motivation Example. Consider the harmonic oscillator model with position measurement described by

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \qquad y = [1 \ \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \end{array} \right. \tag{1.2}$$

Although the above system is both controllable and observable, it cannot be stabilized by a single static output feedback [2]; however, it is stabilizable by a hybrid static output feedback [2], [3]. By letting $u = -y$ and $u = \frac{1}{2}y$, we obtain the following two systems, respectively,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{1.4}$$

Define $V(x) \stackrel{\triangle}{=} x_1^2 + x_2^2$. If the system (1.3) is active in the first and third quadrants, while the system (1.4) is active in the second and fourth quadrants, we will have $\dot{V} < 0$ whenever $x_1 x_2 \neq 0$, which implies that the entire switched system is asymptotically (and hence, for linear systems, exponentially) stable by LaSalle's Principle (e.g., [5]). □

We observe from the above example that when the system (1.1) is not stabilizable by a single static output feedback, it is possible to find a hybrid static output feedback, which is composed of a family of static output feedback controllers and a switching strategy determining which controller should be activated at every instant. There are several existing results concerning such hybrid static output feedback stabilization problem. In [6], it has been shown that if the system (1.1) is controllable and observable, then it admits a stabilizing hybrid output feedback that uses a countable number of discrete states. In [2] as well as [3], the question is proposed whether it is possible to stabilize the system (1.1) by a hybrid static output feedback with a finite number of discrete states. Several specific examples that are included in [2] suggest that the answer to this question may be affirmative, and a sufficient condition based on multiple Lyapunov functions is derived in [3].

In [7], the control problem is considered for two-dimensional LTI systems which are described by the controllable canonical form

$$A = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} , \ \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \ \ C = [1 \ \ 0] \,, \tag{1.5}$$

where $a, b \in \Re$, and $a \geq 0$. It is easy to confirm that although the system (1.1) with (1.5) is both controllable and observable, we can not stabilize this system by a single static output feedback. For this case, a complete solution is presented in [7] for the hybrid static output feedback stabilization problem, which is composed of two important contributions. The first one is to prove that the hybrid control problem for the system (1.1) with (1.5) is solvable, and the second one is to design such a hybrid static output feedback with the number of necessary static output feedbacks being two. That is, we can always construct a 2-state static output feedback to asymptotically stabilize the system (1.1) with (1.5). It is noted that the approach in [7] is based on the so-called conic switching law [8].

We observe that although A and B in (1.5) are general in controllable canonical form, C is not in general form. In fact, we can find other cases of C where the system is not stabilizable by a single static output feedback (for example, the case of $C = [0 \ \ 1]$). Based on this observation, in this paper, we extend our attention to the general case of

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the output matrix C by considering the coefficient matrices described as

$$A = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [c \;\; d], \qquad (1.6)$$

where $a, b, c, d \in \Re$. To ensure that the hybrid static output feedback stabilization problem is meaningful, we need to impose some conditions on the values of these scalars. This will be done precisely in the next section.

For the system (1.1) with (1.6), we propose a hybrid stabilization strategy which is different from the conic switching law based one in [7]. We first characterize the divergence patterns of the closed-loop system when a single static output feedback gain is used, by dividing them into three patterns. If the initial state is in the fourth (or second) quadrant, we design two feedback gains so that two asymptotic lines are obtained in the fourth (or second) quadrant, choose two switching lines inside the area surrounded by the asymptotic lines, and then switch between the two feedback gains using the switching lines. If the initial state is not in the fourth (or second) quadrant, we design one of the two feedback gains so that the system state will enter the fourth (or second) quadrant when activating the feedback gain. As can be seen precisely later, the proposed switching law is composed of output-dependent switching and time-controlled switching. Since the initial state is not available in static output feedback stabilization problem, we use the output information to measure the time instant when the system state reaches the line $y = 0$. After that, we use a time-controlled switching. i.e., activate the two feedback gains alternately with specified time intervals.

The rest of this paper is organized as follows. In Section II, we describe and categorize the control system (1.1) with (1.6), so that the hybrid stabilization problem is meaningful. In Section III, we give precise descriptions on the hybrid stabilization method with necessary proof. The main idea is to characterize the divergence patterns of the closed-loop system when a single static output feedback gain is used. Section IV gives various simulation examples to illustrate the hybrid stabilization method. Finally, we make some concluding remarks in Section V.

II. Problem Description

Since we deal with hybrid stabilization of the systems which are not stabilizable by a single static output feedback, we need to impose some conditions on the values of a, b, c, d in (1.6). First, the closed-loop system composed of the system (1.1) with (1.6) and any static output feedback $u = ky$ is

$$\dot{x} = (A + kBC)x = \begin{bmatrix} 0 & 1 \\ b + kc & a + kd \end{bmatrix} x. \qquad (2.1)$$

In the sequel, we categorize the system under consideration precisely.

1) Case of $cd = 0$: In this case, we further consider the following three subcases.

1–1) $c = d = 0$: It is a trivial case since $C = 0$.

1–2) $c = 0, d \neq 0$: Without loss of generality, assume $c = 0, d = 1$. The characteristic equation of the closed-loop system (2.1) is $s^2 - (a + k)s - b = 0$, and thus the system is stable if and only if

$$a + k < 0, \;\; b < 0. \qquad (2.2)$$

Since k is the feedback gain we can choose, we assume $b \geq 0$ so that the condition (2.2) can not be satisfied with any k.

1–3) $c \neq 0, d = 0$: Without loss of generality, assume $c = 1, d = 0$. Although this case has been considered in [7], we do not exclude it for integrity. The characteristic equation of the closed-loop system (2.1) is $s^2 - as - (b + k) = 0$, and thus the system is stable if and only if

$$a < 0, \;\; b + k < 0. \qquad (2.3)$$

Then, we assume $a \geq 0$ so that the condition (2.3) can not be satisfied with any k.

2) Case of $cd > 0$: The characteristic equation of the closed-loop system (2.1) is $s^2 - (a+kd)s - (b+kc) = 0$, and thus the system is stable if and only if

$$a + kd < 0, \;\; b + kc < 0. \qquad (2.4)$$

It is easy to confirm that since $cd > 0$, we can always find a scalar k such that (2.4) is satisfied. More precisely, when $c > 0, \; d > 0$ choose $k = -M$; when $c < 0, \; d < 0$ choose $k = M$, where M is a sufficiently large scalar. Therefore, the system (1.1) with (1.6) can be stabilized by a single static output feedback, and thus this case is not considered in this paper.

3) Case of $cd < 0$: Without loss of generality, assume $cd = -1$. In this case, we further consider the following two subcases (other subcases can be reduced to these ones).

3–1) $c = 1, d = -1$: The characteristic equation of the closed-loop system (2.1) is

$$s^2 - (a - k)s - (b + k) = 0, \qquad (2.5)$$

and thus the system is stable if and only if

$$a < k < -b. \qquad (2.6)$$

For the same reason as before, we assume $a \geq -b$.

3–2) $c = -1, d = 1$: The characteristic equation of the closed-loop system (2.1) is

$$s^2 - (a + k)s - (b - k) = 0, \qquad (2.7)$$

and thus the system is stable if and only if

$$b < k < -a. \qquad (2.8)$$

For the same reason as before, we assume $b \geq -a$. However, the above two subcases can be dealt with in the same setting by replacing k with $-k$ in the conditions. Thus, we only consider the first subcase.

To summarize, we deal with the following three cases in this paper.

Case A $\quad c = 0, \;\; d = 1, \;\; b \geq 0;$
Case B $\quad c = 1, \;\; d = 0, \;\; a \geq 0;$
Case C $\quad c = 1, \;\; d = -1, \;\; a \geq -b.$

III. Hybrid Stabilization Method

In this section, we present a geometric method for hybrid static output feedback stabilization of the system (1.1) with (1.6) in the cases we summarized in the previous section. We first analyze the divergence patterns of the state trajectory when a single feedback gain is used, and then describe our hybrid stabilization method precisely with part of stability proof.

A. Divergence patterns of state trajectory with single gain

We fix the feedback gain k in the closed-loop system (2.1) to investigate how the state trajectory diverges.

(i) When the characteristic equation of the closed-loop system has complex solutions $\lambda \pm \mu i$ ($\lambda > 0$, $\mu > 0$), the state is computed as

$$\begin{cases} x_1(t) \;=\; C_1 e^{\lambda t} \sin(\mu t + C_2) \\ x_2(t) \;=\; C_3 e^{\lambda t} \sin(\mu t + C_4) \end{cases} \tag{3.1}$$

where $C_1,\ C_2,\ C_3,\ C_4$ are real constants, and the state trajectory diverges in the clockwise direction around the origin. Fig. 1 shows this case where the closed-loop system matrix is $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ and the initial state is $[10 \ \ 10]^T$.

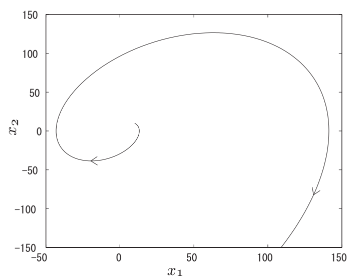

Fig. 1. Divergence pattern (i)

(ii) When the characteristic equation of the closed-loop system has real solutions $-\alpha$, β ($\alpha > 0$, $\beta > 0$), the state is computed as

$$\begin{cases} x_1(t) \;=\; C_1 e^{-\alpha t} + C_2 e^{\beta t} \\ x_2(t) \;=\; -\alpha C_1 e^{-\alpha t} + \beta C_2 e^{\beta t} \end{cases} \tag{3.2}$$

where C_1, C_2 are constants, and the state trajectory diverges in the counterclockwise direction. Further, we obtain

$$\lim_{t \to -\infty} \frac{dx_2}{dx_1} = -\alpha, \qquad \lim_{t \to \infty} \frac{dx_2}{dx_1} = \beta \tag{3.3}$$

which imply that two asymptotic lines $x_2 = -\alpha x_1$ and $x_2 = \beta x_1$ exist. Fig. 2 shows the divergence in this case with the closed-loop system matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and various initial states.

(iii) When the characteristic equation of the closed-loop system has real solutions α, β ($0 < \alpha < \beta$), the state is computed as

$$\begin{cases} x_1(t) \;=\; C_1 e^{\alpha t} + C_2 e^{\beta t} \\ x_2(t) \;=\; \alpha C_1 e^{\alpha t} + \beta C_2 e^{\beta t} \end{cases} \tag{3.4}$$

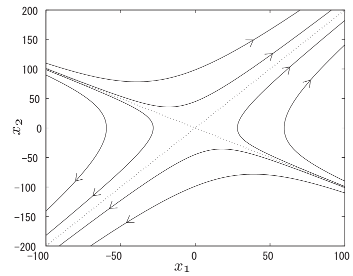

Fig. 2. Divergence pattern (ii)

where C_1, C_2 are constants, and the state trajectory diverges in the counterclockwise direction. Similarly, we obtain

$$\lim_{t \to -\infty} \frac{dx_2}{dx_1} = \alpha\,, \qquad \lim_{t \to \infty} \frac{dx_2}{dx_1} = \beta \tag{3.5}$$

which define two asymptotic lines. Fig. 3 shows the divergence in this case with the closed-loop system matrix $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and various initial states.

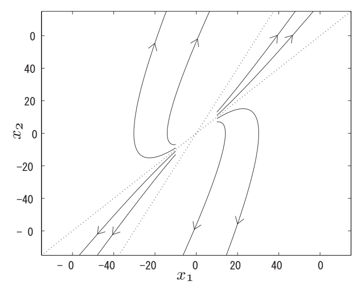

Fig. 3. Divergence pattern (iii)

B. Reachable divergence patterns in Cases A, B and C

Since the main idea in this paper is to connect the divergent state trajectories, obtained in different patterns, so as to get an entirely convergent trajectory, we here discuss which divergence pattern can be reached in the cases we summarized in the end of Section II.

1) Case A: In this case, $c = 0\,,\ d = 1\,,\ b \geq 0$. To study the solution of the characteristic equation $s^2 - (a+k)s - b = 0$, we consider the function $f(x) = x^2 - (a+k)x - b$. Since $f(0) = -b \leq 0$, we see that the characteristic equation has a positive solution and a negative one (when $b = 0$, the solutions are $s = 0,\ a+k$). Thus, no matter how we choose the feedback gain, we can only reach the divergence pattern (ii).

2) Case B: In this case ($c = 1, \ \ d = 0, \ \ a \geq 0$), we consider the function $f(x) = x^2 - ax - (b+k)$. The necessary and sufficient condition for $f(x) = 0$ to have complex solutions is

$$(-a)^2 + 4(b+k) < 0 \Longleftrightarrow k < -\frac{a^2}{4} - b\,. \tag{3.6}$$

Thus, if we choose k such that (3.6) is satisfied, the divergence pattern (i) is obtained. If we choose $k > -b$, then the divergence pattern (ii) is obtained, and if we choose $-\frac{a^2}{4} - b \leq k < -b$ (when $a > 0$), the divergence pattern (iii) is obtained. Thus, in Case B, it is possible to reach all the three divergence patterns.

3) Case C: In this case ($c = 1\,,\ \ d = -1\,,\ \ a \geq -b$), we consider the function $f(x) = x^2 - (a-k)x - (b+k)$. The necessary and sufficient condition for $f(x) = 0$ to have complex solutions is

$$(a-k)^2 + 4(b+k) < 0 \tag{3.7}$$

or equivalently,

$$(k - (a-2))^2 + 4(a+b-1) < 0\,. \tag{3.8}$$

Thus, when $0 \leq a+b < 1$, we can choose k such that (3.8) is satisfied and thus obtain the divergence pattern (i). Using similar discussion as in Case B, it is not difficult to show that whether $a+b < 1$ is satisfied or not, the divergence patterns (ii) and (iii) can always be reached.

To summarize, the reachable divergence patterns in Cases A, B and C are:

Case A: $c = 0\,,\ \ d = 1\,,\ \ b \geq 0$
$\Longrightarrow$ Divergence pattern (ii);
Case B: $c = 1\,,\ \ d = 0\,,\ \ a \geq 0$
$\Longrightarrow$ Divergence patterns (i), (ii) and (iii);
Case C(1): $c = 1\,,\ \ d = -1\,,\ \ 0 \leq a+b < 1$
$\Longrightarrow$ Divergence patterns (i), (ii) and (iii);
Case C(2): $c = 1\,,\ \ d = -1\,,\ \ 1 \leq a+b$
$\Longrightarrow$ Divergence patterns (ii) and (iii).

C. Hybrid stabilization method

The hybrid stabilization method in this paper is motivated by two important observations. The first one is that no matter which pattern of (i), (ii) and (iii) is chosen, if starting in the fourth (or second) quadrant, the value of x_1 decreases and moves towards the origin. The second observation is that in the divergence patterns (ii) and (iii), there are two asymptotic lines and the state diverges to $+\infty$ or $-\infty$ along the asymptotic lines.

Based on these observations, we propose the following hybrid stabilizing method (Fig. 4 gives the illustration) :

- Design two asymptotic lines in the fourth (or second) quadrant by choosing two approapriate gains.
- Set two switching lines inside the conic area which the two asymptotic lines surround in the fourth (or second) quadrant.
- Activate the appropriate feedback gain to drive the initial state into the area surrounded by the two switching lines in the fourth (or second) quadrant.
- Switch to another gain if the system state intersects one of the switching lines, and repeat the procedure.

In the above procedure, we used the system state to describe the switching method. However, since static output feedback stabilization is considered in this paper, state information is not available directly. The initial state is not known either. For this reason, we here propose an output-dependent switching together with a time-controlled switching method. From any starting point, we first activate one static output feedback so that the state trajectory intersects

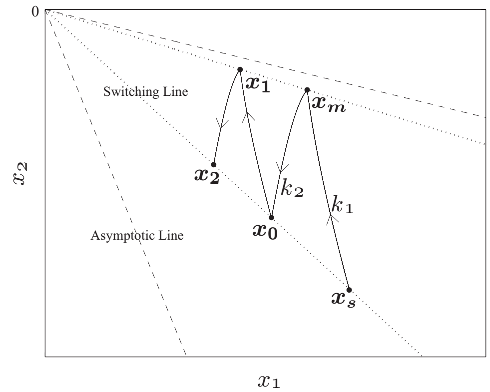

Fig. 4. Illustration of the switching method

$y = Cx = 0$. This is possible since y is measurable. Then, we continue the same feedback gain until the trajectory hits the first switching line. The switching at this stage is output-dependent. Hereafter, we use the procedure in the previous paragragh to proceed. The switching then is indeed time-controlled since we can compute exactly the activation time between the two switching lines using their slopes for the corresponding feedback gains.

Let us use Fig. 4 again to describe the idea more precisely. We choose two asymptotic lines (the hyphenated lines) $x_2 = -\alpha_1 x_1,\ x_2 = -\alpha_2 x_1$ ($\alpha_1 > \alpha_2 > 0$), and then choose two switching lines (the dotted lines) $x_2 = -\alpha_1' x_1,\ x_2 = -\alpha_2' x_1$ inside the area surrounded by the two asymptotic lines, where $\alpha_1 > \alpha_1' > \alpha_2' > \alpha_2 > 0$. Suppose that we have used the output-dependent switching method to drive the initial state to hit the first switching line on $\boldsymbol{x_s}$. Now, we activate the first feedback gain k_1 (suppose the corresponding closed-loop system matrix is A_{k_1}) until it reaches another switching line on $\boldsymbol{x_m}$. Note that the time interval from $\boldsymbol{x_s}$ to $\boldsymbol{x_m}$ is determined by the slopes of the switching lines, not dependent on $\boldsymbol{x_s}$ or $\boldsymbol{x_m}$. From then on, we activate the feedback gain k_2 to drive $\boldsymbol{x_m}$ to $\boldsymbol{x_0}$, and then activate the feedback gain k_1 to drive $\boldsymbol{x_0}$ to $\boldsymbol{x_1}$, and so on. As can be seen later, the time interval for the system trajectory to go from one switching line to another can be computed exactly. Thus, we use the time intervals (NOT the system state) to determine when to switch to another feedback gain. To summarize, the switching method before the system state reaches $y = Cx = 0$ is output-dependent, and the switching method after that is time-controlled.

Now, we discuss the convergence of the system state. Assume that the points' coordinates in Fig. 4 are $\boldsymbol{x_0}(x_{10}, -\alpha_1' x_{10})$, $\boldsymbol{x_1}(x_{11}, -\alpha_2' x_{11})$ and $\boldsymbol{x_2}(x_{12}, -\alpha_1' x_{12})$. If we can show $|\boldsymbol{x_2}| < |\boldsymbol{x_0}|$, then repeating the procedure leads to exponential convergence of the system state.

First, let us see the trajectory $\boldsymbol{x_0}(x_{10}, -\alpha_1' x_{10}) \rightarrow \boldsymbol{x_1}(x_{11}, -\alpha_2' x_{11})$. Without loss of generality, we set the time instant at $\boldsymbol{x_0}$ as $t = 0$. Since it corresponds to the asymptotic line $x_2 = -\alpha_1 x_1$, we obtain

$$\begin{cases} x_1(t) = C_1 e^{-\alpha_1 t} + C_2 e^{\beta_1 t} \\ x_2(t) = -\alpha_1 C_1 e^{-\alpha_1 t} + \beta_1 C_2 e^{\beta_1 t}\,. \end{cases} \tag{3.9}$$

Since the initial state is $\boldsymbol{x_0}(x_{10}, -\alpha_1' x_{10})$,

$$\begin{cases} x_1(0) = C_1 + C_2 = x_{10} \\ x_2(0) = -\alpha_1 C_1 + \beta_1 C_2 = -\alpha_1' x_{10} \end{cases} \tag{3.10}$$

is true. Therefore,

$$C_1 = \frac{\alpha_1' + \beta_1}{\alpha_1 + \beta_1} x_{10}, \qquad C_2 = \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \beta_1} x_{10}\,. \tag{3.11}$$

Denote by t_1 the time interval for the system trajectory to go from $\boldsymbol{x_0}$ to $\boldsymbol{x_1}$, which requires $x_2(t_1) = -\alpha_2' x_1(t_1)$. Substituting these equations into (3.9) leads to

$$\begin{aligned} -\alpha_1 C_1 e^{-\alpha_1 t_1} + \beta_1 C_2 e^{\beta_1 t_1} &= -\alpha_2' C_1 e^{-\alpha_1 t} - \alpha_2' C_2 e^{\beta_1 t} \\ \Longrightarrow e^{(\alpha_1+\beta_1)t_1} &= \frac{\alpha_1 - \alpha_2'}{\alpha_2' + \beta_1}\frac{C_1}{C_2} = \frac{\alpha_1 - \alpha_2'}{\alpha_2' + \beta_1}\frac{\alpha_1' + \beta_1}{\alpha_1 - \alpha_1'} \\ \Longrightarrow t_1 &= \frac{1}{\alpha_1 + \beta_1} \ln\left(\frac{\alpha_1 - \alpha_2'}{\alpha_2' + \beta_1}\frac{\alpha_1' + \beta_1}{\alpha_1 - \alpha_1'}\right). \end{aligned} \tag{3.12}$$

Then, using the obtained t_1, x_{11} is computed as

$$\begin{aligned} x_{11} &= x_1(t_1) = C_1 e^{-\alpha_1 t_1} + C_2 e^{\beta_1 t_1} \\ &= (C_1 + C_2 e^{(\alpha_1+\beta_1)t_1}) e^{-\alpha_1 t_1} \\ &= \left(\frac{\alpha_1' + \beta_1}{\alpha_2' + \beta_1}\right)^{\frac{\beta_1}{\alpha_1+\beta_1}} \left(\frac{\alpha_1 - \alpha_2'}{\alpha_1 - \alpha_1'}\right)^{\frac{-\alpha_1}{\alpha_1+\beta_1}} x_{10}\,. \end{aligned} \tag{3.13}$$

Similarly, the time interval t_2 for the trajectory to go from $\boldsymbol{x_1}(x_{11}, -\alpha_2' x_{11})$ to $\boldsymbol{x_2}(x_{12}, -\alpha_1' x_{12})$ is computed as

$$t_2 = \frac{1}{\alpha_2 + \beta_2} \ln\left(\frac{\alpha_2 - \alpha_1'}{\alpha_1' + \beta_2}\frac{\alpha_2' + \beta_2}{\alpha_2 - \alpha_2'}\right) \tag{3.14}$$

and x_{12} is

$$x_{12} = \left(\frac{\alpha_2' + \beta_2}{\alpha_1' + \beta_2}\right)^{\frac{\beta_2}{\alpha_2+\beta_2}} \left(\frac{\alpha_2 - \alpha_1'}{\alpha_2 - \alpha_2'}\right)^{\frac{-\alpha_2}{\alpha_2+\beta_2}} x_{11}\,. \tag{3.15}$$

Note that t_1 and t_2 are computed by using the system data and the slopes of the switching lines. This implies that the time interval for the system trajectory to go from one switching line to another is constant, and that the proposed hybrid stabilization method is practical.

To proceed, we need the following result.

Lemma. The following two inequalities hold.

$$\left(\frac{\alpha_1' + \beta_1}{\alpha_2' + \beta_1}\right)^{\frac{\beta_1}{\alpha_1+\beta_1}} \left(\frac{\alpha_1 - \alpha_2'}{\alpha_1 - \alpha_1'}\right)^{\frac{-\alpha_1}{\alpha_1+\beta_1}} < 1 \tag{3.16}$$

$$\left(\frac{\alpha_2' + \beta_2}{\alpha_1' + \beta_2}\right)^{\frac{\beta_2}{\alpha_2+\beta_2}} \left(\frac{\alpha_2 - \alpha_1'}{\alpha_2 - \alpha_2'}\right)^{\frac{-\alpha_2}{\alpha_2+\beta_2}} < 1 \tag{3.17}$$

Proof. See Appendix. □

According to this lemma and the above discussions, we obtain the following theorem.

Theorem. Under the proposed hybrid stabilization method, $|\boldsymbol{x_2}| < |\boldsymbol{x_0}|$, and thus the system is exponentially stable. □

IV. Simulation Examples

In this section, we present two simulation examples to demonstrate the hybrid stabilization method proposed in the previous section. Due to space limitation, we give examples for Cases B and C.

A. Example for Case B: $C = [1 \;\; 0]$

Although this case has been studied in [7], the approach proposed in this paper is different from that used in [7]. For comparison and integrity, we give an illustrative example.

We assume that $a = 4$, $b = -13$ in the system (1.1) with (1.6) and the initial state is $[x_1 \;\; x_2]^T = [-10 \;\; 10]^T$. In this case, we can reach all the three divergence patterns. Since it is preferrable to reach the divergence pattern (i) so as to obtain global stabilizability, we choose one feedback gain k such that the characteristic equation of the closed-loop system has complex solutions. According to the previous section, we should choose k satisfying (3.6). Substituting a and b into (3.6) leads to $k < 9$. Here, we choose one feedback gain $k_2 = -2$. No matter where the initial state is, we can always drive the system trajectory into the fourth quadrant by activating this feedback gain.

Next, we design another feedback gain by reaching the divergence pattern (ii). Analyzing the characteristic equation

$$s^2 - as - (b + k) = (s + \alpha)(s - \beta) = 0\,, \tag{4.1}$$

we obtain $-a = \alpha - \beta$, $b + k = \alpha\beta$ and thus

$$k = \alpha^2 + a\alpha - b\,. \tag{4.2}$$

Choose $\alpha = 4$ to obtain the feedback gain $k_1 = 45$. As before, we further choose $x_2 = -2x_1$ and $x_2 = -0.5x_1$ as the switching lines. Fig. 5 depicts the convergence of the system state in this case. Note that in this case the available information for switching is $y = [1 \;\; 0]x = x_1$, we have to drive the system's initial state to x_2-axis, where $x_1 = 0$ can be measured, and then use time-controlled switching method. The precise switching time intervals can be computed as described in the previous section, and are thus omitted. □

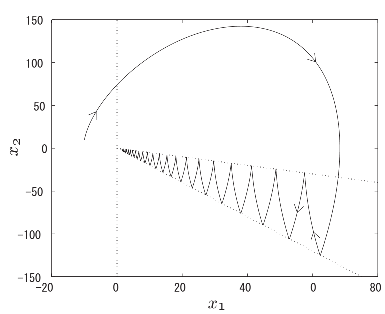

Fig. 5. Example for Case B

B. Example for Case C: $C = [1 \;\; -1]$

Although there are two subcases in this case, for brevity we give the example for the subcase where $C = [1 \;\; -1]$ and $0 \leq a + b < 1$.

We assume that $a = -2.7$, $b = 3.5$, and the initial state is $[x_1 \;\; x_2]^T = [10 \;\; 15]^T$. As explained in the previous section, we can choose one feedback gain so that the divergence pattern (i) is reached and then design another feedback gain to reach the divergence pattern (ii). For the former step, we choose the gain k so as to satisfy (3.8), which turns out to be $(k+4.7)^2 < 0.8$. Thus, we choose $k_2 = -4.4$ so that no matter where the initial state is, we can always drive the trajectory into the fourth quadrant. For the latter step, we use the characteristic equation

$$s^2 - (a-k)s - (b+k) = (s+\alpha)(s-\beta) = 0 \qquad (4.3)$$

to obtain $k = \frac{\alpha^2 + a\alpha - b}{1+\alpha}$. As before, we choose $\alpha = 4$, which requires the feedback gain be $k_1 = 0.34$, and choose $x_2 = -2x_1$ and $x_2 = -0.5x_1$ as the switching lines. Fig. 6 depicts the convergence of the system state in this case. Note that in this case the available information for switching is $y = [1 \quad -1]x = x_1 - x_2$, we have to drive the system's initial state to the line $y = x_1 - x_2 = 0$, and then use time-controlled switching method. The precise switching time intervals after that can be computed as described before. □

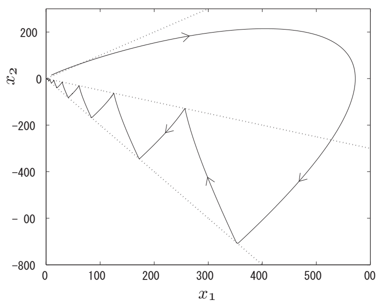

Fig. 6. Example for Case C

V. Concluding Remarks

As an extension to the existing work [7], we have considered a hybrid static output feedback stabilization problem for general two-dimensional LTI systems. The proposed hybrid stabilizing method is composed of two static output feedback gains and a switching law. The main idea is to characterize the divergence patterns of the closed-loop system when a single static output feedback gain is used, and then to design two feedback gains so that two asymptotic lines are obtained in the fourth (or second) quadrant. For benefit of robustness two switching lines are designed inside the area surrounded by the two asymptotic lines. We have used various examples to demonstrate the hybrid stabilization method, and have shown that in some cases the stabilizability depends on the region of the initial state, while in other cases the system is globally stabilizable.

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Appendix

Proof of Lemma. First prove (3.16). Use $\alpha_1\beta_1 = b$ to rewrite (3.16) as

$$\left(\frac{\alpha_1\alpha_1' + b}{\alpha_1\alpha_2' + b}\right)^{\frac{b}{\alpha_1^2+b}} \left(\frac{\alpha_1 - \alpha_2'}{\alpha_1 - \alpha_1'}\right)^{\frac{-\alpha_1^2}{\alpha_1^2+b}} < 1\,. \qquad (A.1)$$

Furthermore, set $\alpha_1' = p\alpha_1,\ \alpha_2' = q\alpha_1 (0 < q < p < 1)$ and $\alpha_1^2 = z$ to rewrite the above inequality as

$$\left(\frac{pz+b}{qz+b}\right)^{\frac{b}{z+b}} \left(\frac{1-q}{1-p}\right)^{\frac{-z}{z+b}} < 1 \qquad (A.2)$$

and equivalently,

$$\frac{b}{z+b}\ln\left(\frac{pz+b}{qz+b}\right) - \frac{z}{z+b}\ln\frac{1-q}{1-p} < 0\,. \qquad (A.3)$$

If we introduce the function

$$f(z) = b\ln\left(\frac{pz+b}{qz+b}\right) - z\ln\frac{1-q}{1-p} \qquad (A.4)$$

and can show $f(z) < 0$, then (3.16) is true.

Differentiating $f(z)$ with respect to z results in

$$f'(z) = \frac{b^2(p-q)}{(pz+b)(qz+b)} - \ln\frac{1-q}{1-p} \qquad (A.5)$$

and

$$f''(z) = -\frac{b^2(p-q)(pqz^2 + b(p+q)z + b^2)}{(pz+b)^2(qz+b)^2}\,. \qquad (A.6)$$

Since $b > 0,\ 0 < q < p < 1$, we obtain $f''(z) < 0$ and that $f'(z)$ is monotone decreasing.

Next, since

$$\begin{aligned} f'(0) &= p - q - \ln\frac{1-q}{1-p} \\ &= (p + \ln(1-p)) - (q + \ln(1-q))\,, \quad (A.7) \end{aligned}$$

we consider the function $g(x) = x + \ln(1-x)$ on the interval $0 \le x < 1$. Due to $g'(x) = -\frac{x}{1-x} \le 0$, we obtain that $g(x)$ is also monotone decreasing, and $g(x) \le 0$ since $g(0) = 0$. Thus, $f'(0) = g(p) - g(q) < 0$ and $f(z)$ is monotone decreasing. Since $f(0) = 0$, we conclude that $f(z) \le 0$. This completes the proof of (3.16).

The proof of (3.17) is easier. It is equivalent to

$$\left(\frac{\alpha_2' + \beta_2}{\alpha_1' + \beta_2}\right)^{\frac{\beta_2}{\alpha_2+\beta_2}} \left(\frac{\alpha_2 - \alpha_2'}{\alpha_2 - \alpha_1'}\right)^{\frac{\alpha_2}{\alpha_2+\beta_2}} < 1\,. \qquad (A.8)$$

Using the fact of $\alpha_1 > \alpha_1' > \alpha_2' > \alpha_2 > 0,\ \beta_1 > 0,\ \beta_2 > 0$, we obtain easily that

$$\begin{cases} 0 < \dfrac{\alpha_2' + \beta_2}{\alpha_1' + \beta_2} < 1, & 0 < \dfrac{\beta_2}{\alpha_2 + \beta_2} < 1, \\ 0 < \dfrac{\alpha_2 - \alpha_2'}{\alpha_2 - \alpha_1'} < 1, & 0 < \dfrac{\alpha_2}{\alpha_2 + \beta_2} < 1\,. \end{cases} \qquad (A.9)$$

Then, according to the above inequalities, (A.8) can be easily shown. □