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An Algorithm for Dynamic Linearization by Generalized Output Injection

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Abstract— **In this paper, a constructive algorithm is proposed for solving a generalized characteristic equation for a single-output nonlinear system. By a reasonable restriction of systems, the proposed method is feasible in the sense that we can obtain the functions satisfying the generalized characteristic equation step by step instead of getting them simultaneously. The proposed algorithm can be used for a dynamic observer error linearization by generalized output injection for a single-output nonlinear system.**

I. INTRODUCTION

The observer error linearization problem posed by [1], [2] has gained much interest and a number of extensions are available in the literature (see [3]–[10] and references therein). One of the main issues of this problem is to develop a constructive algorithm to obtain the transformation. We note that some of the aforementioned results provide fully constructive algorithms, especially for a single-output case. In this respect, the approaches in [7], [8], [10] contain open problems on constructive algorithms, although they considerably enlarged the class of observer error linearizable systems. The main reason is that their characterization is represented by a differential equation (ordinary or partial one) with multiple unknowns.

In this paper, we focus on the result in [10] which covers the works of [7], [8] as well as the conventional observer error linearization problem, and provide a constructive algorithm to check the transformability. More specifically, we consider a special case of systems treated in [10]; the system is in observable form and the nth time derivative of the output is a polynomial in unobservable states. As will be seen later, this restriction is reasonable and makes the problem feasible.

The paper is organized as follows. The problem formulation and notations are given in Section II. Section III contains the main results. An illustrative example is given in Section IV and some conclusions follow in Section V.

II. PROBLEM FORMULATION AND NOTATIONS

Consider a nonlinear system described by

$$\begin{aligned} \dot{\xi} &= f(\xi), \quad \xi \in \mathbb{R}^n \\ y &= h(\xi) \end{aligned} \tag{2.1}$$

with an initial condition $\xi(0) = \xi_0$, where $h(\cdot)$ is a smooth function on $\mathbb{R}$ and $f(\cdot)$ is a smooth vector field on $\mathbb{R}^n$. We will assume $n \geq 2$ throughout the paper.

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In this section, we recall the problem of the dynamic observer error linearization by generalized output injection and introduce some notations which will be used throughout the paper.

Definition 1: ([10]) If there exist an auxiliary dynamic system

$$\begin{aligned} \dot{w} &= \alpha(w, y), \quad w \in \mathbb{R}^q, \\ y_e &= H(w, y), \quad y_e \in \mathbb{R} \end{aligned} \tag{2.2}$$

and an extended state space transformation

$$z = T(w, \xi), \quad z \in \mathbb{R}^{n+q}$$

defined in V_0, a neighborhood of an extended initial state $(0, \xi_0) \in \mathbb{R}^{n+q}$ with $T(0, \xi_0) = 0$ which transforms the extended system

$$\begin{aligned} \dot{x} \triangleq \begin{pmatrix} \dot{w} \\ \dot{\xi} \end{pmatrix} &= F(x) \triangleq \begin{pmatrix} \alpha(w, h(\xi)) \\ f(\xi) \end{pmatrix}, \\ y_e = H(x) &\triangleq H(w, h(\xi)) \end{aligned}$$

into the generalized nonlinear observer canonical form (GNOCF) in $\mathbb{R}^{n+q}$

$$\begin{aligned} \dot{z} &= A_o z + a(w, y), \quad z \in \mathbb{R}^{n+q}, \\ \bar{y}_e &= C_o z, \quad \bar{y}_e \in \mathbb{R} \end{aligned}$$

with

$$A_o \triangleq \begin{bmatrix} 0 & I_{n+q-1} \\ 0 & 0 \end{bmatrix}_{(n+q)\times(n+q)},$$

$$a(w, y) \triangleq \begin{bmatrix} a_{n+q-1}(w, y) \\ \vdots \\ a_0(w, y) \end{bmatrix}_{(n+q)\times 1},$$

$$C_o \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times(n+q)}$$

then the system (2.1) is said to be (locally) dynamically observer error linearizable by generalized output injection (DOELGOI) via the auxiliary dynamic system (2.2). □

It is easy to construct an observer with exponentially stable linear error dynamics for the system which is dynamically observer error linearizable by generalized output injection via some auxiliary dynamic systems.

We will assume that the system (2.1) is locally observable at ξ_0 in the sense that

$$\text{dimspan}\{dh(\xi_0), dL_f h(\xi_0), \cdots, dL_f^{n-1} h(\xi_0)\} = n$$

throughout the paper. Under the observability assumption, there exists U_0, a neighborhood of ξ_0, such that the system

(2.1) can be transformed into the following observable form:

$$\begin{aligned}\dot{\psi}_i &= \psi_{i+1},\ i = 1, \cdots, n-1\\ \dot{\psi}_n &= f_n(\psi_1, \cdots, \psi_n)\\ y &= \psi_1\end{aligned} \tag{2.3}$$

by the state transformation $[\psi_1, \cdots, \psi_n]^T = T_0(\xi)$, $T_0 : U_0 \mapsto T_0(U_0)$ defined by

$$\psi_i = L_f^{i-1} h(\xi),\ i = 1, \cdots, n.$$

Let q be a given nonnegative integer. If we relabel the states of (2.3) by $x_{q+i} = \psi_i$, $i = 1, \cdots, n$ and add q integrators $\dot{x_i} = x_{i+1}$, $i = 1, \cdots, q$, the extended system is of the form

$$\begin{aligned}\dot{x} &= F(x) = F(x_1, \cdots, x_{n+q})\\ &= \sum_{i=1}^{n+q-1} x_{i+1}\frac{\partial}{\partial x_i} + f_n(x_{q+1}, \cdots, x_{n+q})\frac{\partial}{\partial x_{n+q}}\\ y_e &= \bar{h}(x_1, \cdots, x_{q+1}),\end{aligned} \tag{2.4}$$

where $x_1, \cdots, x_q$ are the states of the auxiliary dynamic system and $x_{q+1}, \cdots, x_{n+q}$ are the states of the original system. The initial state of (2.4) is given by $x_0 \triangleq x(0) = [O,\ T_0(\xi_0)]^T$.

The class of the systems that are dynamically observer error linearizable by generalized output injection via integrators was characterized in [10]. Now, the following problem is under consideration:

Dynamic Observer Error Linearization Problem by Generalized Output Injection via Integrators: Given the system (2.4), find functions $a_0, \cdots, a_{n+q}$ of the available states $x_1, \cdots, x_{q+1}$ satisfying the following generalized characteristic equation

$$\Phi(a_0, \cdots, a_{n+q}|F) \triangleq \sum_{j=0}^{n+q} L_F^j a_j = 0 \tag{2.5}$$

such that the transformation defined by

$$\begin{aligned}T(x) &= \left[T_1(x), \cdots, T_{n+q}(x)\right]^T\\ T_i &= -\sum_{j=1}^{i} L_F^{i-j} a_{n+q+1-j}, i = 1, \cdots, n+q\end{aligned}$$

is a diffeomorphism defined on a neighborhood of an extended initial state x_0. □

The notations used in the paper are defined as follows.

- $\triangleq$: defined as.
- $\mathcal{N}_0$: the set of nonnegative integers.
- $X_a \triangleq (x_1, \cdots, x_{q+1})$: available states.
- $X_{ua} \triangleq (x_{q+2}, \cdots, x_{q+n})$: unavailable states.
- $$D^*(g(X_a)) \triangleq \sum_{j=1}^{q} \frac{\partial g}{\partial x_j} x_{j+1},$$
  where $g(\cdot)$ is a smooth function of X_a.
- $P(X_a|X_{ua})$: the ring of polynomials in X_{ua} with coefficients that are C^∞ functions of X_a.
- The weighted degree of x_j in $P(X_a|X_{ua})$: $\text{Wdeg}(x_j) \triangleq \begin{cases} 0 & \text{if } j \le q+1 \\ j-q-1 & \text{if } j > q+1 \end{cases}$.
- The weighted degree of the monomial $x_{j_1} \cdots x_{j_r}$ in $P(X_a|X_{ua})$: $\text{Wdeg}(x_{j_1} \cdots x_{j_r}) \triangleq \text{Wdeg}(x_{j_1}) + \cdots + \text{Wdeg}(x_{j_r})$.
- The weighted degree of the polynomial $f = \sum_{i=1}^{M} d_i(X_a) x_{q+2}^{k_{i1}} x_{q+3}^{k_{i2}} \cdots x_{q+n}^{k_{i(n-1)}}$ in $P(X_a|X_{ua})$ is the maximum of the weighted degree of the monomials $d_i x_{q+2}^{k_{i1}} x_{q+3}^{k_{i2}} \cdots x_{q+n}^{k_{i(n-1)}}$ with $d_i \neq 0$ for $i = 1, \cdots, M$.
- $P^k(X_a|X_{ua})$: the set of the polynomials in $P(X_a|X_{ua})$ of which weighted degree is less than or equal to k.

Unfortunately, the generalized characteristic equation (2.5) is a partial differential equation with $n+q+1$ unknowns and quite complicated to solve. However, solving (2.5) may become a feasible problem, if (2.5) has a special structure defined as follows.

Definition 2: Consider the generalized characteristic equation (2.5) for (2.4). If there exist sets of operators $\mathcal{D}_i = \{D_{i,j}|j = 1, \cdots, \alpha_i\}$, $i = 0, \cdots, n+q$ for positive integers α_i, $i = 0, \cdots, n+q$ and functions $F_{i,j}(\cdot)$ on $\mathbb{R}^{i+1}$ such that

$$D_{i,j}(\Phi) = F_{i,j}(a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i})$$

for $i = 0, \cdots, n+q; j = 1, \cdots, \alpha_i$, then it is said that the generalized characteristic equation (2.5) has a block triangular structure with $\mathcal{D}_i$, $i = 0, \cdots, n+q$. If $\alpha_i = 1$ for each $i = 0, \cdots, n+q$, then it is said that the generalized characteristic equation (2.5) has a triangular structure with $\mathcal{D}_i = \{D_{i,1}\}$, $i = 0, \cdots, n+q$.

III. Main result

Consider the system described by (2.4). We define the following set of exponents with weighted degree $k \in \mathcal{N}_0$ by

$$I_k \triangleq \{(e_1, \cdots, e_{n-1}) \mid \sum_{i=1}^{n-1} i e_i = k, e_i \in \mathcal{N}_0\}.$$

Let $\mu[k] \triangleq \text{card } I_k$. We will denote members of I_k by $e[k, j]$, $1 \le j \le \mu[k]$. $e[k, j]$ can be denoted by

$$e[k, j] = (e[k, j, 1], \cdots, e[k, j, n-1]).$$

Let $e[k, i] = (e[k, i, 1], \cdots, e[k, i, n-1]) \in I_k$ and $e[k, j] = (e[k, j, 1], \cdots, e[k, j, n-1]) \in I_k$. We say that $e[k, i] >_{lex} e[k, j]$ if in the vector difference $e[k, i] - e[k, j]$, the left-most non-zero entry is positive. We say that $e[k, i] =_{lex} e[k, j]$ if $e[k, i, \ell] = e[k, j, \ell]$ for each $\ell = 1, \cdots, n-1$. We will give the following lexicographic ordering to I_k: if $i > j$ then $e[k, i] >_{lex} e[k, j]$. For example, let $n = 4$ and $k = 3$. Then, we have $I_3 = \{(3, 0, 0), (1, 1, 0), (0, 0, 1)\}$ and $\mu[3] = 3$. Since $(3, 0, 0) >_{lex} (1, 1, 0) >_{lex} (0, 0, 1)$, we have $e[3, 1] = (0, 0, 1)$, $e[3, 2] = (1, 1, 0)$ and $e[3, 3] = (3, 0, 0)$. It can be easily seen that $' >'_{lex}$ is a well-defined ordering.

We denote

$$\bar{X}_{ua}^{e[k,j]} \triangleq \prod_{i=1}^{n-1} x_{q+i+1}^{e[k,j,i]}.$$

It can be easily shown that $e[k,j,i]=0$ for $i>k$. Hence it follows that

$$\bar{X}_{ua}^{e[k,j]}=\prod_{i=1}^{k}x_{q+i+1}^{e[k,j,i]} \tag{3.6}$$

for $k\le n-1$. To present the main result, the following preliminary results are needed.

Lemma 3: Consider the system described by (2.4). For any smooth function $a(X_a)$ on $\mathbb{R}^{q+1}$,

$$L_F^k a\in P^k(X_a|X_{ua})$$

for $k=0,\cdots,n-1$.

Proof: The statement is true for $k=0$. As an induction hypothesis, assume that the statement is true for $k=m$ for a nonnegative integer $m<n-1$. Then, there exist smooth functions of X_a on $\mathbb{R}^{q+1}$, $b_{[m:\ell,j]}(X_a)$'s, such that

$$L_F^m a=\sum_{\ell=0}^{m}\sum_{j=1}^{\mu[\ell]}b_{[m:\ell,j]}(X_a)\bar{X}_{ua}^{e[\ell,j]},$$

where $e[\ell,j]\in I_\ell$. From it, we can compute $L_F^{m+1}a$ as follows:

$$L_F^{m+1}a=L_FL_F^m a=\sum_{\ell=0}^{m}\sum_{j=1}^{\mu[\ell]}\Bigg\{\{D^*(b_{[m:\ell,j]}) + \frac{\partial b_{[m:\ell,j]}}{\partial x_{q+1}}x_{q+2}\}\bar{X}_{ua}^{e[\ell,j]}+b_{[m:\ell,j]}L_F\bar{X}_{ua}^{e[\ell,j]}\Bigg\}.$$

Since $D^*(b_{[m:\ell,j]})\in P^0(X_a|X_{ua})$ and $\frac{\partial b_{[m:\ell,j]}}{\partial x_{q+1}}x_{q+2}\bar{X}_{ua}^{e[\ell,j]}\in P^{\ell+1}(X_a|X_{ua})$, we have

$$\sum_{\ell=0}^{m}\sum_{j=1}^{\mu[\ell]}\Bigg\{\{D^*(b_{[m:\ell,j]})+\frac{\partial b_{[m:\ell,j]}}{\partial x_{q+1}}x_{q+2}\}\bar{X}_{ua}^{e[\ell,j]}\Bigg\} \in P^{m+1}(X_a|X_{ua}).$$

From (3.6), we can compute $L_F\bar{X}_{ua}^{e[\ell,j]}$. Then, we have

$$L_F\bar{X}_{ua}^{e[\ell,j]} = \sum_{\substack{r=1\\ e[\ell,j,r]\ge 1}}^{\ell}\Bigg\{\prod_{\substack{i=1\\ i\ne r, i\ne r+1}}^{\ell}x_{q+i+1}^{e[\ell,j,i]}\Bigg\}e[\ell,j,r]x_{q+r+1}^{e[\ell,j,r]-1}x_{q+r+2}^{e[\ell,j,r+1]+1}.$$

Since the weighted degree is increased by one from the contribution of $x_{q+r+1}^{e[\ell,j,r]-1}x_{q+r+2}^{e[\ell,j,r+1]+1}$, it follows that $L_F\bar{X}_{ua}^{e[\ell,j]}\in P^{\ell+1}(X_a|X_{ua})$. Therefore, we can conclude that

$$L_F^{m+1}a\in P^{m+1}(X_a|X_{ua}).$$

Hence, the statement is true for $k=m+1$. ■

In the remaining part of the paper, we will assume that $f_n\in P^M(X_a|X_{ua})$ for $M\in\mathcal{N}_0$, i.e. there exist smooth functions, $\beta_{[\ell,j]}(x_{q+1})$'s, such that

$$f_n=\sum_{\ell=0}^{M}\sum_{j=1}^{\mu[\ell]}\beta_{[\ell,j]}(x_{q+1})\bar{X}_{ua}^{e[\ell,j]}, \tag{3.7}$$

where $e[\ell,j]\in I_\ell$ for each $\ell=0,\cdots,M$. If an n-dimensional nonlinear system in the observable form (2.3) is transformed into the n-dimensional nonlinear observer canonical form, $f_n\in P^n(x_1|x_2,\cdots,x_n)$ (see Proposition 3.3 in [3]). Hence, the assumption (3.7) is not too restrictive. Of course, there exist systems which are dynamically observer error linearizable by generalized output injection but do not satisfy (3.7).

Lemma 4: Consider the integrator-extended system (2.4). Suppose that $f_n\in P^M(X_a|X_{ua})$ for $M\in\mathcal{N}_0$. Then, there exist $q+1$ positive integers $\{\sigma(1),\sigma(2),\cdots,\sigma(q+1)\}$ satisfying $\sigma(1)<\sigma(2)<\cdots<\sigma(q+1)$ such that for any smooth function $a(X_a)$, $L_F^{n+k-1}a\in P^{\sigma(k)}(X_a|X_{ua})$ for $k=1,\cdots,q+1$.

Proof:

Define the integers $\sigma(1),\cdots,\sigma(q+1)$ by

$$\sigma(k)=\begin{cases}n+k-1, M\le n\\ M+(M-n+1)(k-1), M>n\end{cases} \tag{3.8}$$

for $k=1,\cdots,q+1$. Since the rest of the proof is somewhat tedious, it is omitted. The whole proof is available in the homepage: myhome.naver.com/yktak90/goisingle.pdf. ■

In order to simplify notations, the following differential operator is introduced:

$$\Delta_{e[\ell,j]}\triangleq K\cdot\frac{\partial^{e[\ell,j,1]+\cdots+e[\ell,j,n-1]}}{\partial x_{q+2}^{e[\ell,j,1]}\partial x_{q+3}^{e[\ell,j,2]}\cdots\partial x_{q+n}^{e[\ell,j,n-1]}}$$

where K is given by

$$K\triangleq\frac{1}{(e[\ell,j,1]!)(e[\ell,j,2]!)\cdots(e[\ell,j,n-1]!)}$$

for each $e[\ell,j]\in I_\ell$.

The main result of the paper is given as follows.

Theorem 5: Consider the integrator-extended system (2.4). Suppose that $f_n\in P^M(X_a|X_{ua})$ for some integer $M\ge 0$. Then, the generalized characteristic equation (2.5) has a block triangular structure with $\mathcal{D}_i$, $i=0,\cdots,n+q$, which are given by

(1) if $M\le n:\mathcal{D}_i\triangleq\{\Delta_{e[n+q-i,j]}|j=1,\cdots,\mu[n+q-i]\}$ for $i=0,\cdots,n+q$,

(2) if $M>n:\mathcal{D}_i\triangleq\{\Delta_{e[n+q-i,j]}|j=1,\cdots,\mu[n+q-i]\}$ for $i=q+1,\cdots,n+q$,

$\mathcal{D}_i\triangleq\cup_{\ell=1}^{M-n+1}\{\Delta_{e[\sigma(q-i)+\ell,j]}|j=1,\cdots,\mu[\sigma(q-i)+\ell]\}$ for $i=0,\cdots,q$, (3.9)

where $\sigma(i)$'s are given by (3.8) and we set $\sigma(0)=n-1$.

Proof:

(1) Case 1: $M\le n$

From Lemma 3 and Lemma 4, it follows that $L_F^k a_k\in P^k(X_a|X_{ua})$ for $k=0,\cdots,n+q$. Hence we can easily deduce that

$$\Delta_{e[i,j]}(L_F^k a_k)=0 \tag{3.10}$$

for $i > k$. If we operate $\Delta_{e[n+q-i,j]}$ on (2.5), we have

$$\Delta_{e[n+q-i,j]}(\Phi) = \Delta_{e[n+q-i,j]}(\sum_{k=0}^{n+q} L_F^k a_k) \\ = F_{i,j}(a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i}) = 0 \tag{3.11}$$

for $i = 0, \cdots, n+q; j = 1, \cdots, \mu[n+q-i]$.

(2) Case 2: $M > n$

From Lemma 3, it follows that $L_F^k a_k \in P^k(X_a|X_{ua})$ for $k = 0, \cdots, n-1$. If we operate $\Delta_{e[n+q-i,j]}$ on (2.5), we have

$$\Delta_{e[n+q-i,j]}(\Phi) = \Delta_{e[n+q-i,j]}(\sum_{k=0}^{n+q} L_F^k a_k) \\ = F_{i,j}(a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i}) = 0 \tag{3.12}$$

for $i = q+1, \cdots, n+q; j = 1, \cdots, \mu[n+q-i]$. Recall that $\sigma(k) = M + (M-n+1)(k-1)$ for $k = 1, \cdots, q+1$ and $\sigma(0) = n-1$. Hence, it can be easily verified that $\sigma(i+1) - \sigma(i) = M - n + 1$ for each $i = 0, \cdots, q$. From Lemma 4, it follows that $L_F^k a_k \in P^{\sigma(k-n+1)}(X_a|X_{ua})$ for $k = n, \cdots, n+q$. If we operate $\Delta_{e[\sigma(q-i)+\ell,j]}$ on (2.5), we have

$$\Delta_{e[\sigma(q-i)+\ell,j]}(\Phi) = \Delta_{e[\sigma(q-i)+\ell,j]}(\sum_{k=0}^{n+q} L_F^k a_k) \\ = \tilde{F}_{i,j,\ell}(a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i}) = 0 \tag{3.13}$$

for $i = 0, \cdots, q;\ell = 1, \cdots, M-n+1;j = 1, \cdots, \mu[\sigma(q-i)+\ell]$. Renumbering the functions, $\tilde{F}_{i,j,\ell}$ by

$$F_{i,j} \triangleq \tilde{F}_{i,j,1}, \quad j = 1, \cdots, \mu[\sigma(q-i)+1]$$
$$F_{i,j+\mu[\sigma(q-i)+1]} \triangleq \tilde{F}_{i,j,2}, \quad j = 1, \cdots, \mu[\sigma(q-i)+2]$$
$$F_{i,j+\mu[\sigma(q-i)+1]+\mu[\sigma(q-i)+2]} \triangleq \tilde{F}_{i,j,3}, \\ j = 1, \cdots, \mu[\sigma(q-i)+3]$$
$$\vdots$$
$$F_{i,j+\sum_{k=1}^{M-n}\mu[\sigma(q-i)+k]} \triangleq \tilde{F}_{i,j,M-n+1}, \\ j = 1, \cdots, \mu[\sigma(q-i)+M-n+1]$$

for $i = 0, \cdots, q$ completes the proof. ■

Proposition 1: Consider the integrator-extended system (2.4). Suppose that $f_n \in P^M(X_a|X_{ua})$, $M \in \mathcal{N}_0$ and $M \le n$ ($M > n$). Then, the generalized characteristic equation (2.5) has a solution $\{a_0(X_a), \cdots, a_{n+q}(X_a)\}$ if and only if it is a solution to (3.11)((3.12) and (3.13) respectively).

Proof: What we have presented until now is nothing but arranging terms of $L_F^k a_k$ as polynomials in X_{ua} for $k = 0, \cdots, n+q$. Then, we equate all the coefficients of the monomials in X_{ua} of $\sum_{k=0}^{n+q} L_F^k a_k$ to zero. $\Delta_{e[\ell,j]}(\cdot)$ is used to pick up the coefficient of the jth monomial with weighted degree ℓ. Since all the coefficients of the polynomial are zero if and only if the polynomial is zero in the neighborhood of the initial state of X_{ua}, the proposition follows. ■

Using Theorem 5 and Proposition 1, we can solve the generalized characteristic equation (2.5) for (2.4) as follows. Obtain a_{n+q} satisfying $F_{0,j}(a_{n+q}) = 0$ for all possible j ($j = 1, \cdots, \mu[n+q]$ if $M \le n$). Given a_{n+q}, obtain a_{n+q-1} by solving $F_{1,j}(a_{n+q}, a_{n+q-1}) = 0$ for all possible j and so on. If there does not exist a solution in $F_{i,j}$ for any (i,j), the system is not DOELGOI via integrators. This step by step approach is more feasible than direct calculations of the generalized characteristic equation (the number of terms in $L_F^k a$ increases proportional to $(k+1)!$ for $q = 1$). We propose the following algorithm solving Dynamic Observer Error Linearization Problem by Generalized Output Injection via Integrators for the restricted class of the system (2.4) satisfying (3.7).

DOELGOI-Algorithm - the Dynamic Observer Error Linearization by Generalized Output Injection Algorithm Consider the integrator-extended system (2.4) satisfying (3.7).

Step 0: Given integers n, q and M, construct sets of operators as (3.9) and block triangular equations $F_{i,j}$'s as (3.11) if $M \le n$ or (3.12) and (3.13) if $M > n$. Obtain a_{n+q} satisfying $F_{0,j}(a_{n+q}) = 0$ for all possible j.

Step i: $(i = 1, \cdots, n+q)$ Given functions $a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i+1}$, get a_{n+q-i} satisfying $F_{i,j}(a_{n+q}, a_{n+q-1}, \cdots, a_{n+q-i}) = 0$ for all possible j. If $\{dT_1, \cdots, dT_i\}$ is linearly independent at x_0, where $T_k = -\sum_{j=1}^{k} L_F^{k-j} a_{n+q+1-j}$ for $k = 1, \cdots, i$, go to Step $i+1$ until $i = n+q$. If there does not exist a solution making $\{dT_1, \cdots, dT_i\}$ linearly independent at x_0, stop.

Proposition 2: Consider the system (2.4). Suppose that $f_n \in P^M(X_a|X_{ua})$, $M \in \mathcal{N}_0$. Then, the dynamic observer error linearization problem by generalized output injection via integrators is solvable if and only if DOELGOI-A terminates at Step $n+q$.

IV. ILLUSTRATIVE EXAMPLE

We consider the case $n = 3, q = 1$ and $M = 2$ with $x(0) = 0$. Note that $X_a = (x_1, x_2)$ and $X_{ua} = (x_3, x_4)$. We assume that $a_4(x_1, x_2) = -x_1$ and f_3 is given by

$$f_3 = \beta_{[0,1]}(x_2) + \beta_{[1,1]}(x_2)x_3 + \beta_{[2,1]}(x_2)x_4 + \beta_{[2,2]}(x_2)x_3^2.$$

Since the generalized characteristic equation is given by

$$\begin{aligned}
\Phi = \ & b_{[0:0,1]} && : L_F^0 a_0 \\
& + b_{[1:0,1]} + b_{[1:1,1]}x_3 && : L_F^1 a_1 \\
& + b_{[2:0,1]} + b_{[2:1,1]}x_3 + b_{[2:2,1]}x_4 + b_{[2:2,2]}x_3^2 && : L_F^2 a_2 \\
& + b_{[3:0,1]} + b_{[3:1,1]}x_3 + b_{[3:2,1]}x_4 + b_{[3:2,2]}x_3^2 && \\
& + b_{[3:3,1]}x_3x_4 + b_{[3:3,2]}x_3^3 && : L_F^3 a_3 \\
& + b_{[4:0,1]} + b_{[4:1,1]}x_3 + b_{[4:2,1]}x_4 + b_{[4:2,2]}x_3^2 && \\
& + b_{[4:3,1]}x_3x_4 + b_{[4:3,2]}x_3^3 && \\
& + b_{[4:4,1]}x_4^2 + b_{[4:4,2]}x_3^2x_4 + b_{[4:4,3]}x_3^4, && : L_F^4 a_4
\end{aligned}$$

where

$$
\begin{aligned}
&b_{[0:0,1]}(x_1,x_2)=a_0(x_1,x_2),\ b_{[1:0,1]}(x_1,x_2)=x_2\frac{\partial a_1}{\partial x_1},\\
&b_{[1:1,1]}(x_1,x_2)=\frac{\partial a_1}{\partial x_2},\ b_{[2:0,1]}(x_1,x_2)=x_2^2\frac{\partial^2 a_2}{\partial x_1^2},\\
&b_{[2:1,1]}(x_1,x_2)=2x_2\frac{\partial^2 a_2}{\partial x_2\partial x_1}+\frac{\partial a_2}{\partial x_1},\\
&b_{[2:2,1]}(x_1,x_2)=\frac{\partial a_2}{\partial x_2},\ b_{[2:2,2]}(x_1,x_2)=\frac{\partial^2 a_2}{\partial x_2^2},\\
&b_{[3:0,1]}(x_1,x_2)=\frac{\partial^3 a_3}{\partial x_1^3}x_2^3+\frac{\partial a_3}{\partial x_2}\beta_{[0,1]}\\
&b_{[3:1,1]}(x_1,x_2)=3x_2^2\frac{\partial^3 a_3}{\partial x_2\partial x_1^2}+3x_2\frac{\partial^2 a_3}{\partial x_1^2}+\frac{\partial a_3}{\partial x_2}\beta_{[1,1]},\\
&b_{[3:2,1]}(x_1,x_2)=3x_2\frac{\partial^2 a_3}{\partial x_2\partial x_1}+\frac{\partial a_3}{\partial x_1}+\frac{\partial a_3}{\partial x_2}\beta_{[2,1]},\\
&b_{[3:2,2]}(x_1,x_2)=3x_2\frac{\partial^3 a_3}{\partial x_2^2\partial x_1}+3\frac{\partial^2 a_3}{\partial x_2\partial x_1}+\frac{\partial a_3}{\partial x_2}\beta_{[2,2]},\\
&b_{[3:3,1]}(x_1,x_2)=3\frac{\partial^2 a_3}{\partial x_2^2},\\
&b_{[3:3,2]}(x_1,x_2)=\frac{\partial^3 a_3}{\partial x_2^3},\\
&b_{[4:0,1]}(x_1,x_2)=-\beta_{[0,1]},\\
&b_{[4:1,1]}(x_1,x_2)=-\beta_{[1,1]},\\
&b_{[4:2,1]}(x_1,x_2)=-\beta_{[2,1]},\\
&b_{[4:2,2]}(x_1,x_2)=-\beta_{[2,2]},\\
&b_{[4:3,1]}=b_{[4:3,2]}=b_{[4:4,1]}=b_{[4:4,2]}=b_{[4:4,3]}=0
\end{aligned}
$$

we have

$$
\begin{aligned}
&\Delta_{e[3,1]}(\Phi)=F_{1,1}(a_4,a_3)=b_{[4:3,1]}+b_{[3:3,1]}\\
&=b_{[3:3,1]}=3\frac{\partial^2 a_3}{\partial x_2^2}=0\\
&\Delta_{e[3,2]}(\Phi)=F_{1,2}(a_4,a_3)=b_{[4:3,2]}+b_{[3:3,2]}\\
&=b_{[3:3,2]}=\frac{\partial^3 a_3}{\partial x_2^3}=0.
\end{aligned}
\tag{4.14}
$$

Therefore, $a_3(x_1,x_2)$ is of the form

$$
a_3(x_1,x_2)=c_{3,1}(x_1)x_2+c_{3,0}(x_1),
\tag{4.15}
$$

where $c_{3,1}(\cdot)$ and $c_{3,0}(\cdot)$ are smooth functions on $\mathbb{R}$. Since $dT_1=dx_1$ and $dT_2=\frac{\partial a_3}{\partial x_1}dx_1+(1-c_{3,1})dx_2$ should be linearly independent at the origin, we have $c_{3,1}(0)\neq 1$. Next, we need to solve the following equations:

$$
\begin{aligned}
&\Delta_{e[2,1]}(\Phi)=F_{2,1}(a_4,a_3,a_2)=b_{[4:2,1]}+b_{[3:2,1]}+b_{[2:2,1]}\\
&=-\beta_{[2,1]}+3x_2\frac{\partial^2 a_3}{\partial x_2\partial x_1}+\frac{\partial a_3}{\partial x_1}+\frac{\partial a_3}{\partial x_2}\beta_{[2,1]}+\frac{\partial a_2}{\partial x_2}=0\\
&\Delta_{e[2,2]}(\Phi)=F_{2,2}(a_4,a_3,a_2)=b_{[4:2,2]}+b_{[3:2,2]}+b_{[2:2,2]}\\
&=-\beta_{[2,2]}+3x_2\frac{\partial^3 a_3}{\partial x_2^2\partial x_1}+3\frac{\partial^2 a_3}{\partial x_2\partial x_1}+\frac{\partial a_3}{\partial x_2}\beta_{[2,2]}+\frac{\partial^2 a_2}{\partial x_2^2}=0.
\end{aligned}
\tag{4.16}
$$

Substituting (4.15) into (4.16) yields

$$
\begin{aligned}
&4x_2\frac{dc_{3,1}}{dx_1}+\frac{dc_{3,0}}{dx_1}+(c_{3,1}-1)\beta_{[2,1]}+\frac{\partial a_2}{\partial x_2}=0\\
&3\frac{dc_{3,1}}{dx_1}+(c_{3,1}-1)\beta_{[2,2]}+\frac{\partial^2 a_2}{\partial x_2^2}=0.
\end{aligned}
\tag{4.17}
$$

If we differentiate the first equation of (4.17) with respect to x_2 and subtract the second equation of (4.17) from it, we have

$$
\frac{dc_{3,1}}{dx_1}+(c_{3,1}(x_1)-1)\left(\frac{d\beta_{[2,1]}}{dx_2}-\beta_{[2,2]}(x_2)\right)=0.
\tag{4.18}
$$

The differential equation (4.18) for $c_{3,1}(x_1)$ has a solution if and only if $\frac{d\beta_{[2,1]}}{dx_2}-\beta_{[2,2]}(x_2)$ is a constant function. Thus we have the following structural condition:

$$
\frac{d\beta_{[2,1]}}{dx_2}-\beta_{[2,2]}(x_2)=K_1,
\tag{4.19}
$$

where K_1 is a constant. Solving (4.18) under the condition (4.19), we have

$$
c_{3,1}(x_1)=1+K_2\exp(-K_1x_1),
\tag{4.20}
$$

where K_2 is a constant. Hence, $a_3(x_1,x_2)$ is of the form

$$
a_3(x_1,x_2)=(1+K_2\exp(-K_1x_1))x_2+c_{3,0}(x_1).
\tag{4.21}
$$

Substituting (4.20) into the first equation of (4.17) and integrating by x_2 yield

$$
\begin{aligned}
&a_2(x_1,x_2)=2x_2^2K_2K_1\exp(-K_1x_1)-\frac{dc_{3,0}}{dx_1}x_2\\
&-K_2\exp(-K_1x_1)\int\beta_{[2,1]}(x_2)dx_2+c_{2,0}(x_1),
\end{aligned}
\tag{4.22}
$$

where $c_{2,0}(\cdot)$ is a smooth function of x_1. It can be easily verified that dT_1,dT_2 and dT_3 are linearly independent at the origin. We can solve (4.23) for $a_1(x_1,x_2)$

$$
\begin{aligned}
&\Delta_{e[1,1]}(\Phi)=F_{3,1}(a_4,a_3,a_2,a_1)\\
&=b_{[4:1,1]}+b_{[3:1,1]}+b_{[2:1,1]}+b_{[1:1,1]}\\
&=-\beta_{[1,1]}+3x_2^2\frac{\partial^3 a_3}{\partial x_2\partial x_1^2}+3x_2\frac{\partial^2 a_3}{\partial x_1^2}+\frac{\partial a_3}{\partial x_2}\beta_{[1,1]}\\
&+2\frac{\partial^2 a_2}{\partial x_2\partial x_1}x_2+\frac{\partial a_2}{\partial x_1}+\frac{\partial a_1}{\partial x_2}=0
\end{aligned}
\tag{4.23}
$$

by substituting (4.21) and (4.22) into (4.23) and integrating by x_2. Finally, $a_0(x_1,x_2)$ can be obtained by solving

$$
\begin{aligned}
&\Delta_{e[0,1]}(\Phi)=F_{4,1}(a_4,a_3,a_2,a_1,a_0)\\
&=b_{[4:0,1]}+b_{[3:0,1]}+b_{[2:0,1]}+b_{[1:0,1]}+b_{[0:0,1]}\\
&=-\beta_{[0,1]}+\frac{\partial^3 a_3}{\partial x_1^3}x_2^3+\frac{\partial a_3}{\partial x_2}\beta_{[0,1]}+x_2^2\frac{\partial^2 a_2}{\partial x_1^2}\\
&+x_2\frac{\partial a_1}{\partial x_1}+a_0(x_1,x_2)=0.
\end{aligned}
\tag{4.24}
$$

For a numerical example, consider the following system described by

$$
\begin{aligned}
&\dot{\xi}_1=\xi_2,\ \dot{\xi}_2=\xi_3\\
&\dot{\xi}_3=6\xi_2^2+\xi_1^2\xi_2+5\xi_1\xi_3\in P^2(\xi_1|\xi_2,\xi_3)\\
&y=h(\xi)=\xi_1
\end{aligned}
\tag{4.25}
$$

with $\xi(0) = 0$. The system (4.25) is not observer error linearizable via output diffeomorphism since the generalized characteristic equation

$$\Phi(a_0, \cdots, a_3|F) = \sum_{j=0}^{3} L_F^j \bar{a}_j(\xi_1) = 0 \tag{4.26}$$

for (4.25) does not have any solutions. A simple calculation of (4.26) yields

$$\begin{aligned} \mathrm{CE} &= (\bar{a}_2'' + 6\bar{a}_3')\xi_2^2 + (3\xi_3\bar{a}_3'' + \xi_1^2\bar{a}_3' + \bar{a}_1')\xi_2 \\ &+ 5\xi_1\xi_3\bar{a}_3' + \xi_3\bar{a}_2' + \bar{a}_0 = 0. \end{aligned}$$

It is easy to verify that there do not exist functions $\bar{a}_0(\xi_1), \cdots, \bar{a}_3(\xi_1)$ satisfying $\frac{\partial^{3-i}(CE)}{\partial \xi_2^{3-i}} = 0$ for $i = 1, 2, 3$. Moreover, the system (4.25) can not be transformed into the observer form via the output-dependent time-scaling transformation since f_3 is not in the form to which the output-dependent time-scaling transformation method is applicable according to Proposition 1 in [6].

We add a state which is an integral of the output and redefine the system (4.25) as

$$\begin{aligned} &\dot{x_1} = x_2, \ \dot{x_2} = x_3, \ \dot{x_3} = x_4 \\ &\dot{x_4} = 6x_3^2 + x_2^2 x_3 + 5x_2 x_4 \in P_4^{[2]}(x_3) \\ &y_e = \bar{h}(x) = x_1 \end{aligned} \tag{4.27}$$

with an initial state $x(0) = 0$, where $x_1 = \int_0^t \xi_1(\tau)d\tau, \ x_2 = \xi_1, x_3 = \xi_2$ and $x_4 = \xi_3$. If we choose $a_3 = x_2 - e^{x_1}x_2$ and solve the equations for a_2, a_1 and a_0 using DOELGOI-A, we have $a_2 = \frac{9}{2}e^{x_1}x_2^2$, $a_1 = -\frac{31}{6}e^{x_1}x_2^3$ and $a_0 = \frac{5}{3}e^{x_1}x_2^4$. The diffeomorphism

$$z = T(x) = \begin{bmatrix} x_1 \\ e^{x_1}x_2 \\ -\frac{7}{2}e^{x_1}x_2^2 + e^{x_1}x_3 \\ \frac{5}{3}e^{x_1}x_2^3 - 6e^{x_1}x_2x_3 + e^{x_1}x_4 \end{bmatrix}$$

transforms the system (4.27) into the following generalized nonlinear observer canonical form in $\mathbb{R}^4$

$$\begin{aligned} \dot{z} &= A_o z + a(x_1, x_2), \quad z \in \mathbb{R}^4, \\ y_e &= C_0 z = z_1, \quad y_e \in \mathbb{R} \end{aligned}$$

with

$$A_o \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a(x_1, x_2) \triangleq \begin{bmatrix} a_3(x_1, x_2) \\ a_2(x_1, x_2) \\ a_1(x_1, x_2) \\ a_0(x_1, x_2) \end{bmatrix}$$

$$C_o \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, a nonlinear observer for (4.27)

$$\begin{aligned} \dot{\hat{z}} &= A_o\hat{z} + a(x_1, x_2) + K(y_e - C_o\hat{z}), \\ \hat{x} &= T^{-1}(\hat{z}) \end{aligned}$$

with exponentially stable error dynamics can be easily constructed.

V. Conclusion

Recently, it has been found that there exists a class of nonlinear systems which is not dynamically observer error linearizable up to an output injection but is dynamically observer error linearizable by generalized output injection [10]. However, solving the problem of the dynamic observer error linearization by generalized output injection is equivalent to solving a generalized characteristic equation with some constraints. Unfortunately, the generalized characteristic equation is a partial differential equation with $n + q + 1$ unknown functions (n is the number of the states of the original system and q is the number of the states of the auxiliary dynamic system) and it is quite complicated to solve.

In this paper, a constructive algorithm for dynamic observer error linearization problem by generalized output injection has been developed. It is done by imposing a structural restriction on the system: the nth time derivative of the output of the system is a polynomial in the unobservable states. With this restriction, the unknowns of the partial differential equation can be solved one by one. An example was provided to illustrate the proposed algorithm. Further research topic of this paper includes the relaxation of the structural assumption and an algorithm for the problem with more general auxiliary dynamic systems.

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